Size Discovery with Discrete Trading

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ABSTRACT

This paper studies the impact of size-discovery trading protocols on allocative efficiency in markets with discrete trading. I use a dynamic-discrete-trading model with imperfect competition that sequentially offers a size or price-discovery trading session. The frequency at which trade occurs drives the sign of the effect of size-discovery protocols on the market's allocative efficiency. This changes the policy implications of the prior bifurcated academic results and rationalizes why size-discovery protocols only exist in slower markets. Finally, potential conflicts of interest between traders and platform operators are identified but seem unlikely to drive the existence of size-discovery trading protocols.

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I. Introduction

Size-discovery trading protocols allow trade at a *fixed* (frozen) price, regardless of the quantity exchanged, but do not guarantee execution. In contrast, price-discovery trading protocols, such as the popular limit order book, set the price so that the market clears (marketable orders matching with open limit orders). This dichotomy of clearing on price versus quantity is becoming increasingly popular in financial markets[1]. As the electronification of markets increases across many asset classes, Tradeweb and MarketAxess have introduced size-discovery trading protocols in the past three years. Collin-Dufresne et al. (2020) finds that nearly 75% of the interdealer volume of index CDS products is conducted by size-discovery protocols on GFI's SEF.

Despite the growing prevalence of size-discovery trading protocols, recent academic work has called into question the net impact of size-discovery trading protocols on the allocative efficiency of markets. Antill and Duffie (2020) study the allocative efficiency of a market design where size and price-discovery mechanisms occur sequentially, competition is imperfect, and trade is continuous. They find that the existence of a size-discovery trading mechanism strictly lowers the welfare of traders (allocative efficiency of the market). Antill and Duffie (2020) conclude that size-discovery sessions exist from a coordination failure as all traders would be better off colluding to not trade during them (prisoner's dilemma), yet it may still benefit platform operators who may not care directly about the allocative efficiency. However, the empirical reality is that size-discovery trading sessions play an essential role in some asset classes but fail to exist in others.[2] Practitioners and regulators also believe in their usefulness. J. Christopher Giancarlo[3] stated in a white paper that "It is believed that the work-up process increases wholesale trading liquidity in certain OTC swaps by as much as 50 percent." (Giancarlo, 2015).[4] Prior theoretical works fail to explain the existence, or lack thereof, of size-discovery trading protocols across asset classes. This paper will fill that gap

[1] Various forms of size-discovery trading protocols (work-ups, matching sessions, and block-crossing dark pools) can be found in equity, U.S. Treasury, CDS, corporate bond, and interest-rate swap markets.

[2] Duffie and Zhu (2017) find that an initial work-up trading session followed by price-discovery trading sessions would always benefit the allocative efficiency of the market. However, they miss a critical strategic cost of traders delaying trade to wait for better trading terms due to the presence of a size-discovery trading session, which is addressed in Antill and Duffie (2020).

[3] Commissioner and 13^{th} Chairman of the CFTC

[4] Giancarlo further stated as Chairman of the Wholesale Market Brokers' Association, Americas (WMBAA) that "This letter provides additional information on the concept of "work-up" and explains its vital role in liquidity formation in OTC swaps markets. ... The WMBAA believes that prohibiting "work-up" will have significantly harmful implications for OTC swaps markets, to the detriment of market participants in the form of wider bid-ask spreads, less trading volume, and less transparent price discovery."WMBAA (2012)

by providing a rationale for the heterogeneity in existence across different markets.

I use a dynamic model where trade occurs at discrete times, with imperfect competition and adverse selection.[5] Rational traders break up their orders over time to minimize the trading costs incurred when competition is imperfect. Therefore, price-impact avoidance slows trade creating socially costly delays (Vayanos, 1999; Du and Zhu, 2017). Size-discovery trading protocols allow for trading sessions that do not have a price impact, allowing traders to be very aggressive. Nevertheless, the desire to further break up orders during price-discovery trading sessions to wait for size-discovery sessions could offset the allocative efficiency gains from when a size-discovery session does occur. Discretizing trade is the critical economic mechanism for when the allocative efficiency of the markets is improved or harmed by the presence of size-discovery trading protocols. As trading sessions become more frequent, traders have more flexibility in trading, which results in smaller trades per session. However, there is an indirect effect that traders rationally anticipate other traders doing the same. Therefore, market depth is lowered, and price impact increases, which creates a further incentive to break up the orders and wait for better trading terms in a size-discovery session, and this logic repeats. Therefore, welfare loss due to strategic actions increases in the trading frequency and is maximized when trading becomes continuous.[6] A threshold trading frequency always exists such that a size-discovery mechanism improves traders' welfare for any trading frequency slower than the threshold value. This result rationalizes the existence of size-discovery trading sessions on platforms in markets with a slower trading frequency (interdealer segments of U.S. Treasuries, index CDS, interest-rate swaps, and corporate bonds)[7] than markets with a very high trading frequency (equities and futures). Interdealer markets not only tend to have a lower trading frequency, which is associated with a higher allocative efficiency in the model but also tend to have less adverse selection, which is also positively associated with the strategic behavior leading to slower trading and loss in allocative efficiency.

[5] Antill and Duffie (2020) is a limiting case of this model where trade occurs continuously, and there is no adverse selection.

[6] Vayanos (1999) also shows, in a market without size-discovery mechanisms, that the welfare loss of traders due to the strategic avoidance of price impact is at its maximum when trade is continuous. In Du and Zhu (2017), the optimal trading frequency in a market with only price-discovery sessions is only continuous as the number of traders diverges due to this logic.

[7] Collin-Dufresne et al. (2020) document that 5-year IG and HY NA index CDSs average 17 transactions (a transaction every 35 minutes) a day on GFI's Swap Exchange, the leading swap exchange facility for dealers. Fleming and Nguyen (2018) document that the 5-year on-the-run U.S. Treasury averages about 2,700 transactions a day (a transaction every 32 seconds) on BrokerTec, the largest trading platform of the interdealer segment (ADV of $120 billion in 2015).

A second empirical observation that differs across exchanges is what fixed price to use during the size-discovery sessions. Some exchanges use the prior price transacted at during a price-discovery session, as in work-up sessions, and this is the main model of this paper. Others use a price from a different exchange, usually the midpoint. Still, others use what seems to be an exogenous price set by the platform operator that has been conjectured to be set to maximize trade volume. An extension of the model where the price used during the size-discovery session is exogenously supplied, as in matching sessions, which eliminates the strategic effect of extra demand during the price-discovery sessions, adversely affecting the price the size-discovery session operates, shows that welfare is always improved by having the addition of size-discovery sessions. The socially costly strategic delay is more than offset by the ability to be much more aggressive during size-discovery sessions. This provides a new rationale for platform operators to provide the price exogenously instead of relying on a market price. However, exogenously supplying a price may be very costly for platform operators in some asset classes, which helps rationalize why not all markets would add matching sessions despite their strict improvement in allocative efficiency.

In reality, a social planner does not design markets, but a profit-maximizing platform operator whose incentives may not perfectly correlate with those of the traders and, therefore, could rationalize the existence of size-discovery mechanisms despite the potential harm to traders, as conjectured by Antill and Duffie (2020). To help formalize and asses the viability of the potential conflict of interest between traders and a platform operator, a first step is added to the model where the platform operator gets to choose to offer size-discovery trading sessions alongside price-discovery trading sessions or just the price-discovery trading sessions as to maximize the expected volume traded over the expected lifetime of the market. When the size-discovery session is modeled as a work-up session, there is a potential conflict of interest between traders and the platform operator; this conflict of interest does not exist when the size-discovery session is modeled as a matching session. When trade is continuous, as in Antill and Duffie (2020), a platform operator would never offer a size-discovery trading protocol. These new results provide a theoretical rationale to match the reality of size-discovery trading sessions.

This paper uses a dynamic-discrete-trading model with a finite number of traders who have asymmetric information about the asset's fundamental value to study the effects augmenting a price-discovery trading session with randomly occurring size-discovery trading sessions. The baseline model will have the fixed price used in the size-discovery session be the most recent price that cleared the market in the price-discovery session[8]. In subsequent settings, the fixed price used in the size-discovery session is set by a platform operator[9]. As traders know they can trade without price impact when a size-discovery session occurs, they shade their submitted demand schedules in price-discovery sessions than in a market without size-discovery mechanisms. Therefore, the market depth is lowered, which amplifies the price impact associated with price-discovery trading due to the possibility of size-discovery sessions. This gives traders a further incentive to wait for size-discovery sessions to trade. However, the traders trade very aggressively during size-discovery sessions, as they do not internalize any price impact. This extra aggressiveness in trading due to the lack of price impact during size-discovery sessions more than offsets the welfare cost of the induced strategic delay[10]. Therefore, if the price used in size-discovery sessions were unrelated to a trader's trading in price-discovery sessions, as in matching sessions, welfare would always be improved by having randomly occurring size-discovery sessions. However, there is also a secondary negative welfare effect when the price used in a size-discovery trading session is the prior price-discovery trading session, as in work-ups. Given that the size-discovery session uses the price from the previous price-discovery session, traders submit even less aggressive demand schedules during price-discovery sessions to avoid causing the size-discovery session to occur at a less favorable price. However, these incentives are sufficiently small in markets with relatively discrete trading frequencies, where market depth is minimally impacted; yet, traders are still much more aggressive in the size-discovery session. Therefore, when the market is sufficiently slow, adding work-ups to the market design still improves the market's allocative efficiency (traders' welfare). Finally, prices during price-discovery sessions are invariant to adding size-discovery sessions.

In the last part of the paper, I empirically examine the model's results. First, the model is calibrated to different interdealer trading platforms with size-discovery trading sessions. The

[8]This setting best matches size-discovery trading protocols known as work-ups, where volume is "worked up" at the fixed price from the prior transaction on the limit order book for a short amount of time. This type of trading mechanism can be found in the interdealer segment of the secondary market for U.S. government securities, the index CDS, and electronic corporate bond markets.

[9]This setting best matches size-discovery trading protocols known as matching sessions. Matching sessions are initiated by the platform operator, who specifies the trading session's length and the fixed price at which orders are matched. This type of trading mechanism can be found in the interdealer segment of the index CDS and interest-rate swap market.

[10]As shown in Antill and Duffie (2020), when trading becomes continuous, these two welfare effects exactly offset each other.

existence of work-up trading sessions on BrokerTec, in an interdealer segment of on-the-run U.S. Treasuries, shows a decrease in volume and allocative efficiency, all else equal, due to having work-ups because of the high trading frequency and a small number of traders. This matches the fact that BrokerTec stopped offering the work-up protocol in 2021 after being acquired by the CME Group. However, for a platform that trades index CDSs, the allocative efficiency and volume are both increased by the presence of work-up trading sessions. Trading platforms with matching sessions substantially increase the allocative efficiency and total volume traded in market designs that supplement limit-order books or electronic requests for quotes with matching sessions. None of the calibrations suggest a conflict of interest between a platform operator and traders to explain the existence of size-discovery trading sessions, as posited by Antill and Duffie (2020). The calibrations help explain the popularity of size-discovery trading sessions in certain markets due to the large increase in the allocative efficiency and trading volume it induces. Finally, in a case study, when BrokerTec stopped offering a work-up protocol due to being integrated with the CME Group's Globex platform, their main competitor, eSpeed, which offers a work-up protocol, saw a large increase in the average daily volume (ADV), while BrokerTec's ADV substantially decreased. This result is consistent with the hypothesis that traders substituted from BrokerTec to eSpeed to take advantage of the allocative efficiency benefits of the work-up process and inconsistent with a coordination failure between traders.

II. Background[11]

Size-discovery trading sessions have become prominent additions to trading platforms in multiple asset classes. Types of size-discovery trading sessions in practice include work-ups, matching sessions, and block-crossing dark pools. A work-up is a trading session that opens up after a transaction on a limit-order book where the price is fixed at the last executed price, allowing traders to 'work up' volume at that price. Trade continues until there is an idle in trading or a new marketable order appears on the limit-order book. Work-up sessions can be found in the interdealer segment of on-the-run Treasury, index CDS, and interest-rate swap markets, as well as all-to-all trading in corporate bonds. Matching sessions are similar to work-ups but tend to be less frequent and last longer as the assets traded on them are typically thinner markets (corporate bonds, CDSs, interest-rate

[11] Duffie and Zhu (2017) and Antill and Duffie (2020) have detailed overviews of size-discovery trading mechanisms in practice.

swaps). Given the electronification happening in these markets (Riggs et al., 2020; Collin-Dufresne et al., 2020; O'Hara and Zhou, 2021), it is not surprising to find new market mechanisms arising to help liquidity issues. More importantly, matching sessions operate at a price chosen by the platform operator.[12] Block-crossing dark pools are exchanges that trade equities by matching orders at some fixed price, usually between the most current bid and ask on some limit-order book. Block-crossing dark pools tend to operate continuously, and orders can be submitted and fulfilled at any time as long as there is someone on the other side of the trade. Block-crossing dark pools are usually found in equity markets. The key common feature of work-ups, matching sessions, and block-crossing dark pools that are modeled in this paper is that the price is *fixed* by some mechanism that allows unlimited trade at that price, assuming a counterparty is willing to trade. This type of trading session allows trade without any direct price impact but does increase execution risk.

Size-discovery mechanisms are economically important. In 2018, the average daily volume (ADV) for on-the-run US Treasury securities was $362 billion, with 52% traded electronically in the electronic interdealer segment (Brain et al., 2018). In 2015, BrokerTec, which encompasses 75% of the electronic interdealer market, had an ADV of $120 billion with work-ups accounting for 43% to 56% of the trading volume (Fleming and Nguyen, 2018). In 2018, BrokerTec was bought by the CME Group, and when the CME Group merged the BrokerTec platform with their current trading system, CME Globex, for uniformity, they dropped the work-up protocol in January 2021. eSpeed, which is now owned by Tradeweb, the other trading platform in the interdealer market for US Treasuries, continues to use a work-up trading protocol. In the empirical section of this paper, Section VIB, there is a strong shift in volume away from BrokerTec and to eSpeed since 2021. This substitution effect anecdotally shows the importance of the work-up protocol to traders.

In the corporate bond market, for example, MarketAxess started offering in late 2020 bi-weekly matching sessions for US Investment grade, high yield, and European bonds called Mid-X, where trade occurred at a fixed mid-point price determined by a 'composite trading tool' (Barnes, 2020, 2022). Traders submit orders with a direction and size, and then at a specified time, an algorithm matches all orders possible. MarketAxess acknowledges that this protocol saves on cost and helps

[12] GFI's SEF bases the price for a matching session off one or any combination of: "(i) prices of transactions, executable or indicative bids or offers or historical prices on the Trading Platform, other swap execution facilities or markets; (ii) prices derived from transactions, executable or indicative bids or offers or historical prices on the Trading Platform, other swap execution facilities or markets; and (iii) the views of active market participants." GFI (2022)

offload risk in a more efficient way, but at the price of immediacy, as traders must wait for these sessions to occur. This paper will dive into the incentives this creates and the net effect it creates on the allocative efficiency of markets like this one.[13] Matching sessions can also be found on some platforms that trade corporate bonds[14]

Collin-Dufresne et al. (2020) find that 73.5% of the trade volume of CDS high-yield index products is conducted by work-ups and matching sessions on GFI's SEF. Some interest-rate swap markets also use matching sessions, especially when the product is more standardized. There is a stark lack of empirical academic work on the role size-discovery protocols play in the interest-rate swap market.[15] Finally, as of November 2022, Rosenblatt Securities finds that 13.9% of US equity volume was conducted in dark pools.

Given the economic importance of these markets and the dollar volume traded through these differing protocols, academics, regulators, and practitioners have worried and debated the effects of size-discovery protocols on the price formation process and allocative efficiency. However, empirical and theoretical research is still not definitive. Most empirical work focuses on equity markets despite larger percentages of the total trading and dollar volume occurring through size-discovery trading protocols in other asset classes. Degryse et al. (2014) find that a one-standard-deviation increase in trading on dark markets reduces the lit exchange's market depth for that stock by 5.5% for Dutch equities. Other research, such as Farley et al. (2018) and Buti et al. (2022), find that dark trading has zero or a positive effect on the lit market's depth. Consistent with the idea that traders use size-discovery markets to take advantage of private information, Fleming and Nguyen (2018) find that for U.S. Treasuries, on days where private information due to proprietary client order flow information is more likely, work-up trades become more informative. On days when public information is more likely, work-up trades become less informative. Despite the ongoing research, the European Union enacted new rules that cap the volume allowed to be transacted on dark exchanges in January 2018 through the Markets in Financial Instruments Directive II (MiFiD II). However, Johann et al. (2019) found that following this directive, this volume was merely shifted

[13] MarketAxess also recently added work-ups to their limit order book trading for their Live Markets platform (MarketAxess, 2023).

[14] SIFMA (2016) interviewed 19 electronic bond trading platforms and found 8 had matching session protocols (TruMid, GFI, Latium, ICAP, Liquidity Finance, Electronifie, ITG POSIT FI, Codestreet Dealer Pool).

[15] Tradeweb (2014), BGC (2016),Tradition (2021), and GFI (2022) discuss platforms that trade interest rate swaps and WMBAA (2012) and Giancarlo (2015) discuss the importance of these trading protocols in this market.

to “quasi-dark” mechanisms, making the directive mostly ineffective.

From a theoretical viewpoint, several academics have built models to study the effects of size-discovery mechanisms on market efficiency. Zhu (2013) builds a two-period model where, in equilibrium, informed traders prefer lit markets to avoid the execution risk, whereas the liquidity traders prefer the dark market (a size-discovery mechanism) to avoid paying both the spread and the adverse selection of the lit market. Therefore, prices are more informative as all informed traders trade on the lit exchange, and the liquidity traders do not influence the lit price. My model differs by using a dynamic setting, which allows traders to participate in price and size-discovery mechanisms across multiple periods, which minimizes the fear of execution risk in the size-discovery trading sessions and features imperfect competition. My model does not have both trading mechanisms operating simultaneously but sequentially to better fit the market structure of other asset classes than equities. Vayanos (1999) shows that due to traders internalizing their price impact, they become less aggressive in trading, which causes a socially wasteful delay in reaching their desired inventory position, even if trading is continuous. Duffie and Zhu (2017) focus on the allocative efficiency in the interdealer market when considering a size-discovery session at the beginning of trading. They find that this dramatically improves allocative efficiency. However, their model does not consider the strategic delay, and therefore cost, in allocative efficiency that would occur if a size-discovery session happened at any other time than before the price-discovery trading begins. This is addressed by Antill and Duffie (2020), who find that Duffie and Zhu (2017) is the singular special case where allocative efficiency increases when a size-discovery session is included in the model. If a size-discovery session occurred at any time other than before the price-discovery trading begins and trading is continuous, the strategic delay it would induce in traders outweighs the allocative efficiency gains traders receive when size-discovery sessions occur. Antill and Duffie (2020) findings are intuitively troublesome as the existence and widespread prevalence of size-discovery mechanisms in some financial markets seemingly indicate that traders are finding allocative efficiency gains in more than the singular special case unless there is a conflict of interest between platform operators and traders. My paper differs by discretizing trading to better match the trading frequency of the markets of interest, adding asymmetric information, to account for proprietary client flow information, and a platform operator, to study the potential conflict of interest, which will provide a theoretical justification for the existence and rationale of the design of size-discovery trading sessions.

III. Size Discovery Model

This section will develop the main model of interest as well as a benchmark model. The main model will have price or size-discovery sessions randomly occurring in a sequential fashion. A benchmark model (Du and Zhu (2017)), will only have price-discovery sessions. The platform operator's problem is stated but will be further explored in Section IV. The section concludes by exploring the intuition of the equilibrium.

A. Price and Size Discovery

Time, τ, will go from $[0, \infty)$. Trading will happen at clock-times th, where $t \in \{0, 1, 2 \ldots\}$ and $h > 0$ is the length between trading sessions. There are $N \geq 2$ risk-neutral traders trading a divisible asset. The random variable M_t represents the event that a size-discovery trading session occurs at the t^{th} trading session. Therefore, $\mathbb{1}_{M_t}$ is a Bernoulli random variable such that it equals one if a size-discovery session occurs and zero if a price-discovery session occurs at the t^{th} trading session. With probability $P(\mathbb{1}_{M_t} = 0) = 1 - q$, a price-discovery session, modeled by a uniform-price double auction, will occur. With the remaining probability, $P(\mathbb{1}_{M_t} = 1) = q$, a size-discovery session will occur. The realization of event M_t is independent of all other random processes. Given that a price-discovery session will occur in that trading period, each trader submits a demand schedule as a function of the market clearing price. For the uniform-price double auction, a price that clears the market will be selected, and each trader will be allocated the corresponding amount of the asset they demanded at that price. Given the type of equilibrium that will be solved, the market-clearing price will exist and be unique. Each trader then pays (receives) the equilibrium prices times the amount of the asset they were allocated. More formally, trader n at trading period t submits demand schedule $X_{n,th}(p) : \mathbb{R} \to \mathbb{R}$. The market-clearing price, p^*_{th}, is then defined as the price such that

$$\sum_{n=1}^{N} X_{n,th}(p^*_{th}) = 0. \tag{1}$$

If a size-discovery session occurs, traders submit demand quantity $\mu_{n,th} \in \mathbb{R}$ and will be allocated according to a proportional rationing mechanism [16]:

$$Y_{n,th}(\mu) = \mu_{n,th}\left(\mathbb{1}_{\{\mu_{n,th}\sum_j \mu_{j,th}\leq 0\}} + \mathbb{1}_{\{\mu_{n,th}\sum_j \mu_{j,th}>0\}}\left|\frac{\sum_{\{j:\mu_{n,th}\mu_{j,th}\leq 0\}}\mu_{j,th}}{\sum_{\{j:\mu_{n,th}\mu_{j,th}>0\}}\mu_{j,th}}\right|\right). \tag{2}$$

This mechanism is referred to as proportional rationing, as the total allocation is designed to be zero by proportionally rationing the heavy side of demand such that it equals the light side of demand. Importantly, the allocation does not depend on the price, which is the first-order institutional feature of size-discovery trading protocols.

An important institutional feature that will be explored in this paper is what "frozen" price in practice will be used during the size-discovery trading session? Two common approaches are typically used. Work-up trading sessions use the prior price from the price-discovery session. Matching sessions use a price supplied by a platform operator. The choice of which price to use will create different incentives for the traders and, therefore, will affect how they respond to the existence of size-discovery trading protocols. The cash transfer function used for a work-up size-discovery trading session is

$$C^{\text{wu}}_{n,th}(\mu, p_{(t-1)h}) = -p_{(t-1)h}Y_{n,th}(\mu), \tag{3}$$

and the cash transfer function used for a matching size-discovery trading session is

$$C^{\text{ms}}_{n,th}(\mu, p) = -pY_{n,th}(\mu). \tag{4}$$

Since the mechanism has a total allocation of zero, $\sum_{n=1}^{N} Y_{n,th}(\mu) = 0$ and the price paid per unit of asset allocated is constant across traders, $\sum_{n=1}^{N} C_{n,th}(\mu, p) = 0$, the mechanism is strongly budget balanced (there is no surplus or deficit).

Each trader is endowed with some portion of the asset, referred to as the trader's initial inventory. As trade occurs, their inventory will endogenously evolve. The inventory amount held by trader n, prior to the trading session, at clock-time th will be denoted as $z_{n,th}$, where the aggregate time-invariant total amount of the asset will be defined as $Z := \sum_{n=1}^{N} z_{n,th}$ and is known by all traders. The endowment of initial inventory is given exogenously to each trader and denoted $z_{n,0}$. Therefore inventory evolves according to

$$z_{n,(t+1)h} = z_{n,th} + (1 - \mathbb{1}_{M_t})X_{n,th}(p^*_{th}) + \mathbb{1}_{M_t}Y_{n,th}(\mu). \tag{5}$$

[16] An additive rationing mechanism could also be used, $Y_{n,th}(\mu) = \mu_{n,th} - \frac{1}{N}\sum_{n=1}^{N}\mu_{n,th}$. Any allocation mechanism that would allocate the requested quantity in equilibrium would yield the same results. Most size-discovery trading protocols employ time priority instead of rationing if the demanded quantities do not balance, but some do use other algorithms to maximize volume.

The asset will pay a single liquidating dividend at some future random time. The liquidating amount will be the asset's fundamental value at the time of the liquidation, where the fundamental value evolves according to a jump process. The liquidation date will occur at time $\mathcal{T}$ where $\mathcal{T} \sim Exp(r)$, for $r > 0$ and where $\mathbb{E}(\mathcal{T}) = 1/r$. $\mathcal{T}$ is independent of all other random variables in the model. As the expected time for liquidation is finite, the time discounting is normalized to zero for simplicity. Denote information arrival times as $\{T_k\}_{k \geq 1}$, where T_k can be a deterministic or stochastic process.[17] At $T_0 = 0$, the first shock occurs at clock-time zero, and the fundamental value of the dividend will be drawn from and defined as $D_0 \sim N(0, \sigma_D^2)$. Assuming the liquidation time has not occurred yet, the change in the fundamental value of the dividend follows the i.i.d. normal process

$$D_{T_k} - D_{T_{k-1}} \sim N(0, \sigma_D^2). \tag{6}$$

Therefore, the fundamental value of the dividend, assuming the liquidation date has not occurred yet, follows a jump process where

$$D_\tau = D_{T_k}, \text{ if } T_k \leq \tau < T_{k+1}. \tag{7}$$

Each trader at information arrival time T_k receives a noisy signal, S_{n,T_k} about the change in the fundamental value of the dividend where $D_{T_{-1}} := 0$ and

$$S_{n,T_k} = D_{T_k} - D_{T_{k-1}} + \epsilon_{n,T_k}, \text{ where } \epsilon_{n,T_k} \overset{i.i.d.}{\sim} N(0, \sigma_\epsilon^2). \tag{8}$$

Traders also have private values for the dividend, $w_{n,th}$, which is in addition to the fundamental value of the asset. Similarly to the evolution of the fundamental value of the liquidating dividend, the private values are shocked at time $\{T_k\}_{k \geq 0}$, $w_{n,T_{-1}} := 0$ and

$$w_{n,T_k} - w_{n,T_{k-1}} \overset{i.i.d.}{\sim} N(0, \sigma_w^2). \tag{9}$$

Therefore the private values, assuming the liquidation date has not occurred yet, follow a jump process where

$$w_{n,\tau} = w_{n,T_k}, \text{ if } T_k \leq \tau < T_{k+1}. \tag{10}$$

The private value and noisy signal is only observed by the respective trader and never directly disclosed to any other traders, though the aggregate "total" signal will be inferred from prices in

[17]Given prices are martingales, how shocks occur does not affect how a risk-neutral trader behaves today. At equilibrium, the information arrival process only affects a positive scalar of welfare and, therefore, does not affect the sign of the welfare results.

equilibrium. Therefore, the liquidating dividend, if paid at time τ is worth

$$v_{n,\tau} = D_\tau + w_{n,\tau} \tag{11}$$

to trader n per unit of asset held.

Finally, each trader incurs a holding cost of $(\lambda/2)z^2_{n,(t+1)h}$ per unit of clock time if the liquidating dividend has not been paid yet, where all traders know and have the same marginal cost per extra unit of the asset held, $\lambda > 0$. Vives (2011); Rostek and Weretka (2012); Sannikov and Skrzypacz (2016); Du and Zhu (2017); Duffie and Zhu (2017); Antill and Duffie (2020); Chen and Duffie (2021); Chen (2022) all use a similar quadratic holding cost. This can be interpreted as a form of inventory cost, collateral requirements, or risk management, but not technically risk aversion. If the dividend has not been paid at clock-time th, each trader then expects the following duration of the flow cost, $\min(h, \mathcal{T} - th)$ which is expected to be

$$\mathbb{E}\left[\min(h, \mathcal{T} - th) | \mathcal{T} > th\right] = \frac{1 - e^{-rh}}{r} \tag{12}$$

by the memoryless property of the exponential distribution.

Define $H_{n,\tau}$ to be the history (information set) known to by trader at time τ, which is equal to

$$H_{n,\tau} = \{\{S_{n,T_l}, w_{n,T_l}\}_{T_l \le \tau}, \{z_{n,t'h}, X_{n,t'h}, Y_{n,t'h}, \mathbb{1}_{M_{t'}}\}_{t'h < \tau}\}. \tag{13}$$

Each trader knows their noisy signals and private values up to time τ, their inventory path, and the demand schedules that they have submitted. Each trader, from the evolution of inventory and knowing the demand schedule they submitted, can always back out the equilibrium price transacted at any previous period. Note that $H_{n,th}$ does not contain the information of the t^{th} exchange.

Finally, define $V_{n,th}$ as trader n's period-t continuation value right before the t^{th} exchange. Traders then face the following continuation value

$$V_{n,th} = \mathbb{E}\Bigg[\underbrace{\mathbb{1}^c_{M_t}}_{\substack{\text{indicator}\\ \text{for price}\\ \text{discovery}}} \Bigg(\underbrace{-p^*_{th} X^*_{n,th}}_{\substack{\text{cost of}\\ \text{allocation}}} + \underbrace{(1 - e^{-rh})}_{\substack{\text{probability}\\ \text{of liquidation}}} \underbrace{(z_{n,th} + X^*_{n,th}) v_{n,th}}_{\substack{\text{number of shares}\\ \times \text{traders value per share}}} - \underbrace{\frac{(1 - e^{-rh})\lambda}{2r} (z_{n,th} + X^*_{n,th})^2}_{\text{expected inventory cost}} \Bigg) + \tag{14}$$

$$\underbrace{\mathbb{1}_{M_t}}_{\substack{\text{indicator}\\ \text{for size}\\ \text{discovery}}} \Bigg(\underbrace{-p^*_{(t-1)h} Y^*_{n,th}}_{\substack{\text{cost of}\\ \text{allocation}}} + \underbrace{(1 - e^{-rh})}_{\substack{\text{probability}\\ \text{of liquidation}}} \underbrace{(z_{n,th} + Y^*_{n,th}) v_{n,th}}_{\substack{\text{number of shares}\\ \times \text{traders value per share}}} - \underbrace{\frac{(1 - e^{-rh})\lambda}{2r} (z_{n,th} + Y^*_{n,th})^2}_{\text{expected inventory cost}} \Bigg) +$$

$$\underbrace{e^{-rh} V_{n,(t+1)h}}_{\substack{\text{probability liquidation doesn't} \\ \text{occur before next period} \\ \times \text{next period's continuation value}}} \Bigg| \underbrace{H_{n,th}}_{\substack{\text{information} \\ \text{up to time } th}} \Bigg]$$

where $X^*_{n,th}$ is the concise way of writing $X_{n,th}(p^*_{th})$, the equilibrium allocation when a price-discovery session occurs, and $Y^*_{n,th}$ is the equilibrium allocation when a size-discovery trading session occurs. The first term is the trader's net cash flow of acquiring $X^*_{n,th}$ units of the asset at a price p^*_{th}. The second term is the probability that the dividend will liquidate before the next trading session times the trader's new inventory amount times the asset's value per share to the trader. The third term is the expected quadratic holding cost. The middle line has the same interpretation as the top, except for when a size-discovery session occurs. The final term is the probability that the dividend does not liquidate before the next trading period times the next period's continuation value. Iterating this forward, the continuation value for trader n at clock-time th is

$$V_{n,th} = \mathbb{E}\Bigg[\sum_{t'=t}^{\infty} e^{-rh(t'-t)}\Bigg(\mathbb{1}^c_{M_{t'}}\Bigg(-p^*_{t'h}X^*_{n,t'h} + (1-e^{-rh})\Bigg(v_{n,t'h}(X^*_{n,t'h}+z_{n,t'h}) - \frac{\lambda}{2r}(X^*_{n,t'h}+z_{n,t'h})^2\Bigg)\Bigg) \quad (15)$$

$$+\mathbb{1}_{M_{t'}}\Bigg(-p^*_{(t'-1)h}Y^*_{n,t'h} + (1-e^{-rh})\Bigg(v_{n,t'h}(z_{n,t'h}+Y^*_{n,t'h}) - \frac{\lambda}{2r}(z_{n,t'h}+Y^*_{n,t'h})^2\Bigg)\Bigg)\Bigg)|H_{n,th}\Bigg].$$

Definition 1 (Perfect Bayesian Equilibrium)**.** A perfect Bayesian equilibrium is a strategy profile $\{Z_{n,th}\}_{1\leq n\leq N, t\geq 0}$ where each $Z_{n,th}$ depends only on $H_{n,th}$ such that for every trader n and at every path of their information set $H_{n,th}$, trader n has no incentive to deviate from $\{Z_{n,th}\}_{t'\geq t}$. Mathematically, this is expressed as, for every other possible strategy profile $\{\tilde{Z}_{n,th}\}_{t'\geq t}$, it is the case that

$$V_{n,th}\big(\{Z_{n,t'h}\}_{t'\geq t}, \{Z_{i,t'h}\}_{i\neq n, t'\geq t}\big) \geq V_{n,th}\big(\{\tilde{Z}_{n,t'h}\}_{t'\geq t}, \{Z_{i,t'h}\}_{i\neq n, t'\geq t}\big).$$

Note that $Z_{n,th}$ is the pair $(X_{n,th}, Y_{n,th})$, where $X_{n,th}$ is the strategy profile at time th if a price-discovery trading session occurs and $Y_{n,th}$ is the strategy profile at time th if a size-discovery trading session occurs.

To maintain tractability and follow prior literature, the equilibrium strategy that will be the focus of this paper is that which is linear, symmetric, and stationary. The equilibrium demand schedule traders submit for price-discovery sessions is conjectured to be of the following form

$$X_{n,th}(p) = as_{n,th} - bp + dz_{n,th} + fZ. \quad (16)$$

The demand schedule submitted is a function of the total signal[18], a linear combination of the noisy signal about the fundamental value and private value of the asset, the equilibrium price, inventory at the start of the period, and the aggregate inventory amount. When size-discovery trading sessions occur, traders submit a demand quantity that results in the desired inventory position amount[19]:

$$\mu_{n,th} = a_s s_{n,th} + d_s z_{n,th}. \tag{17}$$

In equilibrium, it will be shown that traders are allocated the competitive demand. Therefore, their post-mechanism allocation is such that they do not desire to trade in any future periods unless the information environment changes. This is not surprising as by removing the price impact of trades, traders are able to achieve the first-best allocation (First Welfare Theorem). This will be the desired inventory level that traders will be trying to achieve, which will be used in studying how quickly this is achieved later under different types of market designs. It will be shown later that the equilibrium allocation from the price-discovery trading sessions will be a scaled version of the size-discovery trading mechanisms allocation, which is the maximum aggressiveness a trader optimally can take, as there is no price impact.

Note that α is a measure of the adverse selection the trader faces in the market and the weight a trader places on their private information versus what they infer from the market price. When adverse selection is low (α is high), the trader places the majority of the weight of his expectation

[18]Traders learn about the asset's fundamental value from prices but have to account for their private values and noise in other traders' signals. By defining the "total signal" as

$$s_{n,th} := \frac{\chi}{\alpha} \sum_{l=0}^{k} S_{n,T_l} + \frac{1}{\alpha} w_{n,T_k},$$

where

$$s_{n,\tau} = s_{n,T_k}, \text{ where } \tau \in [T_k, T_{k+1})$$

and by symmetry and the projection theorem for multivariate normals, solving for χ which is the unique solution in $(0,1)$ to

$$\chi = \frac{1/(\chi^2 \sigma_e^2)}{1/(\chi^2 \sigma_D^2) + 1/(\chi^2 \sigma_e^2) + (N-1)/(\chi^2 \sigma_e^2 + \sigma_w^2)}.$$

Then α can be expressed as

$$\alpha = \frac{\chi^2 \sigma_e^2 + \sigma_w^2}{N \chi^2 \sigma_e^2 + \sigma_w^2} > \frac{1}{N}.$$

For further details, see Du and Zhu (2017).

[19]This is the allocation traders would receive at a price-discovery-only model where traders are competitive and treat prices as given (Du and Zhu (2017)). For a finite number of traders, this is "schizophrenic", as pointed out by Hellwig (1980). However, this is fully rational in a size-discovery trading session as traders do not influence price and can be supported in equilibrium given the allocation function.

on the value of their private signals. But as the number of traders increases, or the volatility in the signal's noise about the fundamental value of the asset increases, then $\alpha \to 0$, and the trader puts all the weight on the inferred signal. From prices, traders then update their beliefs about the value of the asset to them as

$$\mathbb{E}(v_{n,th}|H_{n,th}) = \alpha s_{n,th} + \big(1-\alpha\big)\frac{\sum_{i\neq n}^{N} s_{i,th}}{N-1}. \tag{18}$$

Therefore traders weigh their total signal by α and the inferred total signals of the other $N-1$ traders from prices by $1-\alpha$.

Given the conjectured functional form and amount to be submitted in price and size discovery trading sessions, respectively, using equations (16) and (17), the PBE can be solved for and is summarized in Proposition (1).[20]

PROPOSITION 1: *When there are enough traders, there exists a PBE where each trader does as follows. They submit the following demand schedule for price-discovery sessions*[21]*:*

$$X_{n,th}^{s,j}(p) = -d_{s,j}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-p\big) - z_{n,th} + \frac{1-\alpha}{N-1}Z\right) \tag{19}$$

where $-1 \leq d_{s,j} \leq 0$ *is given by equations* (67) *and* (79) *in Appendix A, and* $j \in \{\text{wu}, \text{ms}\}$ *classifies if the size-discovery mechanism is modeled as a work-up or matching session. If a price-discovery trading session occurs, the period-*t *equilibrium price is*

$$p_{th}^{*} = \frac{1}{N}\sum_{n=1}^{N} s_{n,th} - \frac{\lambda}{rN}Z. \tag{20}$$

[20]Note the equilibrium is solved by backward induction. The traders, taking the market structure (q) as given, choose their strategies, and then the platform operator chooses if there is a size-discovery trading mechanism offered or not.

[21]There are two solutions to this model. I will focus on the equilibrium with higher market depth and the demand schedule limits to the competitive trader's demand schedule, $d_{s,j} \to -1$, as the number of traders grows to infinity ($N \to \infty$). The other equilibrium limits to zero trade in equilibrium, which is counterfactual to the real world.

Traders submit and are allocated the following demand quantity for size-discovery sessions[22]

$$\mu_{n,th} = Y_{n,th}(\mu) = \frac{r(N\alpha - 1)}{\lambda(N-1)}\left(s_{n,th} - \bar{s}_{th}\right) - z_{n,th} + \frac{Z}{N}. \tag{21}$$

See Appendixes A and B for the proofs. Note that if the equilibrium price, (20), is plugged into (19), the demand schedule submitted is a fraction of the competitive demand schedule, $X^{s,j}_{n,th}(p^*_{th}) = -d_{s,j}Y_{n,th}(\mu)$, where (21) is the competitive demand quantity. Therefore, having imperfect competition or size-discovery trading sessions only scales the allocated amount in equilibrium. In a competitive model, the coefficient out front of the demand schedule submitted during price-discovery sessions, d, would be equal to -1, which is also the coefficient on (21). It can be shown that $-1 < d_{s,j} < 0$, and, therefore, traders are less aggressive in the price-discovery sessions than in the size-discovery sessions or a competitive market. This is optimal as they know with probability q, a size-discovery session will appear, where they can trade with no price impact. Some trade is still optimal in the price-discovery period as it is extra costly for traders to hold inventory in excess of their desired amounts. But as traders can be very aggressive in the size-discovery sessions and are slightly less aggressive in the price-discovery session, a natural question is which effect dominates and how does this relate to the rate of convergence to the trader's desired inventory amount and, therefore, the allocative efficiency of these market designs. It is also worth noting that the equilibrium price, (20), does not depend on the probability of size-discovery sessions occurring, q.

To start to provide insight into how quickly the noncompetitive inventory allocations converge to the competitive inventory allocations, given no new information has arrived, we need proposition 2.

PROPOSITION 2: *Let $0 \leq \underline{t} \leq \bar{t}$ where $s_{n,th} = s_{n,\underline{t}h}$ for every n and $t \in \{\underline{t}, \underline{t}+1, \ldots \bar{t}\}$. Then for every trader, n, the equilibrium inventories satisfy:*

$$z_{n,(t+1)h} - z^c_{n,(\underline{t}+1)h} = \Big(\prod_{j=1}^{t+1} \mathbb{1}^c_{M_{\underline{t}+j-1}}(1 + d_{s,j})\Big)\big(z_{n,\underline{t}h} - z^c_{n,(\underline{t}+1)h}\big),\ \forall t \in \{\underline{t}, \underline{t}+1, \ldots \bar{t}\}. \tag{22}$$

[22] For now, it is assumed that for matching sessions, the platform operator chooses to offer the equilibrium price, (20). This will be shown to be true in an extension where the platform operator chooses the price to maximize the expected volume in section VA.

Values closer to -1 for $d_{s,j}$ will be interpreted as more aggressive as traders are selling (buying if $z_{n,th} < 0$) off the vast majority of their prior inventory. In contrast, coefficients near 0 are very unaggressive as they are holding onto most of their prior inventory. Intuitively, when traders sell (buy) off the majority of their prior inventory, their end-of-period inventory is made up of more of the desired inventory amount. When traders bid shade more ($d_{s,j}$ is closer to 0), the majority of the ending inventory in that period will be from the prior inventory. Taking an expectation, Proposition 2 shows that the expected rate of convergence between traders' desired inventory amount and their current amount closes at an exponential rate, $(1-q)(1+d_{s,j})$, which depends on how frequently size-discovery sessions happen explicitly and through how aggressive traders are in the price-discovery sessions through the $d_{s,j}$ coefficient from the demand schedules they submit.

B. Price Discovery Benchmark

As a benchmark, this subsection characterizes the equilibrium if the social planner bans the existence of size-discovery trading mechanisms. If trade is not perfectly competitive, and, therefore, traders internalize the price impact of their trades, and there are no size-discovery sessions ($q = 0$), then the following corollary characterizes the equilibrium.

COROLLARY 1: *When there are enough traders, $N\alpha > 2$, and no size-discovery sessions, $q = 0$, there exists a PBE where each trader submits the following demand schedule in the price-discovery sessions*[23]*:*

$$X^p_{n,th}(p) = -d_p\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{n,th}-p\right) - z_{n,th} + \frac{1-\alpha}{N-1}Z\right) \tag{23}$$

where

$$d_p = \frac{-1}{2e^{-rh}}\left((N\alpha-1)(1-e^{-rh}) + 2e^{-rh} - \sqrt{(N\alpha-1)^2(1-e^{-rh})^2 + 4e^{-rh}}\right). \tag{24}$$

The period-t equilibrium price is

$$p^*_{th} = \frac{1}{N}\sum_{n=1}^{N} s_{n,th} - \frac{\lambda}{rN}Z. \tag{25}$$

Corollary (1) is Proposition (1) but shuts down the size-discovery sessions by letting $q = 0$. It

[23] This model is the model used in Du and Zhu (2017)

can be shown that $-1 = d_c < d_p < d_{s,\text{ms}} < d_{s,\text{wu}} < 0$. Therefore by comparing equations (19), (21) and (23), traders reduce their demand (bid shade) more when they internalize their price impact and even further when they strategically delay when waiting for size-discovery sessions. They further strategically delay their demand when their current demand may affect the price used next period if a size-discovery trading session occurs. There will always be trade in every period, regardless of if there is new information or not. This creates a socially costly delay in inventory reallocation.

COROLLARY 2: *Let $0 \leq \underline{t} \leq \bar{t}$ where $s_{n,th} = s_{n,\underline{t}h}$ for every n and $t \in \{\underline{t}, \underline{t}+1, \dots \bar{t}\}$. Then for every trader, n, the equilibrium inventories satisfy:*

$$z_{n,(t+1)h} - z^c_{n,(\underline{t}+1)h} = (1 + d_p)^{t+1-\underline{t}}\big(z_{n,\underline{t}h} - z^c_{n,(\underline{t}+1)h}\big),\ \forall t \in \{\underline{t}, \underline{t}+1, \dots \bar{t}\}. \quad (26)$$

Corollary 2 is directly from Proposition 2, by shutting down the size-discovery sessions by setting $q = 0$. Corollary 2 shows that when new information does not arrive, the inventory level in the price-discovery-only model converges to the desired inventory level exponentially at a rate of $1 + d_p$. This results from the bid shading, which is delaying the traders from achieving their desired inventory level because of the price impact of their trades. The following sections will show, under certain parameterizations, that the addition of size-discovery sessions may help speed up the rate at which the inventory levels converge to the efficient levels.

C. *Illustrative Examples*

To help build intuition for the model, Figure 1 plots an illustrative example inventory path of a single trader under different market designs for a few trading sessions. Figure 1(a) is the base case. It plots the inventory position in a price-discovery-only market design, the red dot-and-dashed line, with that of a market design that can have size-discovery trading sessions, the blue dashed line, occur too. The solid black line is the desired inventory position of the trader. In this example, there is a shock to the desired inventory position just after the second trading session, and it just happens that the in a market design that allows the possibility of a size-discovery session, one occurs at the third trading session. It is clear that when there is a possibility of a size-discovery trading mechanism, traders are less aggressive during the price-discovery sessions, which causes the inventory position to be further away from the desired amount than if there was never a possibility of a size-discovery trading mechanism. Traders are closing a percent gap between their current and desired inventory position. Therefore, the lack of aggressiveness compounds over time when size-discovery sessions can occur, but have not yet. When the size-discovery session does occur, the trader is able to completely close the gap and is now holding their desired inventory position. The welfare question is simply, on average, are the extra gaps prior to size-discovery sessions due to traders strategically waiting for the better trading terms get offset by the gains realized when a size-discovery session does occur.

The final three plots in Figure 1 are of the same exact example, but one primitive parameter is changed. In Figure 1(b), the number of traders doubles. This lowers price impact per trading session, which allows traders to close a larger amount of the excess inventory gap per trading session. There is also less strategic delay waiting for size-discovery sessions as the current trading opportunity is not as costly, which leads to more trade, which increases liquidity and lowers price impact, which leads to more trade etc. In Figure 1(c), the probability of a size-discovery session increase by 50%. This lowers the expected excess inventory costs incurred in waiting for a size-discovery session. This induces traders to wait further for a size-discovery session, which lowers liquidity and raises price impact during a price-discovery session. Traders respond by further bid shading and so on. Therefore, there is an even wider gap between the current and desired inventory position until a size-discovery session occurs.

Finally, in Figure 1(d), the trading frequency per unit of clock time is doubled. There are now more trading opportunities to split orders over, which increases the flexibility traders have. But there is an indirect effect. As traders further break up their orders, liquidity drops, and price impact increases per trading session. This further induces traders to shade their demands to minimize execution costs. Now that the price impact is even worse, the incentive to wait for a size-discovery session is further increased, which also further drives up the current price impact for a price-discovery session. This logic repeats and more than offsets the gains from the flexibility of trade by increasing the trading frequency.

D. Equilibrium Properties

Comparing (19), (21), and (23), it can be seen that the equilibrium allocation is the same across up to a scaling factor, d. The scaling factor controls how aggressive traders are in the current trading session and as we will see later, how the allocative efficiency of the market designs are for the traders and choice for the platform operator. Note the closer d is to -1, the quicker traders reach their desired inventory position. This results in a higher allocative efficiency and larger trading volume, which is better for both traders and the platform operator. Figure 2 plots the aggressiveness of traders, d, as a function of different exogenous parameters to help build intuition.

The results in this equilibrium are driven by the interaction of three different forces. When there is a possibility of a size-discovery session, traders bid shade their demands as they are willing to incur some holding costs for the possibility of trading large amounts with no price impact. This extra aggressiveness due to better trading terms is the second force. The tradeoff between the delay in trading closer to the desired inventory position during price-discovery sessions versus the gain in allocative efficiency when traders are able to be extremely aggressive during size-discovery trading sessions will be analyzed below when studying the welfare of the market design. Finally, the third force, which is only present in work-up trading sessions, is the fact that the equilibrium price from the price-discovery trading session may be the price used if a size-discovery trading session occurs next period. Therefore, demanding a little bit more in a price-discovery session not only makes the current trade more expensive but would make a subsequent size-discovery trade more costly. In matching sessions, the price used is exogenous, and, therefore, this force does not exist.

The left column of Figure 2[24] models a size-discovery trading mechanism as a work-up session, the prior price is the fixed price used, whereas the right column models a matching session, the fixed price is exogenously given. The top row compares how aggressive traders are during price-discovery trading sessions as a function of the number of traders in the market. Given the third force, of not having the price be used again, does not exist for the right column, the equilibrium exists in a larger part of the parameter space, and results in slightly more aggressive demand during price-discovery sessions than a market with work-up sessions. The second and third columns compare demand in

[24] Gray areas are where parameter regions where the type of equilibrium solved for fails to exist. This is usually due to the fact that trade on the price-discovery breaks down as it becomes too costly for small markets or the possibility of better trading terms becomes too large.

price-discovery sessions as a function of both the time between trading sessions, h, and the relative frequency, γ, of a size-discovery session occurring, $q = \gamma h$, with time between trading sessions as a fraction of the trading day on a $\log_{10}$ scale. As γ increases, traders become less aggressive, and eventually, there is too large of an incentive to delay trade, and the price-discovery trading session breaks down. As h decreases, demand in each price-discovery session also decreases as the implicit probability a size-discovery trading session occurs in the next unit of clock-time is increasing. There is also less time before any type of trading session, and therefore, traders can minimize the costs of trades by further breaking up orders over time. The second and third rows are for a small and larger market size, respectively. The larger market sizes minimize these forces as the price impact per trade is much lower as the market depth increases in the number of traders.

E. Volume

An interesting quantity to look at that drives the platform operator's decision in the model is how these different market designs affect the overall volume traded. The unconditional expected volume traded in equilibrium is

$$\text{Volume} = \sum_{n=1}^{N} \mathbb{E}\left[\sum_{t=0}^{\infty} e^{-rht}\left(\mathbb{1}_{M_t}^{c}\left|X_{n,th}^{*}\right| + \mathbb{1}_{M_t}\left|Y_{n,th}^{*}\right|\right)\right]. \tag{27}$$

Plugging in the equilibrium allocations for both the price and size-discovery trading sessions and taking the expectation gives the simplified formula for the expected volume traded over the life of the asset given by equation (194) in Appendix E, which will be used later when studying the potential conflict of interest between traders and a platform operator. If you set $q = 0$, shut down the size-discovery trading sessions, the scalar becomes d_p. To understand the percent of trading volume that goes through a size-discovery trading protocol relative to the total expected volume, it simply is

$$\begin{array}{c}\text{Fraction of Volume through}\\ \text{Size-Discovery Protocol}\end{array} = \frac{q}{q-(1-q)d_s}. \tag{28}$$

Figure 3 plots this statistic, where the left column is for work-ups and the right is for matching sessions. As the relative frequency of a size-discovery trading session, γ, increases, a larger percentage of the volume goes through that trading mechanism. As these better trading sessions occur more frequently, which naturally will increase the volume through that type of session, traders are also less aggressive in the price-discovery trading sessions as they do not expect as long of a delay before the better trading terms. Likewise, if the time between trading sessions, h, decreases the relative likelihood of better trading terms increases, which drives down their aggressiveness in trading in price-discovery trading sessions. The second row increases the size of the market relative to the first row. Like in Figure 2, this mitigates these effects as the price-discovery market is much more liquid. This is also shown in the third row, where h and q are fixed, but the size of the market increases.

IV. Welfare

This section will study the welfare (allocative efficiency) of a market with size-discovery sessions versus a market with only price-discovery sessions. This section will first consider the allocative efficiency of the market designs from a trader's perspective. The focus will then shift to the platform operator's problem. This will naturally lead to when do the traders and platform operator agree or disagree on the optimal market design from their perspectives.

A. Traders' Welfare

Following Du and Zhu (2017), trader welfare is defined as the sum of ex-ante expected utilities over all traders.

$$W_j^s = (1-e^{-rh})\sum_{n=1}^{N}\mathbb{E}\left[\sum_{k=0}^{\infty}e^{-rhk}\left(\mathbb{1}_{M_k}\left(v_{n,kh}(z_{n,kh}+Y^*_{n,kh})-\frac{\lambda}{2r}(z_{n,kh}+Y^*_{n,kh})^2\right)+\right.\right. \tag{29}$$

$$\left.\left.\mathbb{1}^c_{M_k}\left(v_{n,kh}(X^*_{n,kh}+z_{n,kh})-\frac{\lambda}{2r}(X^*_{n,kh}+z_{n,kh})^2\right)\right)\middle|H_{n,kh}\right]$$

This is a measure of the allocative efficiency of the market. Naturally, the price component sums to zero across traders. In this section, this welfare is compared to the "competitive" welfare, which is a market where traders treat prices as given. In the competitive market design, traders reach their desired inventory position after any trading session ($d_c = -1$). Comparing the allocation efficiency to this benchmark allows us to focus on the costs in allocative efficiency from traders internalizing price impact or delaying for better trading terms. We will also see that this will simplify the welfare equations later as the information arrival process, size of the shocks to the desired inventory position, and cost per unit of inventory held will only be proportional to the loss in allocative efficiency.

Plugging in the equilibrium strategies and simplifying (29), the expected allocative efficiency of the market equals

$$W_j^s = \underbrace{\sum_{n=1}^{N}}_{\substack{\text{sum over}\\ \text{all traders}}} \mathbb{E}\left[\sum_{k=0}^{\infty} \underbrace{e^{-rhk}}_{\substack{\text{probability of}\\ \text{no liquidation}}} \left(\underbrace{(1-e^{-rh})}_{\substack{\text{probability}\\ \text{of liquidation}}} \underbrace{v_{n,kh} z^c_{n,(k+1)h}}_{\substack{\text{desired value}\\ \text{at liquidation}}} - \underbrace{(1-e^{-rh})\frac{\lambda}{2r}(z^c_{n,(k+1)h})^2}_{\substack{\text{expected holding cost}\\ \times\text{desired inventory squared}}} \right)\right] \tag{30}$$

$$- \underbrace{\frac{\lambda(1-e^{-rh})}{2r}}_{\text{expected holding cost}} \underbrace{\frac{(1-q)(1+d_{s,j})^2}{1-e^{-rh}(1-q)(1+d_{s,j})^2}}_{\substack{\text{cost of strategic}\\ \text{trading}}} \underbrace{\left[\sum_{n=1}^{N} \mathbb{E}[(z_{n,0}-z^c_{n,h})^2] + \sum_{n=1}^{N}\sum_{k'=1}^{\infty} \mathbb{E}[(z^c_{n,(k'+1)h} - z^c_{n,k'h})^2] e^{-rhk'}\right]}_{\substack{\text{expected cost of}\\ \text{inventory shocks}}}.$$

If $q = 0$, then (30) reduces to the allocative efficiency of a market design with just price-discovery trading sessions. Given the results of Antill and Duffie (2020), a natural first question is: does there exist a trading frequency where the allocative efficiency is improved by adding size-discovery sessions?

PROPOSITION 3: *A) If the size-discovery mechanism is modeled as a work-up session and $q > 0$, then there exists a trading frequency $h^* \in (0, \infty)$ such that the allocative efficiency is improved by having both size and price-discovery trading sessions relative to just have price-discovery trading sessions.*

$$W^s_{wu}(h^*) - W^p(h^*) > 0$$

B) If the size-discovery mechanism is modeled as a matching session and $q > 0$, then the allocative efficiency is always improved by having both size and price-discovery trading sessions relative to just having price-discovery trading sessions.

$$W^s_{ms} - W^p > 0$$

Proposition 3 proves that the addition of a work-up session can improve and the addition of a matching session always improves the traders' welfare (allocative efficiency). For work-up sessions, there always exists a trading frequency such that the extra aggressiveness achieved during the work-up sessions outweighs the two costs during price-discovery sessions. When the second cost, the price from the price-discovery session being used again if a size-discovery session occurs, is taken away when the size-discovery session is modeled as a matching session, then the extra aggressiveness during size-discovery sessions always outweighs the strategic delays it induces. The proof for Proposition 3 and special cases of the above formulas for when trading is continuous, as in Antill and Duffie (2020), is found in Appendix E.5. When trading is continuous, the allocative efficiency is strictly harmed by the addition of a work-up session and unaffected (the costs and benefits balance out) for matching sessions. This is the central finding of Antill and Duffie (2020). It is worth noting that the costs to allocative efficiency relative to a competitive benchmark, $W^c - W_j^s$, is maximized as trade becomes continuous, Appendix E.4. Therefore, Antill and Duffie (2020) unintentionally studied the case where the strategic effects would be the most costly.

For the rest of the section, I will compare the allocation efficiencies of the different market designs to that of the competitive benchmark. In the competitive benchmark, W^c, the second line of (30) is zero, as $d_c = -1$. Therefore by looking at the quantity $W^c - W_j^s$, the first line disappears, which depends on aggregate inventory, signals, and private values. Note that $W^c - W_j^s > 0$ and, therefore, this quantity can be attributed to the loss in allocative efficiency due to strategic trading due to imperfect competition and the presence of two types of trading mechanisms, where the lower, the better. Finally, if the size-discovery trading sessions are shut down, $q = 0$, then we can study the welfare loss due purely to $W^c - W^p > 0$ due to strategic trading due to imperfect competition. The main quantity of interest going forward will be the percentage difference between these two quantities, which will quantity the welfare gain or loss due to the presence of a size-discovery trading mechanism relative to the total inefficiency of the market:

$$\frac{\left(W^c - W^p\right) - \left(W^c - W_j^s\right)}{W^c - W_j^s} = \frac{W_j^s - W^p}{W^c - W_j^s} = \frac{\frac{(1+d_p)^2}{1-e^{-rh}(1+d_p)^2} - \frac{(1-q)(1+d_{s,j})^2}{1-e^{-rh}(1-q)(1+d_{s,j})^2}}{\frac{(1-q)(1+d_{s,j})^2}{1-e^{-rh}(1-q)(1+d_{s,j})^2}}. \tag{31}$$

The beauty of studying the above expression is in all of the variables that do not appear. No variable relating to the information arrival process, which could be deterministic or stochastic, or marginal inventory cost per unit of clock time (λ) appears. The variables controlling the noise of the shocks only appear in the adverse selection parameter (α), which only appears as a scaling of the number of traders (N) in the formulas for d. Therefore, the quantity in (31) will allow us to only focus on the direct strategic effects of interest in this paper.

To better understand how the allocative efficiency is affected as a function of the different market designs, Figure 4 plots (31) as a function of different primitive parameters. The left column is for a work-up trading protocol, whereas the right column is for a matching session.

Starting with the market design with work-up sessions, j = wu, the first noteworthy result is that depending on the frequency of trading and relative frequency of a size-discovery trading session, the ex-ante allocative efficiency can be either increased or decreased due to the presence of a work-up trading mechanism. This is in strong contrast to Antill and Duffie (2020), who find that allocative efficiency can only be harmed. For small markets, the allocative efficiency can only be increased when trading is very infrequent, as this increases the liquidity during price-discovery trading sessions and helps minimize the third channel. On the surface, this seems to line up with the type of assets that are traded this way. This will be explored more fully in a calibration below.

The right column does the same exercise, but now for matching sessions, j = ms. For this type of market design, the allocative efficiency can only be increased due to the presence of size-discovery trading sessions. One may wonder why every asset does not trade with the addition of matching sessions, as their addition can only increase allocative efficiency. This is rationalizable as it is very difficult, risky, and costly for the platform operator to determine the price at which trading should occur for most asset classes. This is potentially simple enough to do for index CDS's or corporate bonds, which is one reason it may exist in reality for that asset class. In contrast to Antill and Duffie (2020), they find that under continuous trading and this market design, the allocative efficiency is irrelevant to the addition of a size-discovery trading session.

V. Model Extensions

A. *Adding a Platform Operator*

In reality, a social planner does not design markets, but a profit-maximizing firm does. This profit-maximizing firm may have differing incentives from the traders on the platform. Antill and Duffie (2020) point to this potential conflict of interest and coordination failure by traders as a reason for the existence of size-discovery trading protocols, despite their finding that they strictly harm the welfare of the traders. To formally study this possibility, I add a platform operator who chooses whether to offer both price and size-discovery trading protocols or just a price-discovery trading protocol to maximize the expected trading volume over the life of the exchange. Most platforms charge trading fees to both sides of the trade as a fraction of the amount traded. Therefore, maximizing volume will maximize profit. If the platform operator chooses to have size-discovery sessions, the probability is still exogenous. The platform operator makes this decision prior to any trading. If the size-discovery session is modeled as a matching session, the platform operator also chooses what price to offer during the matching session. Therefore, for work-ups, they maximize

$$\underset{o\in\{0,1\}}{max}\ \mathbb{E}\left[\sum_{t=0}^{\infty} e^{-rht}\sum_{n=1}^{N} o\Big(\mathbb{1}_{M_t}\big|Y^*_{n,th}\big| + \mathbb{1}^c_{M_t}\big|X^*_{n,th}(q)\big|\Big) + (1-o)\big|X^*_{n,th}(0)\big|\right], \tag{32}$$

and for matching sessions, they maximize

$$\underset{\substack{o\in\{0,1\}\\ \{p_{th}\}_{t\geq 0}}}{max}\ \mathbb{E}\left[\sum_{t=0}^{\infty} e^{-rht}\sum_{n=1}^{N} o\Big(\mathbb{1}_{M_t}\big|Y^*_{n,th}(p_{th})\big| + \mathbb{1}^c_{M_t}\big|X^*_{n,th}(q)\big|\Big) + (1-o)\big|X^*_{n,th}(0)\big|\right]. \tag{33}$$

In an abuse of notation, denote that $X^*_{n,th}(q)$ means the equilibrium demand schedule played by trader n if the size-discovery trading mechanisms occur with probability q, and $Y^*_{n,th}(p_{th})$ is the equilibrium demand quantity submitted by trader n if p_{th} is the price offered in the matching session.

PROPOSITION 4: *When the size-discovery session is modeled as a matching session where the platform operator chooses the price to maximize the expected volume as a linear function of the total signal and aggregate inventory, the platform operator will always choose to offer the equilibrium price,* $p^*_{th} = \bar{s}_{th} - \frac{\lambda}{rN}Z$.

See Appendixes B.1 and C for the proof. This confirms the assumption in Proposition 1 that the 'correct' price offered during a matching session is innocuous. The basic intuition for this result is that the platform operator does not have any additional information other than the total signal. Also, by offering the correct price, each trader, in equilibrium, is as aggressive as possible. Offering any other price would only lower the aggression by some traders without any offsetting effect due to the fact that traders do not wish to hold beyond their desired position given no new information, despite the cheaper price. Knowing the platform operator will offer the correct price, traders during price-discovery sessions do not have to worry about pushing the price in a direction that may be costly to them in the following period, which increases their aggressiveness in the price-discovery sessions.

The prior section studied when the allocative efficiency is improved by having size-discovery trading sessions. The rest of this subsection studies the choice of offering a size-discovery trading session, $o \in \{0, 1\}$, and when both, neither, or only one of the two would prefer the existence of the size-discovery trading sessions. Figure 5 examines this. The left column is for a work-up size-discovery trading session, and the right is for a matching session size-discovery trading session.

When work-up trading sessions are present, depending on the frequency of occurrence and trade, the market design could be good, bad, or conflicted for both parties. When the trading frequency is sufficiently low, the addition of size-discovery trading sessions increases volume and allocative efficiency. But due to the quadratic costs of holding inventory, it hurts the allocative efficiency of traders before it hurts the expected volume as trading frequency increases. If the trading frequency is too frequent, both are better off only having price-discovery trading sessions. Antill and Duffie (2020) suggest that this conflict of interest may explain the existence of size-discovery trading sessions when they find that they only harm allocative efficiency. Given the way that I model the platform operator, in that they choose to maximize ex-ante trading volume, this can't be true as they would never offer a size-discovery trading mechanism if trading were continuous. This conflict of interest is possible for a small range of the parameter space for work-up trading sessions. It seems potentially more probable that the existence may be driven by their usefulness for both parties.

In the right column of Figure 5, the plots examine the conflict of interest when the size-discovery trading session is modeled as a matching session. Given the welfare result that these can only increase the allocative efficiency for traders, it is not surprising that there is no conflict of interest and both parties would prefer the existence of a matching session. This analysis does ignore the potentially substantial cost to the platform operator to come up with a price at which these matching sessions will occur.

B. Exogenous Execution Risk

During size-discovery sessions, there is execution risk in that it is not guaranteed that a trader will be allocated the full amount demanded. However, in equilibrium, all traders receive the full amount demanded and reach their desired inventory positions when a size-discovery session occurs.

This is not very surprising as the main friction in the model keeping it away from the competitive equilibrium is the internalization of the price impact of a trade by traders, which when "removed", reverts back to the competitive equilibrium. Some readers may be skeptical that the positive effect of size-discovery sessions is due to the fact that there is no execution risk in equilibrium. While this would also strengthen the incentives to wait for size-discovery sessions, which harms the allocative efficiency, maybe the lack of execution risk outweighs that strategic cost, and, therefore, this equilibrium is a special case and not a more general finding.

In this subsection, exogenous execution risk is introduced during size-discovery sessions. The execution risk is not micro-founded in any particular way but provides an interesting channel to study execution risk in size-discovery sessions and bid-shading during price-discovery sessions. Let ξ_t be a random variable defined on the unit interval, $[0, 1]$. ξ_t represents the fraction of the demanded quantity during a size-discovery session that a trader receives at the t^{th} trading session. ξ_t can be any distribution defined on $[0, 1]$.[25] In the standard setting that is the main focus of this paper, I take $\xi_t = 1$ almost surely. If a size-discovery session occurs at the t^{th} trading session, and trader n submits a demand quantity $\mu_{n,th}$, then they are allocated $\xi_t Y_{n,th}(\mu)$, where $Y_{n,th}$ is defined by (2), and the cash-transfer function is $-p\xi_t Y_{n,th}$, where p is the prior price for work-ups and an exogenous price for matching sessions. The equilibrium solved for is of the same form as before.

PROPOSITION 5: *When there are enough traders, there exists a PBE where each trader does as follows. They submit the following demand schedule for price-discovery sessions*[26]*:*

$$X^{s,j}_{n,th}(p) = -d^{\xi}_{s,j}\left(\frac{r(N\alpha - 1)}{\lambda(N-1)}\big(s_{n,th} - p\big) - z_{n,th} + \frac{1-\alpha}{N-1}Z\right) \tag{34}$$

where $-1 \leq d^{\xi}_{s,j} \leq 0$ *is given by equations* (67) *and* (79) *in Appendix A, and* $j \in \{\text{wu, ms}\}$ *classifies if the size-discovery mechanism is modeled as a work-up or matching session. If a price-discovery trading session occurs, the period-t equilibrium price is*

$$p^*_{th} = \frac{1}{N}\sum_{n=1}^{N} s_{n,th} - \frac{\lambda}{rN}Z. \tag{35}$$

[25] It can be a discrete, continuous, or even a mixed distribution. In equilibrium, the traders only care about the first two non-central moments and characterize their demand as a function of just those two moments.

[26] There are two solutions to this model. I will focus on the equilibrium with higher market depth and the demand schedule limits to the competitive trader's demand schedule, $d_{s,j} \to -1$, as the number of traders grows to infinity ($N \to \infty$). The other equilibrium limits to zero trade in equilibrium, which is counterfactual to the real world.

Traders submit and are allocated the following demand quantity for size-discovery sessions

$$\mu_{n,th} = Y_{n,th}(\mu) = \frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha - 1)}{\lambda(N-1)} s_{n,th} - z_{n,th}\right), \tag{36}$$

where $\mathbb{E}[\xi_t] = \xi_\mu,$ *and* $\mathbb{E}[\xi_t^2] = \xi_\sigma.$

See Appendixes A and B for the proofs. The welfare of the traders, when the size-discovery trading protocol has exogenous execution risk, is given by

$$W_j^{s_\xi} = \underbrace{W^c}_{\substack{\text{welfare in}\\ \text{competitive market}}} - \underbrace{\frac{\lambda(1-e^{-rh})}{2r}}_{\text{expected holding cost}} \times \tag{37}$$

$$\underbrace{\frac{(1-q)(1+d_{s,j}^{\xi})^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}\big((1-q)(1+d_{s,j}^{\xi})^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}}_{\substack{\text{cost of strategic}\\ \text{trading}}} \underbrace{\left[\sum_{n=1}^{N} \mathbb{E}[(z_{n,0} - z_{n,h}^c)^2] + \sum_{n=1}^{N}\sum_{k'=1}^{\infty} \mathbb{E}[(z_{n,(k'+1)h}^c - z_{n,k'h}^c)^2] e^{-rhk'}\right]}_{\substack{\text{expected cost of}\\ \text{inventory shocks}}}.$$

Only the first two non-central moments of ξ play a role in demand during both types of trading sessions. It is simplest to first look at how the size-discovery demand quantity changes. Traders scale their demand by $\frac{\xi_\mu}{\xi_\sigma}$, which is always greater than or equal to 1. The fraction a trader expects to get, relative to the competitive quantity, is $\mathbb{E}[\xi_t \frac{\xi_\mu}{\xi_\sigma}] = \frac{\xi_\mu^2}{\xi_\sigma} = \frac{\mathbb{E}(\xi)^2}{\text{Var}(\xi)+\mathbb{E}(\xi)^2} \in [0, 1]$. Therefore, any execution risk leads to a less efficient size-discovery allocation. The efficiency loss decreases in the expected allocation amount and increases in the second non-central raw moment, which accounts for uncertainty in the allocation quantity. Traders demand more than the desired amount knowing that in the expectation they will not receive the full amount. As the uncertainty in the allocation quantity grows, demanding more increases the probability the trader will end up on the other side of excess inventory.

The top row of Figure 6 is the same illustrative example as Figure 1 but adds execution risk to the size-discovery allocation. Due to the execution risk, the incentive to delay trade to size-discovery sessions is lowered, which increases the amount of trade during a price-discovery session. In the illustrative example, when a size-discovery session occurs, a large gap of the excess inventory is closed, but not fully anymore. The bottom four rows of Figure 6 plots $d_{s,j}^{\xi}$ for different values of

$\mathbb{E}(\xi)$, and $\mathrm{sd}(\xi)$. Traders rationally anticipate execution risks involved in size-discovery sessions that may occur in the future. Therefore, they adjust their demand during price-discovery sessions. The bottom right corner, when $\mathbb{E}(\xi) = 1$, and $\mathrm{sd}(\xi) = 0$, is the value of $d^{\xi}_{s,j}$ when there is no exogenous execution risk. As the expected fraction received decreases or the volatility in the fraction received increases, the size-discovery trading session becomes less efficient, which traders attempt to offset by being more aggressive in the price-discovery sessions. These general forces are true when the size-discovery protocol is modeled as a work-up or matching session. If the size-discovery session is very inefficient, near the top left corner of the plot, traders become even more aggressive than if there was no size-discovery session ($d_p = -0.153$), as some trade is better than none.

The bottom row of Figure 6 plots the percent change in the welfare loss relative to a competitive benchmark of a market with and without size-discovery sessions. For this specific set of values, the addition of work-ups is harming the allocative efficiency. As the exogenous execution risk increases, traders become less aggressive during the size-discovery sessions, and more aggressive during price-discovery sessions. This shrinks the incentives to delay trade, which results in slightly higher allocative efficiency. When the size-discovery session is a matching session, the increase in the exogenous execution risk harms the benefits of the size-discovery sessions, until it makes it so inefficient, that traders are worse off by having a matching session, though the magnitude is small.

VI. Empirical Counterpart

A. Calibration

The prior section made predictions on when the alternative market design of having both price and size-discovery trading sessions versus just price-discovery trading sessions would be beneficial for traders and a platform operator. This section will attempt to bring the model to the real world by calibrating the model to a few different asset classes. The calibrations will closely follow the aggregate market statistics from Fleming and Nguyen (2018) for the U.S. Treasury asset class and Collin-Dufresne et al. (2020) for index CDSs. Both of the trading platforms that will be calibrated to are interdealer trading platforms, which both handle the majority of order flow in this section of these markets. The U.S. Treasury market will be calibrated to the BrokerTec platform, which used to have a work-up trading mechanism. The index CDS market will be calibrated to the GFI SEF, which has work-up and matching session trading mechanisms, which will be calibrated separately.

The parameters used to calibrate the model are taken as directly as possible from the above-mentioned papers. α is set to 1 as the adverse selection is not a substantial feature of these markets. Following Du and Zhu (2017), I set $r = 1/30$. N is taken from Joint Staff Report (2015) for the U.S. Treasury market as the sum of Proprietary Trading Firms and Bank-Dealers, and only the Bank-Dealer count is used for the interdealer market for the index CDS market. h is set to the reciprocal of the number of trades in a day times 2 and divided by the number of traders in the market, so that the model and data have the same number of transactions. Finally, q is solved such that it matches the volume in a size-discovery trading session relative to total volume, given all the other parameters. The results are in Table I.

Panel A is for when the size-discovery trading session is modeled as a work-up trading session. The first three rows are for on-the-run U.S. Treasury securities trading on BrokerTec. The two columns on interest are the last two, which quantify the percentage change in traders' welfare relative to $W^c - W^s_j$, and platform operator's volume relative to a market design of just price-discovery trades. For U.S. Treasuries, there are a fair amount of trades that occur per day, which implies a fast trading frequency. Also, a very high percentage of volume goes through the work-up mechanism. Therefore, due to the fast trading frequency and a small amount of traders on the platform, traders strategically delay very heavily, which leads to a large decrease in the allocative efficiency and volume relative to a market that only had a price-discovery trading mechanism. This matches nicely with the fact that while BrokerTec used to offer work-up trading sessions until 2021, after being acquired the the CME Group, they stopped offering the work-up trading protocol. According to the callibration, this should have increased the allocative efficiency and volume, all else equal. However, for the index CDS market that models GFI's SEF, there is a much slower trading frequency, and a much smaller percentage of volume goes through the work-up mechanism. Despite having a fewer number of traders, liquidity is concentrated in the fewer trading opportunities, which then minimizes the strategic costs. This results in real gains in allocative efficiency.

Panel B is for when the size-discovery trading session is modeled as a matching session. The SEF run by GFI for the index CDS market is one exchange that has matching sessions. When the model is calibrated to this trading platform, there is a large increase in both the allocative efficiency for traders and trading volume. This helps rationalize the existence of matching sessions in the index CDS space. In fact, the gain in welfare numbers are close to those estimated by Giancarlo (2015)

(50%). The large increase in volume and, therefore, fees, may reasonably offset the additional costs of determining the timing and price of matching sessions incurred by the platform operator. A similar logic may explain the use of matching sessions for some corporate bonds of interest-rate swaps.

B. Case Study

BrokerTec and eSpeed are the two largest electronic interdealer trading platforms accounting for over $200 billion a day in U.S. Treasury trading volume. In November 2018, the BrokerTec platform was acquired by the CME Group from the NEX Group. In order to create uniformity with its current trading platform, CME Globex, the CME Group took away the work-up protocol in January 2021 (CME Group, 2021). Yet, eSpeed, which is owned by Tradeweb now, the platform that has been the main competitor to BrokerTec, continues to offer both a limit-order book and a work-up protocol. While the model in this paper does not directly model the competition between platforms, if the existence of a work-up protocol was due to a coordination failure between traders to not trade during those sessions, one would expect to see a migration of trading volume from eSpeed to BrokerTec following the removal of the work-up protocol. If a work-up protocol helped increase the allocative efficiency of trading, then one may expect an increase in trading volume on eSpeed following the removal of the work-up protocol on BrokerTec. This section empirically tests these hypotheses.

Using monthly volume data from both platforms from January 2017 to July 2023, I set up a differences-in-differences framework to test if and, potentially, in which direction was there a shift in trading volume when BrokerTec removed the work-up protocol. The results are in Table II. In the pre-period, the ADV of BrokerTec was $145.697 billion in January 2017 dollars. After BrokerTec stops offering the work-up protocol, there is a relative increase in trading volume on eSpeed by 85%. This effect is decomposed into eSpeed increasing its volume by 45.6% (standard error of 11.5%) and BrokerTec's volume decreasing by 39.9%. Despite eSpeed having much less trading volume on average before 2021, they average a 10.7% higher ADV in the post, though statistically insignificant from zero (standard error of 11.5%). These results are consistent with a substitution hypothesis that traders value the allocative efficiency gain from having the work-up protocol and, therefore, substitute trade towards platforms that offer it, which is also inconsistent with a coordination failure hypothesis.

VII. Conclusion

This paper shows that a market design with discrete and sequential trading between price and size-discovery trading mechanisms can improve the allocative efficiency of the market. The allocative efficiency is always improved if the price is exogenously supplied to the size-discovery trading mechanism, as is common for matching sessions. Calibrations imply that it is unlikely that size-discovery trading mechanisms exist due to a conflict of interest between traders and platform operators. These theoretical results help rationalize the existence and design of interdealer segments in multiple asset classes.

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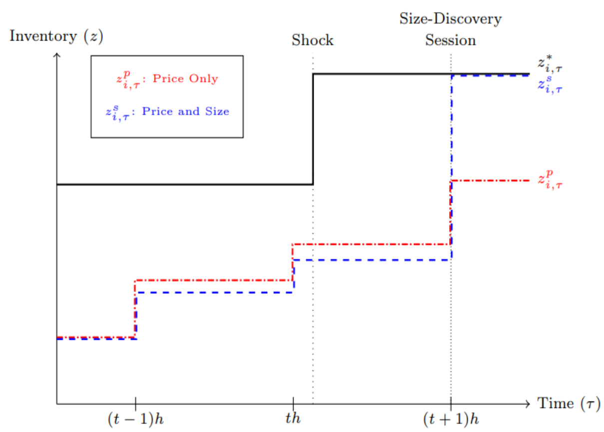
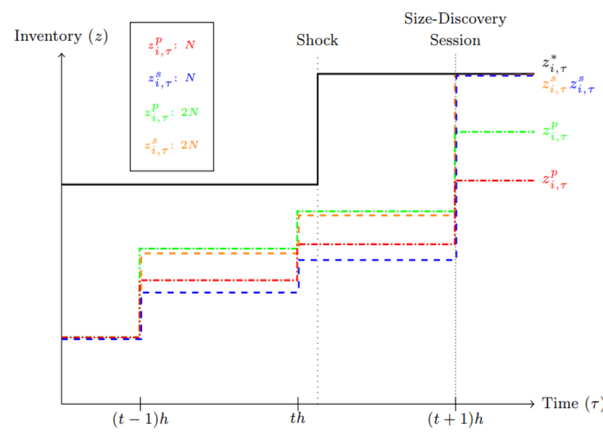
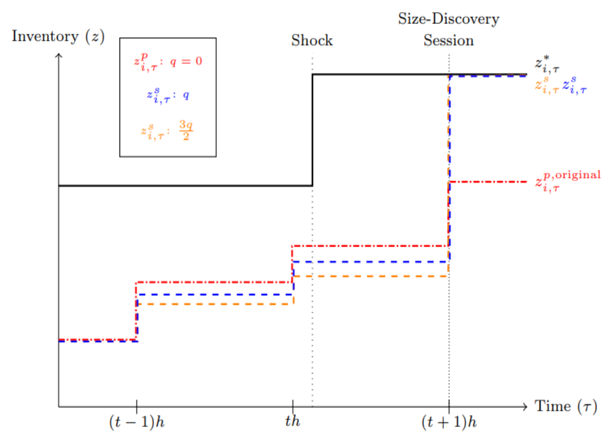


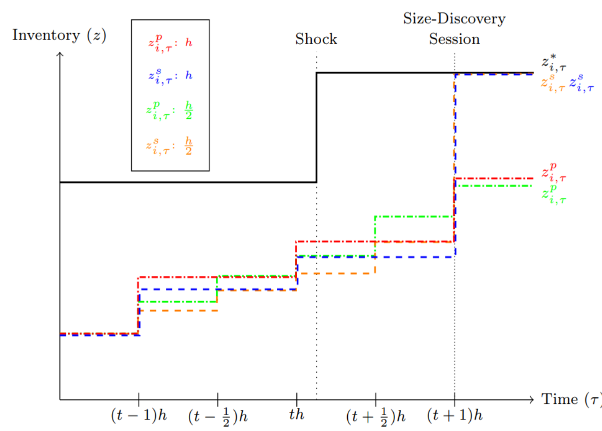
(a) Base example illustration of the two different market designs.



(b) Comparative static if the number of traders doubled.



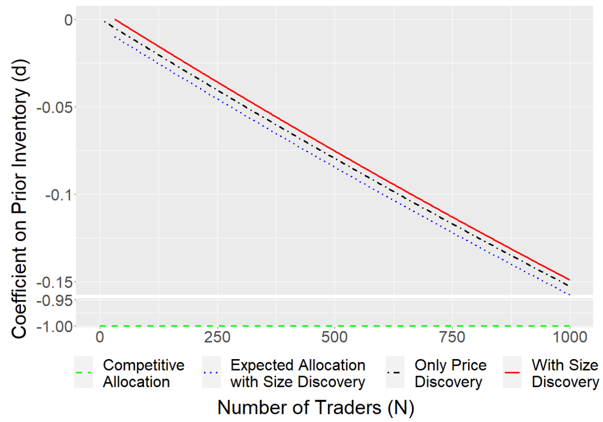
(c) Comparative static if the probability of a size-discovery session increased by 50%.



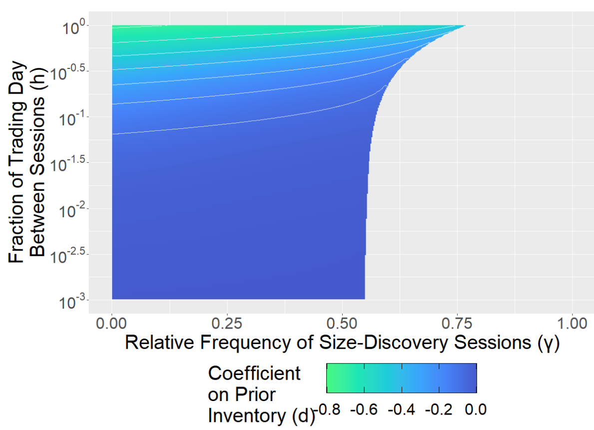
(d) Comparative static if the trading frequency doubled.

Figure 1. These figures are illustrative example plots of a single trader's inventory position across a few trading opportunities. All four plots are the same example, where the red dot-and-dashed line is a model with only price-discovery sessions, the blue dashed line has the possibility of a size-discovery session, and the solid black line is the desired inventory position. The orange and green lines are the inventory paths under the same example simulation, except for one primitive parameter that is changed. The dotted vertical line is when there is a shock to the desired inventory position. The finely dotted vertical line is when a size-discovery session occurs. The base values are $\alpha = 1$, $q = .1$, $h = .1$, $N = 100$ and $r = .1$.

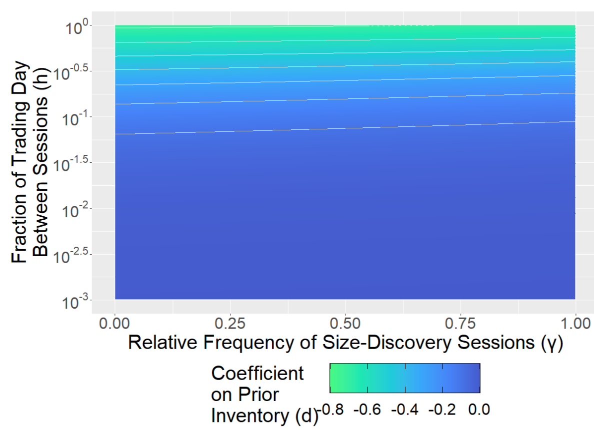
(a) A market with work-up sessions as a function of the size of the market



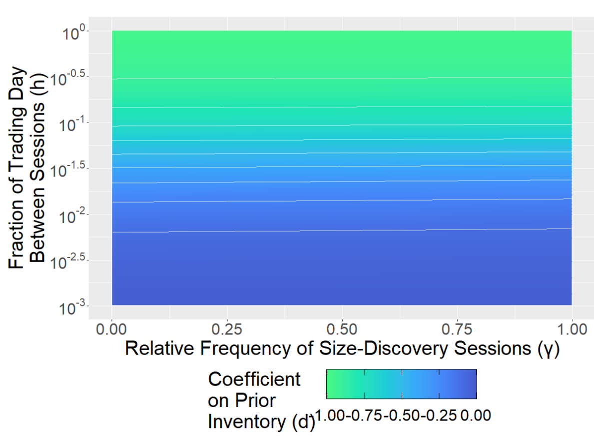
(b) A market with matching sessions as a function of the size of the market



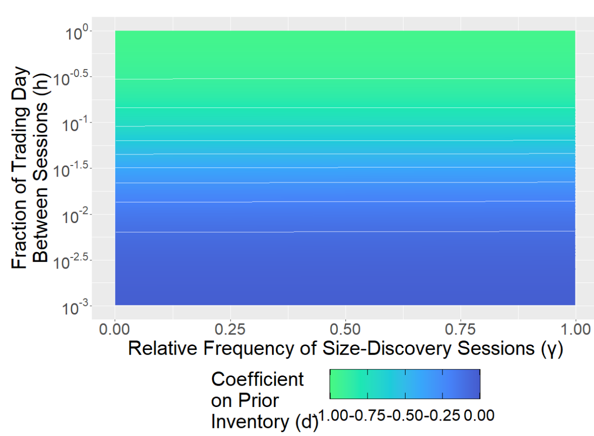
(c) Small market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)



(d) Small market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)

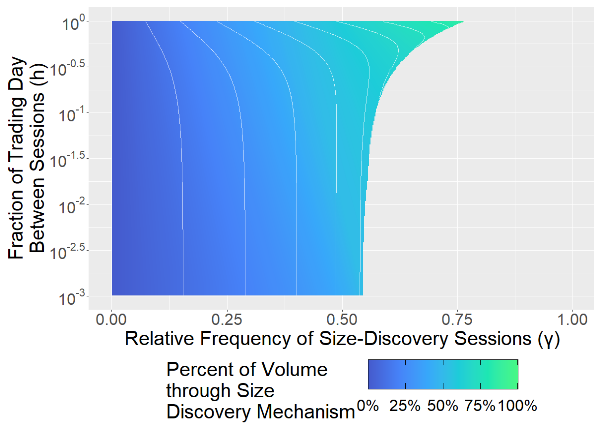


(e) Large market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

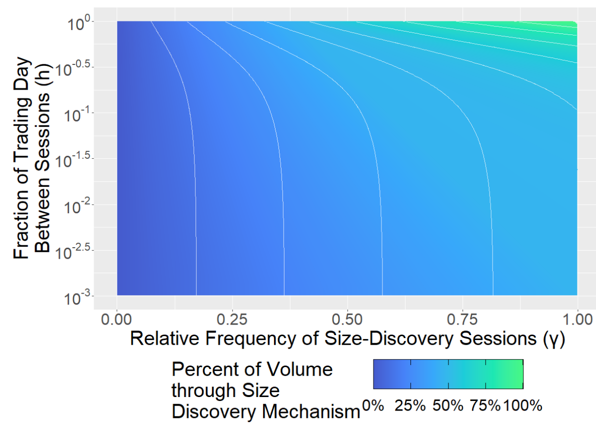


(f) Large market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

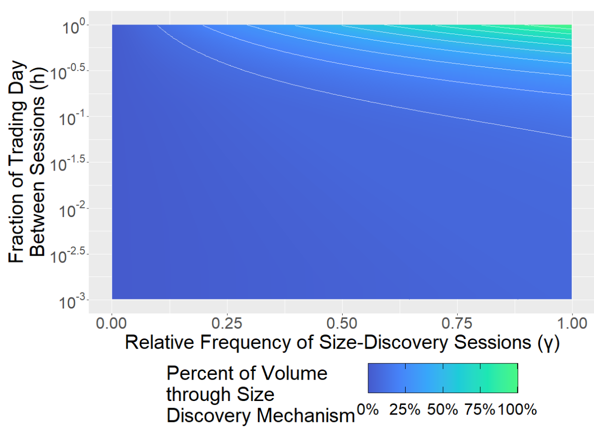
Figure 2. These figures show how much of the prior inventory is sold as a function of different parameters when a price discovery session occurs. The lower the coefficient on prior inventory, the more aggressive traders are in their demand. The left column is for a market design that has work-ups, while the right column is when there are matching sessions. To avoid a scaling issue, the relative frequency parameter, γ, is such that $q = \gamma h$. The base values, unless otherwise specified, are $\alpha = 1$, $r = 1/30$, $h = .01$, and $q = .01$.



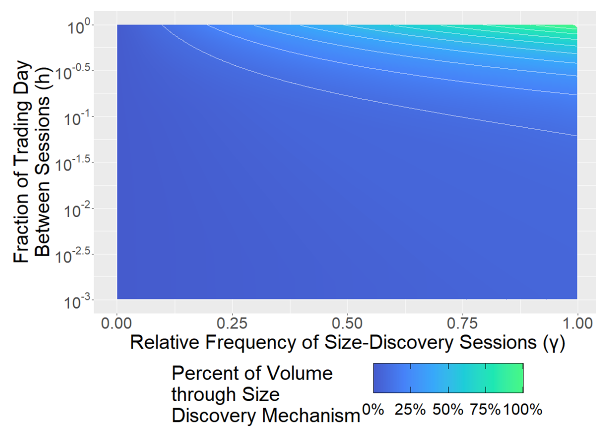
(a) Small market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)



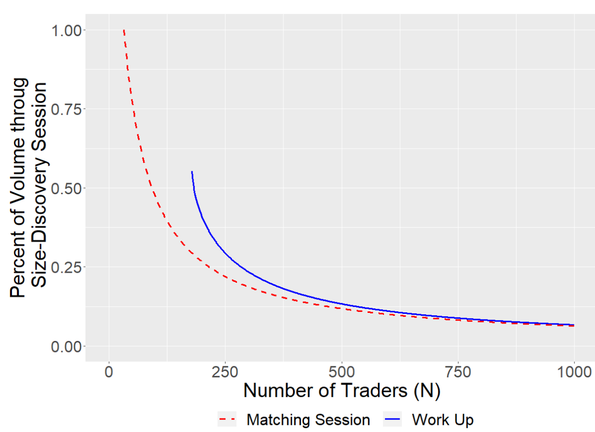
(b) Small market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)



(c) Large market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

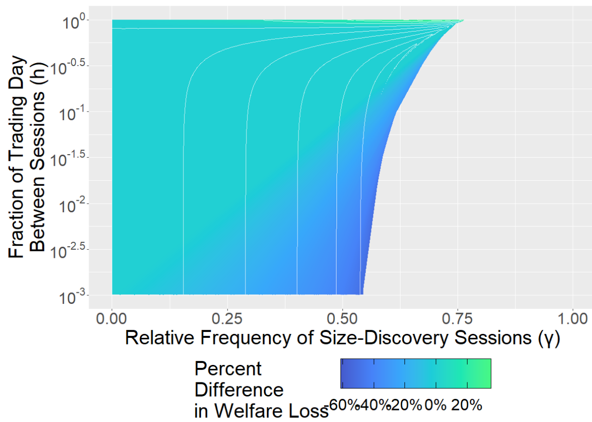


(d) Large market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

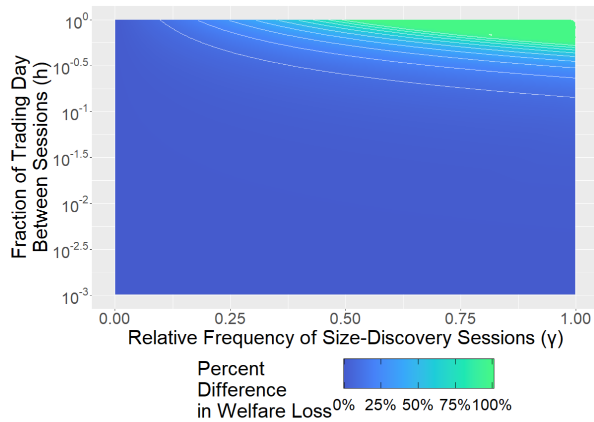


(e) A market with size-discovery sessions as a function of the size of the market

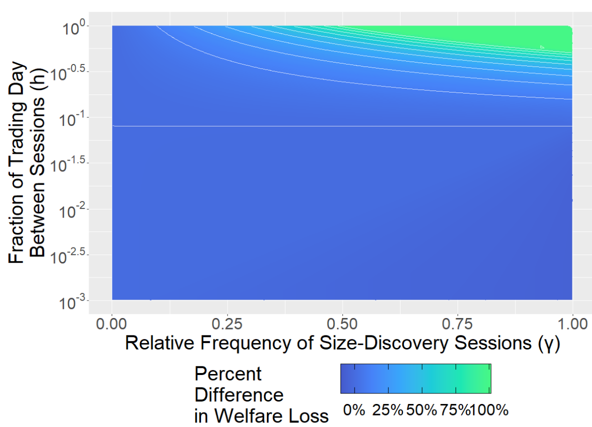
Figure 3. These figures are of the unconditional expected percent of trading volume that goes through a size-discovery trading session relative to the total trading volume. The left column is for a market design that has work-ups, while the right column is when there are matching sessions. To avoid a scaling issue, the relative frequency parameter, γ, is such that $q = \gamma h$. The base values, unless otherwise specified, are $\alpha = 1$, $r = 1/30$, $h = .01$, and $q = .01$.



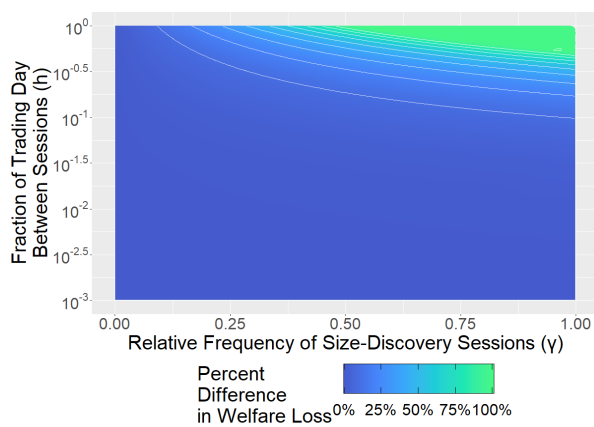
(a) Small market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, $(N = 100)$



(b) Small market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, $(N = 100)$

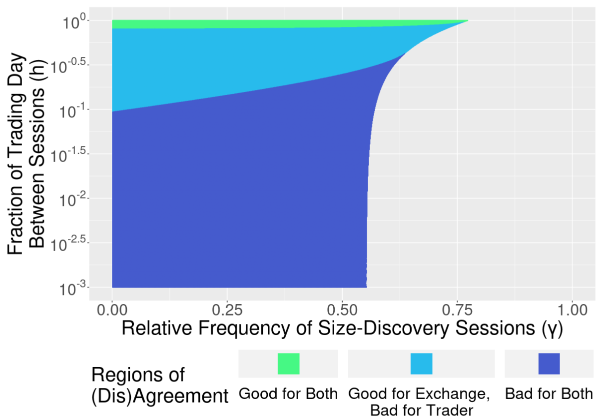


(c) Large market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, $(N = 1,000)$

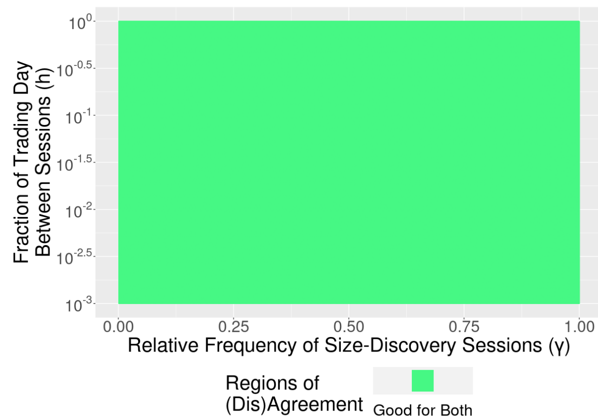


(d) Large market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, $(N = 1,000)$

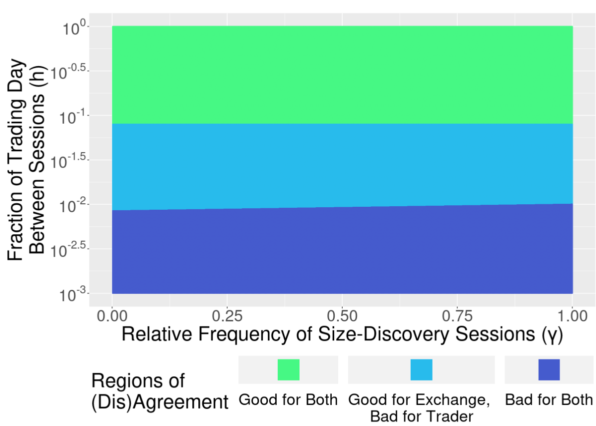
Figure 4. These figures are of the percent welfare loss relative to an efficient benchmark between a market design with both size and price-discovery trading mechanisms versus just a price-discovery trading mechanism. The left column is for a market design that has work-ups, while the right column is when there are matching sessions. To avoid a scaling issue, the relative frequency parameter, γ, is such that $q = \gamma h$. The base values, unless otherwise specified, are $\alpha = 1$, and $r = 1/30$.



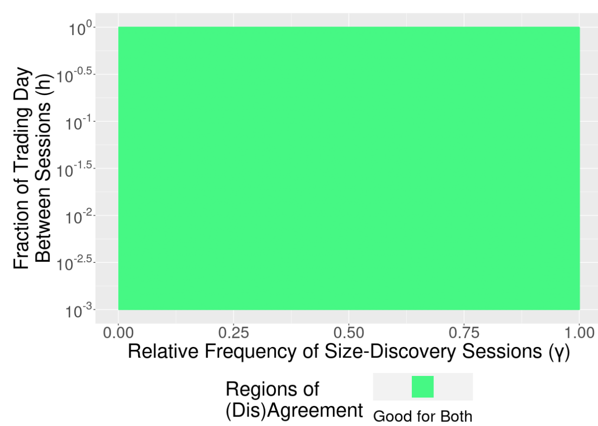
(a) Small market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)



(b) Small market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 100$)

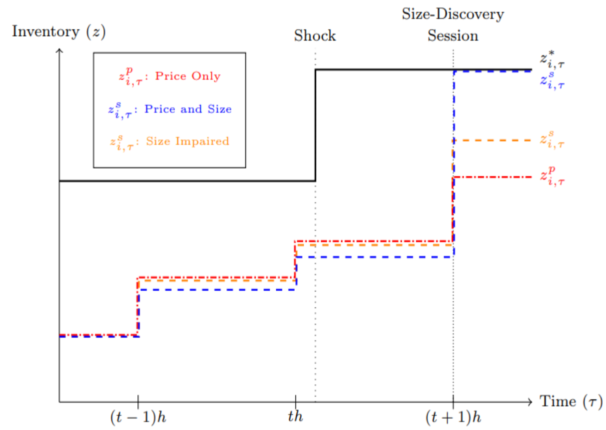


(c) Large market with work-up sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

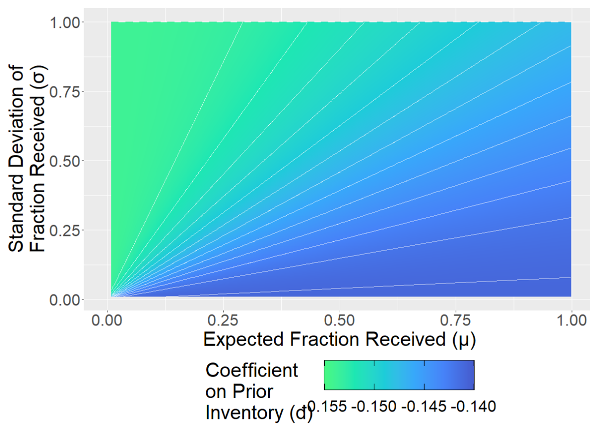


(d) Large market with matching sessions as a function of the time between trades and frequency of size-discovery sessions, ($N = 1,000$)

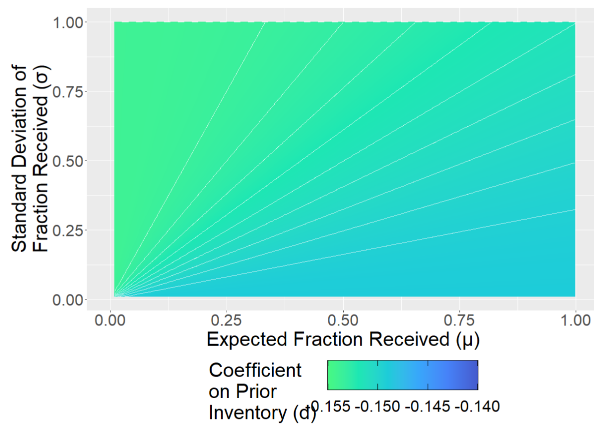
Figure 5. These figures are of regions where traders and platform operators agree or disagree on the market design. The left column is for a market design that has work-ups, while the right column is when there are matching sessions. To avoid a scaling issue, the relative frequency parameter, γ, is such that $q = \gamma h$. The base values, unless otherwise specified, are $\alpha = 1$, and $r = 1/30$.



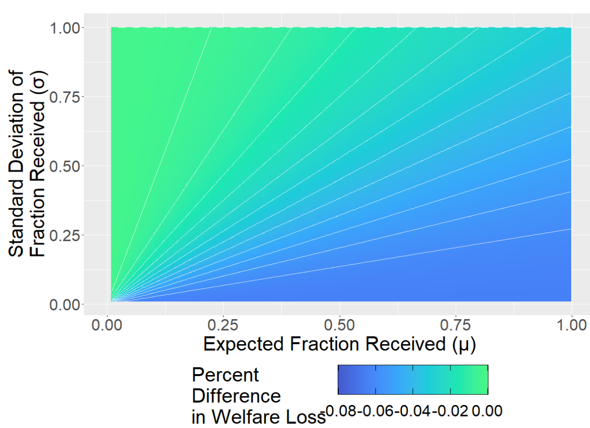
(a) Comparative static if the size-discovery allocation is impaired.



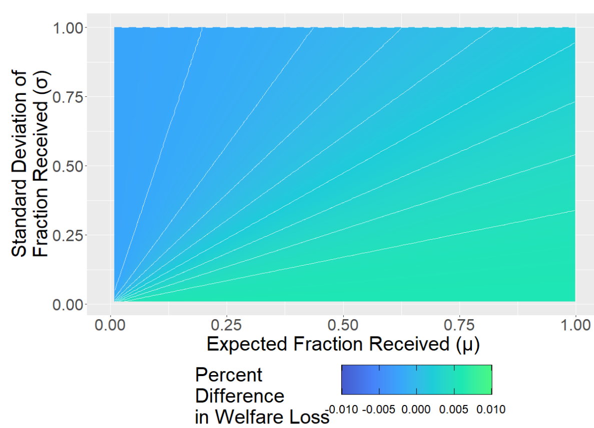
(b) The base value for a market with size-discovery sessions is $d_p = -.153$, and for unimpaired size-discovery sessions $d_{s,\text{wu}} = -.141$.



(c) The base value for a market with size-discovery sessions is $d_p = -.153$, and for unimpaired size-discovery sessions $d_{s,\text{ms}} = -.149$



(d) The welfare loss of a market design with work-ups that are unimpaired is −6.48%.



(e) The welfare gain of a market design with matching sessions that are unimpaired is 0.58%.

Figure 6. The top figure is the same as Figure 1, but impairs the size-discovery allocation. The bottom four figures are of the coefficient on the prior inventory position during price-discovery sessions and the percent change in welfare lost relative to a competitive benchmark between a market with and without size-discovery sessions. The left column is for a market design that has work-ups, while the right column is when there are matching sessions. The top row is of $d^{\xi}_{s,j}$, and the bottom row is of the percent change in (37) when $q = 0$ and $q = .01$. The bottom right corner of each plot is the unimpaired value. The base values are $\alpha = 1$, $q = .01$, $h = .01$, $N = 1,000$ and $r = 1/30$.

Table I
Calibration

Panels A and B calibrate the model to exchanges that operate with both price and size-discovery trading mechanisms. Panel A calibrates the exchanges that use the work-up mechanism for the size-discovery trading mechanism, while panel B does the calibration for exchanges that uses matching sessions for the size-discovery trading mechanism. The U.S. Treasury exchange the data is calibrated to is BrokerTec, which is studied by Fleming and Nguyen (2018). The CDS exchange is GFI's SEF, as studied in Collin-Dufresne et al. (2020). $\hat{q}$ is calibrated by solving for the q that matches the volume to (28). $\%\Delta_{\text{trader}}$ in Welfare Loss is calculated as $\frac{W^s - W^p}{W^c - W^s}$, and $\%\Delta_{\text{exchange}}$ in Welfare is the percent change in the future expected volume between a market design with both trading mechanisms versus one with only a price-discovery trading mechanism.

Panel A: Work-Up Protocol

Asset Class	Security	N	% of Volume	$\frac{\text{\# of Trades}}{\text{\# of Days}}$	$\hat{q}$	$\%\Delta_{\text{trader}}$ in Welfare Loss	$\%\Delta_{\text{exchange}}$ in Welfare
U.S. Treasury							
	2-Year	85	51.4%	6,254.31	0.0031	-51.4%	-10.9%
	10-Year	85	60.1%	20,982.56	0.0009	-57.5%	-14.4%
	30-Year	85	48.0%	5,911.76	0.0032	-48.0%	-9.4%
Index CDS							
	5-Year NA IG	40	19.9%	5.14	0.1638	10.5%	3.1%
	5-Year NA HY	40	15.5%	4.50	0.1300	9.0%	2.2%

Panel B: Matching Protocol

Asset Class	Security	N	% of Volume	$\frac{\text{\# of Trades}}{\text{\# of Days}}$	$\hat{q}$	$\%\Delta_{\text{trader}}$ in Welfare Loss	$\%\Delta_{\text{exchange}}$ in Welfare
Index CDS							
	5-Year NA IG	40	52.2%	10.03	0.4081	61.48%	18.0%
	5-Year NA HY	40	58.7%	10.97	0.4604	74.7%	22.1%

Table II
Substitution towards Workups

This table tests a substitution hypothesis between BrokerTec and eSpeed when BrokerTec stopped using a work-up protocol to integrate with the CME Globex platform in January 2021, while eSpeed continued to offer a work-up protocol. January 2021 is dropped from the analysis. The dependent variable is the natural logarithm of the inflation-adjusted average daily volume (ADV) in January 2017 dollars of each platform at a monthly frequency from January 2017 to July 2023. Post is an indicator if the observation occurs after January 2021, and the variable eSpeed takes a value of one if the platform is eSpeed, and zero if the platform is BrokerTec. Standard errors are Newey-West standard errors with two lags.

	log(ADV)		
	(1)	(2)	(3)
Constant	25.7048***		
	(0.0547)		
Post	-0.3992***		
	(0.0715)		
$\mathbb{1}$(eSpeed)	-0.7477***	-0.7477***	-0.7477***
	(0.0729)	(0.0691)	(0.0598)
Post×$\mathbb{1}$(eSpeed)	0.8552***	0.8552***	0.8552***
	(0.0895)	(0.0820)	(0.0682)
Month FE	No	Yes	No
Year FE	No	Yes	No
Month×Year FE	No	No	Yes
Observations	156	156	156
Adj. R^2	0.68	0.73	0.65

Appendix

Appendix A Derivation of Demand Schedule for Price-Discovery Sessions

This section derives and solves for an equilibrium demand schedule that traders use when they submit linear, symmetric, and stationary demand schedules during price-discovery sessions when size-discovery sessions may be impaired (a realization of a random variable in the unit interval that is an exogenous fraction of what is allocated). Traders are rational about how the impairment in the size-discovery trading session affects the amount allocated when that session occurs when forming a demand schedule to submit during price-discovery trading sessions. Following Definition 1, I solve for a Perfect Bayesian Equilibrium, in which I focus on demand schedules that are linear, symmetric, and stationary. The equilibrium demand schedule conjectured, and I confirm to exist, will take the following linear form, (16):

$$X_{n,th}(p) = as_{n,th} - bp + dz_{n,th} + fZ. \tag{38}$$

Given that price-discovery sessions will operate as a uniform-price double auction, (1), and the above-conjectured demand schedule used during price-discovery trading sessions, the equilibrium price will take the following form:

$$p^*_{th} = \frac{a}{Nb}\sum_{n=1}^{N} s_{n,th} + \frac{d+Nf}{Nb}Z. \tag{39}$$

Define $\mathbb{1}_{M_k}$ to be an indicator function equal to one if a size-discovery session occurs in period k and zero otherwise (a price discovery session occurs in period k). Define ξ_t to be a random variable on the unit interval with CDF F. ξ_t represents the fraction of the allocation that a trader receives when there is a size-discovery trading session. In the standard discrete-time setting that is the main focus of this paper, I take the function $\xi_t = 1$ almost surely. Given that the traders will be allocated the competitive demand quantities when size-discovery sessions occur, up to a scaling for the potential impairment which is proved in Appendix B, the inventory of trader n evolves according to

$$z_{n,(t+1)h} = z_{n,th} + \mathbb{1}_{M_t}\xi_t\frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{n,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) - z_{n,th} + \frac{Z}{N}\right) + \mathbb{1}_{M_t^c}\left(as_{n,th} - bp^*_{th} + dz_{n,th} + fZ\right). \tag{40}$$

The evolution of inventory, (40), will be used multiple times throughout this derivation and in the below appendices when studying the effect of the presence of size-discovery sessions on the welfare (allocation efficiency) of the market.

To prove that a trader's optimal strategy is the conjectured linear function (38), I will use the one-stage deviation principle to verify the equilibrium strategy (Fudenberg and Tirole, 1991). Define $X_{n,th} : \mathbb{R} \to \mathbb{R}$ to be the demand schedule submitted by trader n at period t, which occurs at clock-time th. X is a function from the price that clears the market to the demanded quantity. Traders are rational in that they take into account their demand on the price faced. Trader n faces the residual demand curve $X_{n,th} = -\sum_{i\neq n}^{N} X_{i,th}$, as the market clearing condition implies $\sum_{i=1}^{N} X_{i,th}(p^*_{th}) = 0$, equation (1). As proved in Appendix B, $Y_{n,th}$ is the competitive demand allocation, scaled by the $\frac{\xi_\mu}{\xi_\sigma}$, where $\mathbb{E}(\xi_t) = \xi_\mu$ and $\mathbb{E}(\xi_t^2) = \xi_\sigma$. Therefore taking the first order condition of trader n's continuation utility, (15), with respect to price and evaluating at the market clearing price, the first order condition becomes:

$$\frac{\partial}{\partial p_{th}} V_{n,th}\bigg|_{p_{th}=p^*_{th}} =$$

$$\frac{\partial}{\partial p_{th}}\mathbb{E}\Bigg[\sum_{k=0}^{\infty} e^{-rhk}\Bigg(\mathbb{1}_{M_{t+k}}\bigg(-p_{(t+k-1)h}\xi_{t+k}Y_{n,(t+k)h} + (1-e^{-rh})v_{n,(t+k)h}(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h}) - \frac{\lambda}{2r}(1-e^{-rh})(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})^2\bigg)$$

$$+\mathbb{1}^c_{M_{t+k}}\bigg(-p_{(t+k)h}X_{n,(t+k)h} + (1-e^{-rh})v_{n,(t+k)h}(X_{n,(t+k)h} + z_{n,(t+k)h}) - \frac{\lambda}{2r}(1-e^{-rh})(X_{n,(t+k)h} + z_{n,(t+k)h})^2\bigg)\Bigg)\bigg|H_{n,th}\Bigg]\Bigg|_{p_{th}=p^*_{th}}.$$

Under technical regularity conditions, we can bring the differentiation inside the expectation and distribute it across the infinite sum:

$$= \mathbb{E}\Big[\sum_{k=0}^{\infty} e^{-rhk}\Big(\mathbb{1}_{M_{t+k}}\Big(-\frac{\partial(p_{(t+k-1)h}\xi_{t+k}Y_{n,(t+k)h})}{\partial p^*_{th}}+(1-e^{-rh})v_{n,(t+k)h}\frac{\partial(\xi_{t+k}Y_{n,(t+k)h}+z_{n,(t+k)h})}{\partial p^*_{th}}-\frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(\xi_{t+k}Y_{n,(t+k)h}+z_{n,(t+k}}{\partial p^*_{th}}$$

$$\mathbb{1}^c_{M_{t+k}}\Big(-\frac{\partial(p_{(t+k)h}X_{n,(t+k)h})}{\partial p^*_{th}}+(1-e^{-rh})v_{n,(t+k)h}\frac{\partial(X_{n,(t+k)h}+z_{n,(t+k)h})}{\partial p^*_{th}}-\frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(X_{n,(t+k)h}+z_{n,(t+k)h})^2}{\partial p^*_{th}}\Big)\Big)\Big|H_{n,th}\Big]. \tag{41}$$

In order to evaluate the partial derivatives in (41), we will need to look at the evolution of inventory to see the functional relationship between inventory in period $t+k$ and the demand in period t. Iterating (40) forward k times, the evolution of inventory in period $t+k$ can be written as function of the equilibrium demand allocation $X_{n,th}$ as

$$z_{n,(t+k)h} = \sum_{j=1}^{k-1}\mathbb{1}^c_{M_{t+k-j}}\Big(\prod_{i=1}^{j-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\Big(as_{n,(t+k-j)h}-bp^*_{t+k-j}+fZ\Big)+$$

$$\sum_{j=1}^{k-1}\mathbb{1}_{M_{t+k-j}}\Big(\prod_{i=1}^{j-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\xi_{t+k-j}\frac{\xi_\mu}{\xi_\sigma}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big(s_{n,(t+k-j)h}-\frac{1}{N}\sum_{n=1}^{N}s_{n,(t+k-j)h}\Big)+\frac{Z}{N}\Big)+$$

$$\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)(z_{n,th}+X_{n,th}). \tag{42}$$

Note that, without loss of generality, we can assume that $\mathbb{1}^c_{M_t}=1$ ($\mathbb{1}_{M_t}=0$) as otherwise, the demand schedule submitted is irrelevant. Using (38), (39), (42) and the residual demand curve trader n faces, we can derive the partial derivatives in (41) explicitly. First, by the market-clearing condition, the equilibrium price is only a function of the aggregate signal and the total amount of inventory. Therefore, a change in price in period t has no effect on prices in the future in equilibrium.

$$\frac{\partial p^*_{(t+k)h}}{\partial p_{th}} = 0, \text{ for } k \geq 1 \tag{43}$$

From the residual demand curve with (38) and the market clearing condition, we get the change in the optimal demand with respect to a change in price is

$$\frac{\partial X_{n,th}}{\partial p_{th}} = b(N-1). \tag{44}$$

Using (42), (43), and (44), we can derive the relationship of the change in future inventory as the price changes in period t for $k \geq 1$ as

$$\frac{\partial z_{n,(t+k)h}}{\partial p_{th}} = \Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)b(N-1). \tag{45}$$

With the above partial, (45), we can then see how future demand changes as a the price today changes.

$$\frac{\partial X_{n,(t+k)h}}{\partial p_{th}} = \Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)db(N-1) \tag{46}$$

From appendix B and (45) we know that for $k \geq 1$ (as $\frac{\partial Y^*_{n,th}}{\partial p_{th}}=0$)

$$\frac{\partial Y_{n,(t+k)h}}{\partial p_{th}} = \frac{\xi_\mu}{\xi_\sigma}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)b(N-1). \tag{47}$$

Finally, from (45), (46) and (47) it is simple to see how the post-trade inventory after price and size-discovery trading sessions change with respect to price as

$$\frac{\partial(z_{n,(t+k)h}+X_{n,(t+k)h})}{\partial p_{th}} = \Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)(1+d)b(N-1), \tag{48}$$

$$\frac{\partial(z_{n,(t+k)h}+\xi_{t+k}Y_{n,(t+k)h})}{\partial p_{th}} = \Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)(1-\frac{\xi_\mu}{\xi_\sigma}\xi_{t+k})b(N-1). \tag{49}$$

Now that the partials have been derived, equations (43), (44), (46), (47), (48), and (49) can be plugged into (41) which when set equal to zero will yield the optimal demand schedule for trader n to play in period t.

$$\mathbb{E}\Bigg[-e^{-rh}\mathbb{1}_{M_{t+1}}\xi_{t+1}Y_{n,(t+1)h}+b(N-1)\frac{\xi_\mu}{\xi_\sigma}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}\xi_{t+k}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)p_{(t+k-1)h}+ \quad (50)$$

$$(1-e^{-rh})b(N-1)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}(1-\frac{\xi_\mu}{\xi_\sigma}\xi_{t+k})\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\Big(v_{n,(t+k)h}-\frac{\lambda}{r}(z_{n,(t+k)h}+\xi_{t+k}Y_{n,(t+k)h})\Big)+$$

$$-X^*_{n,th}-b(N-1)p^*_{th}-db(N-1)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)p_{(t+k)h}+$$

$$(1-e^{-rh})b(N-1)\Big(v_{n,th}-\frac{\lambda}{r}(z_{n,th}+X_{n,th})\Big)+$$

$$(1-e^{-rh})b(N-1)(1+d)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\Big(v_{n,(t+k)h}-\frac{\lambda}{r}(z_{n,(t+k)h}+X_{n,(t+k)h})\Big)\Big|H_{n,th}\Bigg]=0$$

To simplify further so that we can solve for the optimal functional form for trader n's demand schedule in period t, we need to take the conditional expectations of all processes that have a random variable in them. The first thing to note is that the equilibrium price is a martingale, as total signals are martingales. Therefore we have for all $k \geq 1$,

$$\mathbb{E}[p^*_{(t+k)h}|H_{n,th}] = p^*_{th}. \quad (51)$$

Next, as the probability that a size-discovery session occurs is independent of all other processes and $\mathbb{1}_{M_{t+k}} \sim$ Bernoulli(q) for all $k \geq 0$, then

$$\mathbb{E}[\mathbb{1}^w_{M_{t+k}}|H_{n,th}] = q, \text{ where } w \in \mathbb{Z}_{++}. \quad (52)$$

Note that $\mathbb{1}^c_{M_{t+k}} \sim$ Bernoulli$(1-q)$ for all $k \geq 0$, and therefore the expectation works the same way. For the next conditional expectation, first note that the fundamental value process is a martingale. Then see Lemma 1 from Du and Zhu (2017) for the derivation of the second equality, which is done by normal-normal updating. The conditional expectation is then a weighted average between trader n's total signal and the other $N-1$ traders' total signals, inferred from the equilibrium price, where the weight is α, a measure of the amount of adverse selection in the market.

$$\mathbb{E}(v_{n,(t+k)h}|H_{n,th}) = \mathbb{E}(v_{n,th}|H_{n,th}) = \alpha s_{n,th} + (1-\alpha)\frac{\sum_{j\neq n} s_{j,th}}{N-1} \quad (53)$$

The conditional expectation of the future inventory process relies on (42), (51), (52), the fact that total signals are a martingale and the law of the unconscious statistician. After distributing the product of indicator random variables, we are left with the double sum

$$\sum_{k=1}^{\infty}e^{-rhk}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\sum_{l=1}^{k-1}\mathbb{1}^c_{M_{t+k-l}}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big). \quad (54)$$

Combining similar indicators and rearranging yields the following expression

$$\sum_{k=1}^{\infty}e^{-rhk}\sum_{l=1}^{k-1}\mathbb{1}^c_{M_{t+k-l}}\Big((1-\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-l}}d)\Big)\Big(\prod_{j=l+1}^{k-1}(1-\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-j}}d)\Big)\Big(\prod_{i=1}^{l-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)^2.$$

Taking expectations of the random variables, which are independent across time and of any other random process in the model, gives

$$(1-q)(1+d)\sum_{k=1}^{\infty}e^{-rhk}\sum_{l=1}^{k-1}\Big((1-q)(1+d)+q(1-\frac{\xi^2_\mu}{\xi_\sigma})\Big)^{k-l-1}\Big((1-q)(1+d)^2+q(1-\frac{\xi^2_\mu}{\xi_\sigma})\Big)^{l-1}. \quad (55)$$

Finally, the double sum can be evaluated to be equal to

$$\frac{e^{-2rh}(1-q)(1+d)}{\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}. \tag{56}$$

The conditional expectation of the future inventory process relies on (42), (51), (52), the fact that total signals are a martingale and the law of the unconscious statistician. The result is different if a size-discovery session happens.

$$\sum_{k=1}^{\infty} e^{-rhk}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\sum_{l=1}^{k-1}\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big). \tag{57}$$

Combining similar indicators and rearranging yields the following expression

$$\sum_{k=1}^{\infty} e^{-rhk}\sum_{l=1}^{k-1}\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\Big(\big(1-\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-l}}d\big)\Big)\Big(\prod_{j=l+1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-j}}d\big)\Big)$$
$$\times\Big(\prod_{i=1}^{l-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)^2. \tag{58}$$

First, take the expectations of the random variables, which are independent across time and of any other random process in the model. It can be seen that $\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\big(1-\mathbb{1}_{M_{t+k-l}}\xi_{t+k-l}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-l}}d\big)=0$, and, therefore, this whole term equals zero.

The equilibrium demand allocation from Appendix B for size-discovery sessions, we have the expected allocation next period if a size-discovery session happens is

$$\mathbb{E}(Y_{n,(t+1)h}|H_{n,th})=\frac{\xi_\mu}{\xi_\sigma}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big(s_{n,th}-\frac{1}{N}\sum_{n=1}^{N}s_{n,th}\Big)+\frac{Z}{N}-z_{n,th}-X_{n,th}\Big). \tag{59}$$

Note that we can also condition on $\mathbb{1}_{M_t}=0$, as if a size-discovery session occurs in period t, then the demand schedule we play in period t is irrelevant. Plugging (51), (52), (53), and (57) into (50), and evaluating the infinite sums, as they are geometric sums and assuming for and verifying later that $|(1-q)(\xi+d)+1-\xi|<e^{rh}$, the first-order condition becomes

$$-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big(s_{n,th}-\frac{1}{N}\sum_{n=1}^{N}s_{n,th}\Big)-z_{n,th}+\frac{Z}{N}\Big)+e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}X_{n,th}+\frac{b(N-1)q\frac{\xi_\mu^2}{\xi_\sigma}e^{-rh}p_{th}}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}+$$
$$\frac{(1-e^{-rh})b(N-1)(1-\frac{\xi_\mu^2}{\xi_\sigma})qe^{-rh}}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\mathbb{E}[v_{n,th}|H_{n,th}] \tag{60}$$
$$-\frac{\lambda(1-e^{-rh})b(N-1)(1-\frac{\xi_\mu^2}{\xi_\sigma})q}{r}\Big(\frac{e^{-2rh}(1-q)(1+d)}{\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}(as_{n,th}-bp^*_{th}+fZ)+$$
$$\frac{e^{-rh}}{1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}(z_{n,th}+X_{n,th})\Big)+$$
$$-X_{n,th}-b(N-1)p_{th}-\frac{db(N-1)(1-q)e^{-rh}p_{th}}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}+(1-e^{-rh})b(N-1)\Big(\mathbb{E}[v_{n,th}|H_{n,th}]-\frac{\lambda}{r}(z_{n,th}+X_{n,th})\Big)+$$
$$\frac{(1-e^{-rh})b(N-1)(1+d)(1-q)e^{-rh}}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\Big(\mathbb{E}[v_{n,th}|H_{n,th}]-\frac{\lambda}{r}\big(as_{n,th}-bp^*_{th}+fZ\big)\Big)$$
$$-\frac{\lambda(1-e^{-rh})b(N-1)(1+d)^2(1-q)}{r}\Big(\frac{e^{-2rh}(1-q)(1+d)}{\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}(as_{n,th}-bp^*_{th}+fZ)+$$

$$\frac{e^{-rh}}{1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\big(z_{n,th}+X_{n,th}\big)\Big)=0$$

The rest is rearranging and minor simplification of (60) to match coefficients of the optimal demand schedule, $X_{n,th}$, submitted for trader n at trading period t at clock-time th given all other traders submit (16). As we will see, the optimal demand schedule, $X_{n,th}$, will take the same functional form as (16). From this relationship, we will match coefficients from the conjectured functional form, (16) evaluated at the equilibrium market clearing price, and that found in the below equation to pin down the coefficients in equilibrium. The main algebraic steps that happen are moving all terms that involve $X_{n,th}$ to the other side of the equation and combining all terms that are a function of $s_{n,th}-\bar{s}_{th}$, $z_{n,th}$, and $\frac{Z}{N}$ together. I start the algebraic work by combining all like terms

$$\begin{aligned}
&-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big(s_{n,th}-\bar{s}_{th}\Big)+\frac{Z}{N}\Big)+\frac{(1-e^{-rh})b(N-1)}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\mathbb{E}[v_{n,th}|H_{n,th}]+ \qquad (61)\\
&-\frac{b(N-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}p_{th}^*-\frac{(1-e^{-rh})b(N-1)(1+d)(1-q)e^{-rh}\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\big(as_{n,th}-bp_{th}^*+fZ\big)\\
&-\frac{\lambda(1-e^{-rh})b(N-1)\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}{r}\frac{e^{-2rh}(1-q)(1+d)(as_{n,th}-bp_{th}^*+fZ)}{\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}+\\
&\Big(e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}-\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)z_{n,th}=\Big(1-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}+\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)X_{n,th}.
\end{aligned}$$

Next, plug in the equilibrium price $p_{th}^*=\frac{a}{b}\bar{s}_{th}+\frac{d+Nf}{Nb}Z$. Note that it is needed that $a=b$, such that the left-hand side can be written as a linear function of $s_{n,th}-\bar{s}_{th}$.

$$\begin{aligned}
&-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big(s_{n,th}-\bar{s}_{th}\Big)+\frac{Z}{N}\Big)+\frac{(1-e^{-rh})b(N\alpha-1)}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\big(s_{n,th}-\bar{s}_{th}\big)+ \qquad (62)\\
&-\frac{(N-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}\frac{d+Nf}{N}Z-\frac{(1-e^{-rh})b(N-1)(1+d)(1-q)e^{-rh}\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\big(a(s_{n,th}-\bar{s}_{th})-d\frac{Z}{N}\big)\\
&-\frac{\lambda(1-e^{-rh})b(N-1)\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}{r}\frac{e^{-2rh}(1-q)(1+d)\big(a(s_{n,th}-\bar{s}_{th})-d\frac{Z}{N}\big)}{\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}+\\
&\Big(e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}-\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)z_{n,th}=\Big(1-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}+\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)X_{n,th}.
\end{aligned}$$

Finally, combine all like terms of Z and $s_{n,th}-\bar{s}_{th}$ to get the final

$$\begin{aligned}
&\Big[-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}-\frac{(N-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}(d+Nf)+ \qquad (63)\\
&\frac{e^{-rh}\lambda(1-e^{-rh})b(N-1)(1-q)(1+d)d}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big]\frac{Z}{N}\\
&\Big[\frac{(1-e^{-rh})b(N\alpha-1)}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}-\frac{\xi_\mu^2}{\xi_\sigma}q\frac{r(N\alpha-1)}{\lambda(N-1)}e^{-rh}-\\
&\frac{\lambda(1-e^{-rh})b(N-1)e^{-rh}a(1-q)(1+d)}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big]\big(s_{n,th}-\bar{s}_{th}\big)\\
&\Big(e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}-\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)z_{n,th}=\\
&\Big(1-e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}+\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Big)X_{n,th}.
\end{aligned}$$

Taking (16) and plugging in the conjectured market clearing price gives the equilibrium allocation $X^*_{n,th}(p^*_{th}) = a(s_{n,th} - \bar{s}_{th}) + dz_{n,th} - \frac{d}{N}Z$. Therefore, by matching coefficients, three equations with three unknowns can be set up to pin down b, d, and f, given we already know how $a = b$. The analytical solution is a solution to a cubic equation, and while technically has a closed-form solution, it is unwieldy. I, therefore, omit it as given the length of the formula, there is little economic intuition one can learn directly from it. I only focus on one of the three solutions to the cubic. One solution implies a $d \notin (-1, 0)$ and therefore the infinite sums done earlier will not converge. Therefore, that solution can not be one in equilibrium. The other two solutions are solutions of the cubic and are essentially mirrors of each other and are always in $(-1, 0)$. As the number of traders diverges to infinity, $N \to \infty$, one solution converges to trade vanishing, and the other converges to the competitive equilibrium. I therefore only focus on the one that converges to the competitive equilibrium as the number of traders becomes large, as that matches economic reality and intuition. Similar logic is used in Du and Zhu (2017) and Antill and Duffie (2020) when choosing which equilibrium to focus on. After algebra, the four parameters can be shown to satisfy

$$a = -d\frac{r(N\alpha - 1)}{\lambda(N-1)}, \tag{64}$$

$$b = a, \tag{65}$$

$$f = -\frac{d}{N} - \frac{a\lambda}{Nr}, \tag{66}$$

$$e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}(1+d) = d - \frac{(1+d)d(N\alpha-1)(1-e^{-rh})}{1 - e^{-rh}\big((1-q)(1+d)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}, \tag{67}$$

where d is the solution discussed above to the bottom equation. Therefore, the equilibrium demand schedule submitted, price, and allocation are

$$X^*_{n,th}(p) = a\Big[s_{n,th} - p - \frac{\lambda(N-1)}{r(N\alpha-1)}z_{n,th} + \frac{\lambda(1-\alpha)}{r(N\alpha-1))}Z\Big], \tag{68}$$

$$p^*_{th} = \bar{s}_{th} - \frac{\lambda}{rN}Z, \tag{69}$$

$$X^*_{n,th}(p^*_{th}) = -d_{\text{wu}}\Big[\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th} - \bar{s}_{th}\big) - z_{n,th} + \frac{Z}{N}\Big]. \tag{70}$$

The next step is to check that the second order condition is negative in equilibrium, and therefore the d we solve for is a minimum. Differentiating (41) again with respect to price yields the following equation needed to be evaluated at the solved for d.

$$\mathbb{E}\Big[\sum_{k=0}^{\infty} e^{-rhk}\Big(\mathbb{1}_{M_{t+k}}\Big(-\frac{\partial^2(p_{(t+k-1)h}\xi_{t+k}Y_{n,(t+k)h})}{(\partial p^*_{th})^2} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial^2(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})}{(\partial p^*_{th})^2} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial^2(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})^2}{(\partial p^*_{th})^2}\Big) +$$
$$\mathbb{1}^c_{M_{t+k}}\Big(-\frac{\partial^2(p_{(t+k)h}X_{n,(t+k)h})}{(\partial p^*_{th})^2} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial^2(X_{n,(t+k)h} + z_{n,(t+k)h})}{(\partial p^*_{th})^2} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial^2(X_{n,(t+k)h} + z_{n,(t+k)h})^2}{(\partial p^*_{th})^2}\Big)\Big)\Big|H_{n,th}\Big]. \tag{71}$$

In (71) many terms disappear when differentiating a second time with respect to the price as they are linear in price. Therefore (71) becomes

$$\mathbb{E}\Big[2e^{-rh}\mathbb{1}_{M_{t+1}}\xi_{t+1}\frac{\xi_\mu}{\xi_\sigma}b(N-1) \tag{72}$$
$$-(1-e^{-rh})b^2(N-1)^2\frac{\lambda}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1-\frac{\xi_\mu}{\xi_\sigma}\xi_{t+k})^2\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma} + \mathbb{1}^c_{M_{t+k-i}}d\big)\Big)^2 +$$
$$-2b(N-1) - (1-e^{-rh})b^2(N-1)^2\frac{\lambda}{r} +$$
$$-(1-e^{-rh})b^2(N-1)^2(1+d)^2\frac{\lambda}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma} + \mathbb{1}^c_{M_{t+k-i}}d\big)\Big)^2\Big|H_{n,th}\Big]$$

Clearly, every term is negative except for the first term. The expectation of the first term is $2e^{-rh}q\frac{\xi_\mu^2}{\xi_\sigma}b(N-1)$, which is less than the negative term $2b(N-1)$ in the third line, as by Jensen's inequality, $\xi_\mu^2 < \xi_\sigma$, and $q < 1$.

Therefore, (72) is negative when the equilibrium $d \in (-1, 0)$. This implies that the demand schedule we solved for is a maximum.

Appendix A.1 Proof of Price-Discovery Optimality for Matching Sessions with Impaired Size-Discovery

When the size-discovery session is modeled as a matching session, the equilibrium demand schedule is slightly different. During a work-up session, the prior price is used as the price for the size-discovery session. In a matching session, the price used for size-discovery trading is provided by the platform operator to maximize the expected total volume, which is proved to be the correct price in Appendix C. Therefore, the FOC is the exact same except for the first term, as demand today does not affect the price used in the next trading session if a size-discovery session occurs. Therefore, the FOC is

$$\mathbb{E}\Bigg[b(N-1)\frac{\xi_\mu}{\xi_\sigma}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}\xi_{t+k}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)p_{(t+k-1)h}+ \tag{73}$$

$$(1-e^{-rh})b(N-1)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}(1-\frac{\xi_\mu}{\xi_\sigma}\xi_{t+k})\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\Big(v_{n,(t+k)h}-\frac{\lambda}{r}(z_{n,(t+k)h}+\xi_{t+k}Y_{n,(t+k)h})\Big)+$$

$$-X^*_{n,th}-b(N-1)p^*_{th}-db(N-1)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)p_{(t+k)h}+$$

$$(1-e^{-rh})b(N-1)\Big(v_{n,th}-\frac{\lambda}{r}(z_{n,th}+X_{n,th})\Big)+$$

$$(1-e^{-rh})b(N-1)(1+d)\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma}+\mathbb{1}^c_{M_{t+k-i}}d)\Big)\Big(v_{n,(t+k)h}-\frac{\lambda}{r}(z_{n,(t+k)h}+X_{n,(t+k)h})\Big)\Bigg|H_{n,th}\Bigg]=0$$

Skipping the algebra, which is almost the exact same, after taking the expectation and evaluating the sums, the FOC can be written as

$$\Bigg[-\frac{(N-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}(d+Nf)+ \tag{74}$$

$$\frac{e^{-rh}\lambda(1-e^{-rh})b(N-1)(1-q)(1+d)d}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Bigg]\frac{Z}{N}$$

$$\Bigg[\frac{(1-e^{-rh})b(N\alpha-1)}{1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}-$$

$$-\frac{\lambda(1-e^{-rh})b(N-1)e^{-rh}a(1-q)(1+d)}{r\big(1-e^{-rh}\big((1-q)(1+d)+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Bigg](s_{n,th}-\bar{s}_{th})$$

$$\Bigg(-\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Bigg)z_{n,th}=$$

$$\Bigg(1+\frac{(1-e^{-rh})b(N-1)\lambda}{r\big(1-e^{-rh}\big((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)\big)}\Bigg)X_{n,th}.$$

Taking (16) and plugging in the conjectured market clearing price gives the equilibrium allocation $X^*_{n,th}(p^*_{th}) = a(s_{n,th}-\bar{s}_{th})+dz_{n,th}-\frac{d}{N}Z$. Therefore, by matching coefficients, three equations with three unknowns can be set up to pin down b, d, and f, given we already know how $a = b$. There are two solutions that are essentially mirrors of each other and are always in $(-1, 0)$. As the number of traders diverges to infinity, $N \to \infty$, one solution converges to trade vanishing, and the other converges to the competitive equilibrium. I therefore only focus on the one that converges to the competitive equilibrium as the number of traders becomes large, as that matches economic reality and intuition.

Similar logic is used in Du and Zhu (2017) and Antill and Duffie (2020) when choosing which equilibrium to focus on. After algebra, the four parameters can be shown to satisfy

$$a = -d\frac{r(N\alpha - 1)}{\lambda(N-1)}, \tag{75}$$

$$b = a, \tag{76}$$

$$f = -\frac{d}{N} - \frac{a\lambda}{Nr}, \tag{77}$$

$$\frac{(1+d)(N\alpha-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)} = 1. \tag{78}$$

Solving the quadratic equation for d_{ms} and focusing on the more economically intuitive equilibrium yields

$$d_{\text{ms}} = -1 + \frac{\sqrt{(N\alpha-1)^2(1-e^{-rh})^2 - 4e^{-rh}(1-q)\big(e^{-rh}q(1-\frac{\xi_\mu^2}{\xi_\sigma}) - 1\big)} - (N\alpha-1)(1-e^{-rh})}{2e^{-rh}(1-q)}. \tag{79}$$

The equilibrium demand schedule submitted, price, and allocation are

$$X^*_{n,th}(p) = a\Big[s_{n,th} - p - \frac{\lambda(N-1)}{r(N\alpha-1)} z_{n,th} + \frac{\lambda(1-\alpha)}{r(N\alpha-1))} Z\Big], \tag{80}$$

$$p^*_{th} = \bar{s}_{th} - \frac{\lambda}{rN} Z, \tag{81}$$

$$X^*_{n,th}(p^*_{th}) = -d_{\text{ms}}\Big[\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th} - \bar{s}_{th}\big) - z_{n,th} + \frac{Z}{N}\Big]. \tag{82}$$

The next step is to check that the second order condition is negative in equilibrium, and therefore the d we solve for is a minimum. Differentiating (41) again with respect to price yields the following equation needed to be evaluated at the solved for d.

$$\mathbb{E}\Big[\sum_{k=0}^{\infty} e^{-rhk}\Big(\mathbb{1}_{M_{t+k}}\Big(-\frac{\partial^2(p_{(t+k-1)h}\xi_{t+k}Y_{n,(t+k)h})}{(\partial p^*_{th})^2} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial^2(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})}{(\partial p^*_{th})^2} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial^2(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})^2}{(\partial p^*_{th})^2}\Big) +$$
$$\mathbb{1}^c_{M_{t+k}}\Big(-\frac{\partial^2(p_{(t+k)h}X_{n,(t+k)h})}{(\partial p^*_{th})^2} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial^2(X_{n,(t+k)h} + z_{n,(t+k)h})}{(\partial p^*_{th})^2} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial^2(X_{n,(t+k)h} + z_{n,(t+k)h})^2}{(\partial p^*_{th})^2}\Big)\Big)\Big|H_{n,th}\Big]. \tag{83}$$

In (71) many terms disappear when differentiating a second time with respect to the price as they are linear in price. Therefore (71) becomes

$$\mathbb{E}\Big[-(1-e^{-rh})b^2(N-1)^2\frac{\lambda}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1-\frac{\xi_\mu}{\xi_\sigma}\xi_{t+k})^2\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma} + \mathbb{1}^c_{M_{t+k-i}}d\big)\Big)^2 + \tag{84}$$
$$-2b(N-1) - (1-e^{-rh})b^2(N-1)^2\frac{\lambda}{r} +$$
$$-(1-e^{-rh})b^2(N-1)^2(1+d)^2\frac{\lambda}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}\big(1-\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}\frac{\xi_\mu}{\xi_\sigma} + \mathbb{1}^c_{M_{t+k-i}}d\big)\Big)^2\Big|H_{n,th}\Big]$$

Clearly, every term is negative. Therefore, the second derivative is negative when the equilibrium $d_{\text{ms}} \in (-1, 0)$. This implies that the demand schedule solved for is a maximum.

Appendix B Proof of the Impaired Work-Up Optimality of Demand Quantities

In this appendix section, I derive that (17) is the optimal demand quantity to submit for size-discovery sessions in the class of linear, symmetric, and stationary demand quantities for when size-discovery sessions occur. The proof is unaffected if the demand schedule played in price-discovery sessions accounts for the size-discovery session as impaired or not (the coefficients used during the price-discovery sessions differ but do not affect the solution). Define ξ_t to be a random variable on the unit interval with CDF F. ξ_t represents the fraction of the competitive allocation that a trader receives when there is a size-discovery trading session. In the standard discrete-time setting that is the main focus of this paper, I take $\xi_t = 1$ almost surely. The demand quantity conjectured will take the following form:

$$\mu_{n,th} = a_s\big(s_{n,th} - \bar{s}_{th}\big) + d_s\big(z_{n,th} - \frac{Z}{N}\big). \tag{85}$$

The exact functional form for the allocation mechanism, $Y_{n,th}(\mu)$, as a function of the quantities demanded, does not play a role in the equilibrium demanded quantity as long as it generates the competitive allocation when all traders submit (101). The cash transfer function will be (3). If $\xi_t \neq 1$ almost surely, then both $Y_{n,th}$ the cash transfer function will be scaled by ξ_t. I will use the one-stage deviation principle to verify the equilibrium strategy. As a reminder, let $\mathbb{1}_{M_k}$ be an indicator function equal to one if a size discovery session occurs in period k and zero otherwise (a price discovery session occurs in period k). The inventory of a trader evolves according to

$$z_{n,(t+1)h} = z_{n,th} + \mathbb{1}_{M_t}\xi_t\Big(a_s\big(s_{n,th} - \bar{s}_{th}\big) + d_s\big(z_{n,th} - \frac{Z}{N}\big)\Big) + \mathbb{1}_{M_t^c}\Big(as_{n,th} - bp^*_{th} + dz_{n,th} + fZ\Big). \tag{86}$$

Plugging in the equilibrium demand schedule solved for in Appendix A when price-discovery sessions occur gives the slightly simplified equation

$$z_{n,(t+1)h} = \mathbb{1}_{M_t}\xi_t\Big(a_s\big(s_{n,th} - \bar{s}_{th}\big) - d_s\frac{Z}{N}\Big) - \mathbb{1}^c_{M_t}d\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th} - \bar{s}_{th}\big) + \frac{Z}{N}\Big) + \big(1 + \mathbb{1}_{M_t}\xi_t d_s + \mathbb{1}^c_{M_t}d\big)z_{n,th}. \tag{87}$$

Therefore taking the first order condition of trader n's continuation utility with respect to trader n's demand quantity, the first order condition becomes

$$\frac{\partial}{\partial\mu_{n,th}}V_{n,th}\Big|_{\mu_{n,th}} = \tag{88}$$

$$\mathbb{E}\Bigg[\sum_{k=0}^{\infty}e^{-rhk}\Bigg(\mathbb{1}_{M_{t+k}}\Bigg(-\frac{\partial(p_{(t+k-1)h}\xi_{t+k}Y_{n,(t+k)h})}{\partial\mu_{n,th}} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})}{\partial\mu_{n,th}} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})^2}{\partial\mu_{n,th}}\Bigg)+$$
$$\mathbb{1}^c_{M_{t+k}}\Bigg(-\frac{\partial(p_{(t+k)h}X_{n,(t+k)h})}{\partial\mu_{n,th}}+(1-e^{-rh})v_{n,(t+k)h}\frac{\partial(X_{n,(t+k)h} + z_{n,(t+k)h})}{\partial\mu_{n,th}} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(X_{n,(t+k)h} + z_{n,(t+k)h})^2}{\partial\mu_{n,th}}\Bigg)\Bigg)\Bigg|H_{n,th}\Bigg].$$

To study the evolution of inventory, iterate equation (104) forward k times. Then the inventory of trader n in period $t+k$ can be written as

$$z_{n,(t+k)h} = \sum_{j=1}^{k-1}\Bigg(\prod_{i=1}^{j-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h} - \bar{s}_{(t+k-j)h}\big) + \frac{Z}{N}\Big)+$$
$$\sum_{j=1}^{k-1}\Bigg(\prod_{i=1}^{j-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s\big(s_{n,(t+k-j)h} - \bar{s}_{(t+k-j)h}\big) - d_s\frac{Z}{N}\Big)+ \tag{89}$$
$$\Bigg(\prod_{i=1}^{k-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\big(z_{n,th} + \xi_t Y_{n,th}\big).$$

Note that, without loss of generality, we can assume that $\mathbb{1}_{M_t} = 1$ ($\mathbb{1}^c_{M_t} = 0$) as otherwise the demand quantity submitted is irrelevant. Using (101), (39), (105) and the residual quantity trader n faces, we can derive the partial derivatives in (104). First, by the market-clearing condition, the equilibrium price is only a function of the aggregate signal and the total amount of inventory. Therefore, a change in quantity demanded in period t has no effect on prices in equilibrium.

$$\frac{\partial p^*_{(t+k)h}}{\partial\mu_{n,th}} = 0, \text{ for } k \geq 0 \tag{90}$$

Finally, the change in the allocated quantity with respect to the quantity demanded is the subderivative of $\frac{\partial Y_{n,th}}{\partial \mu_{n,th}} := Y_\mu$, as $Y_{n,th}$ may or may not be differentiable everywhere. If $Y_{n,th}$ is the proportional rationing mechanism, then it is not differentiable at $\mu_{n,th} = 0$, but the subdifferential is in a bounded range that does not include 0, but it is differentiable everywhere else. If $Y_{n,th}$ is the additive rationing mechanism, then the subderevative is the derivative. Using the above equations, we can derive the relationship of the change in future inventory as the demanded quantity in period t as

$$\frac{\partial z_{n,(t+k)h}}{\partial \mu_{n,th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t, & \text{if } k = 1 \\ Y_\mu \xi_t \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{91}$$

With the above partial, (107), we can see how future demand in price-discovery sessions changes as the demand in size-discovery sessions in period t changes.

$$\frac{\partial X_{n,(t+k)h}}{\partial \mu_{th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t d, & \text{if } k = 1 \\ Y_\mu \xi_t d \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{92}$$

From (107) and the residual demand quantity, we know that for $k \geq 1$

$$\frac{\partial Y_{n,(t+k)h}}{\partial \mu_{n,th}} = \begin{cases} Y_\mu \xi_t d_s, & \text{if } k = 1 \\ Y_\mu \xi_t d_s \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{93}$$

Finally, from (107), (108), and (109) it is simple to see how the post-trade inventory for price and size-discovery sessions change with respect to quantity demanded in period t as

$$\frac{\partial (z_{n,(t+k)h} + X_{n,(t+k)h})}{\partial \mu_{n,th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t (1+d), & \text{if } k = 1 \\ Y_\mu \xi_t (1+d) \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2, \end{cases} \tag{94}$$

$$\frac{\partial (z_{n,(t+k)h} + \xi_{t+k} Y_{n,(t+k)h})}{\partial \mu_{n,th}} = \begin{cases} Y_\mu \xi_t, & \text{if } k = 0 \\ Y_\mu \xi_t (1 + d_s \xi_{t+k}), & \text{if } k = 1 \\ Y_\mu \xi_t (1 + d_s \xi_{t+k}) \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2, \end{cases} \tag{95}$$

Now that the partials have been derived, equations (106), (107), (108), (109), (110), and (111) can be plugged into (104) which when set equal to zero will yield the optimal demand quantity for trader n to play in period t when a size-discovery session occurs.

$$\mathbb{E}\Bigg[-\xi_t p_{(t-1)h} Y_\mu - d_s \xi_t Y_\mu \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}_{M_{t+k}} \xi_{t+k} p_{(t+k-1)h} \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big) + (1 - e^{-rh}) \xi_t v_{n,th} Y_\mu$$

$$-\frac{Y_\mu (1 - e^{-rh}) \lambda \xi_t}{r} (z_{n,th} + \xi_t Y_{n,th}) + \xi_t Y_\mu (1 - e^{-rh}) \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}_{M_{t+k}} (1 + d_s \xi_{t+k}) v_{n,(t+k)h} \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big)$$

$$-\frac{(1 - e^{-rh}) \lambda Y_\mu \xi_t}{r} \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}_{M_{t+k}} (1 + d_s \xi_{t+k}) (z_{n,(t+k)h} + \xi_{t+k} Y_{n,(t+k)h}) \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big)$$

$$-d Y_\mu \xi_t \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}^c_{M_{t+k}} p_{(t+k)h} \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big)$$

$$+(1+d) Y_\mu \xi_t (1 - e^{-rh}) \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}^c_{M_{t+k}} v_{n,(t+k)h} \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big)$$

$$-\frac{(1-e^{-rh})\lambda Y_\mu \xi_t(1+d)}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}(z_{n,(t+k)h}+X_{n,(t+k)h})\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big|H_{n,th}\Big]=0 \tag{96}$$

To start the simplification, factor out the common term Y_μ, which is never zero. Then, plug in the inventory process to get

$$\mathbb{E}\Big[-\xi_t p_{(t-1)h}-d_s\xi_t\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}\xi_{t+k}p_{(t+k-1)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)+(1-e^{-rh})\xi_t v_{n,th}$$

$$-\frac{(1-e^{-rh})\lambda\xi_t}{r}(z_{n,th}+\xi_t Y_{n,th})+\xi_t(1-e^{-rh})\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})v_{n,(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$-\frac{(1-e^{-rh})\lambda\xi_t}{r}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big[\Big(a_s\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)-d_s\frac{Z}{N}\Big)$$

$$+(1+\xi_{t+k}d_s)\Big(\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)+\frac{Z}{N}\Big)+$$

$$\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)-d_s\frac{Z}{N}\Big)+$$

$$\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\big(z_{n,th}+\xi_t Y_{n,th}\big)\Big)\Big]$$

$$-d\xi_t\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}p_{(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$+(1+d)\xi_t(1-e^{-rh})\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}v_{n,(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big) \tag{97}$$

$$-\frac{(1-e^{-rh})\lambda(1+d)\xi_t}{r}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big[(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)-\frac{Z}{N}\Big)$$

$$+(1+d)\Big(\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)+\frac{Z}{N}\Big)+$$

$$\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)-d_s\frac{Z}{N}\Big)+$$

$$\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\big(z_{n,th}+\xi_t Y_{n,th}\big)\Big)\Big]\Big|H_{n,th}\Big]=0$$

Now, we're in a form where the expectation can be taken. Note that ξ_{t+k}, $\mathbb{1}_{M_{t+k}}$, and $v_{n,(t+k)h}$ are independent of each other, and ξ_{t+k} and $\mathbb{1}_{M_{t+k}}$ are also independent across time (k). Finally, define $\mathbb{E}(\xi_{t+k})=\xi_\mu$ and $\mathbb{E}(\xi^2_{t+k})=\xi_\sigma$. Finally, recall that the total signal process and, therefore, prices are martingales. Also, factor out a ξ_μ after taking the expectations. Using this, (113) can be written as

$$-p_{(t-1)h}-q\xi_\mu d_s p_{(t-1)h}\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}+(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]$$

$$-\frac{(1-e^{-rh})\lambda}{r}(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th})+q(1+\xi_\mu d_s)(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}$$

$$-\frac{(1-e^{-rh})\lambda}{r}\sum_{k=1}^{\infty}e^{-rhk}\Big[q(1+\xi_\mu d_s)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\Big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\Big)$$

$$+q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)\Big(\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{j=1}^{k-1}(-d)(1-q)(1+d)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+$$

$$
\begin{aligned}
&\Big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\Big)\sum_{j=1}^{k-1} q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{k-1}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Big)\Big]\\
&-d(1-q)p_{(t-1)h}\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\\
&+(1+d)(1-q)\,\mathbb{E}[v_{n,th}|H_{n,th}](1-e^{-rh})\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1} \qquad (98)\\
&-\frac{(1-e^{-rh})\lambda(1+d)(1-q)(-d)}{r}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\\
&-\frac{(1-e^{-rh})\lambda(1+d)^2(1-q)}{r}\sum_{k=1}^{\infty}e^{-rhk}\Big((1-q)(1+d)(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{j=1}^{k-1}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)\Big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\Big)\sum_{j=1}^{k-1}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{k-1}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Big)=0
\end{aligned}
$$

In the next step, I will evaluate the partial sums and then the infinite sums. Assume, which can be verified after solving the equilibrium, that $|e^{-rh}(q(1+\xi_\mu d_s)+(1-q)(1+d))|<1$ and $|e^{-rh}(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2)|<1$. Then (114) can be simplified and written as

$$
\begin{aligned}
&-p_{(t-1)h}-\frac{q\xi_\mu e^{-rh}d_s}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}p_{(t-1)h}+(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]\\
&-\frac{(1-e^{-rh})\lambda}{r}(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th})+\frac{q(1+\xi_\mu d_s)e^{-rh}(1-e^{-rh})}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\,\mathbb{E}[v_{n,th}|H_{n,th}]\\
&-\frac{(1-e^{-rh})\lambda}{r}\Big[\frac{q(1+\xi_\mu d_s)e^{-rh}}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\Big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\Big)\\
&+q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)\Big(\frac{(1-q)(1+d)e^{-2rh}(-d)\big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)e^{-2rh}\big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{e^{-rh}}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Big)\Big]\\
&-\frac{d(1-q)e^{-rh}}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}p_{(t-1)h}\\
&+\frac{(1+d)(1-q)e^{-rh}(1-e^{-rh})}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\,\mathbb{E}[v_{n,th}|H_{n,th}] \qquad (99)\\
&-\frac{(1-e^{-rh})\lambda(1+d)(1-q)e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\\
&-\frac{(1-e^{-rh})\lambda(1+d)^2(1-q)}{r}\Big[\frac{(1-q)(1+d)e^{-2rh}(-d)\big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)e^{-2rh}\Big(a_s\big(s_{n,th}-\bar{s}_{th}\big)-d_s\frac{Z}{N}\Big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{e^{-rh}}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Big]=0
\end{aligned}
$$

From here, it is all algebra to show that the $Y_{n,th}$ is linear in $s_{n,th}$ and $z_{n,th}$ in equilibrium. Then we will match the coefficients from the conjectured linear form to pin down the coefficients in equilibrium. Start by combining the like terms and simplify out a $1-e^{-rh}$ of (115) to get

$$-\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}p_{(t-1)h}+\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\mathbb{E}[v_{n,th}|H_{n,th}]$$

$$-\frac{\lambda e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big)\big(s_{n,th}-\bar{s}_{th}\big)+$$

$$+\frac{\lambda e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(d_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu+d(1+d)(1-q)\big)\frac{Z}{N}+$$

$$=\frac{\lambda}{r\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)$$

For now, assume that there is no shock. Multiply through by $\frac{r}{\lambda}$ Plug in the equilibrium price and expected value and combine to get:

$$-\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\left(\frac{r}{\lambda}\bar{s}_{th}-\frac{1}{N}Z\right)+\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\frac{r}{\lambda}\left(\frac{N\alpha-1}{N-1}s_{n,th}-\frac{N(1-\alpha)}{N-1}\bar{s}_{th}\right)$$

$$-\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big)\big(s_{n,th}-\bar{s}_{th}\big)+$$

$$+\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(d_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu+d(1+d)(1-q)\big)\frac{Z}{N}+$$

$$=\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big).$$

After combining like terms one last time, it can be seen the optimal allocation is now a linear function of the deviation from the average total signal, $s_{n,th}-\bar{s}_{th}$, the prior inventory level, $z_{n,th}$, and the perfect risk-sharing amount, $\frac{Z}{N}$.

$$\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)$$

$$-\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big)\big(s_{n,th}-\bar{s}_{th}\big)+$$

$$+\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\frac{Z}{N}+$$

$$-\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}z_{n,th}=\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}.$$

Therefore, any $Y_{n,th}$ that allocates $\mu_{n,th}$ when all traders play (101) yields

$$\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)$$

$$-\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big)\big(s_{n,th}-\bar{s}_{th}\big)+$$

$$+\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\frac{Z}{N}+$$

$$-\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}z_{n,th}=\frac{1}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}\left(a_s\big(s_{n,th}-\bar{s}_{th}\big)+d_s\big(z_{n,th}-\frac{Z}{N}\big)\right).$$

Matching coefficients on $z_{n,th}$ yields the equation

$$0=1+\frac{\xi_\sigma}{\xi_\mu}d_s,$$

which has the unique solution

$$d_s=-\frac{\xi_\mu}{\xi_\sigma}.$$

Plug this in for the above equation to get

$$\frac{1}{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)$$
$$+\frac{e^{-rh}}{\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big(s_{n,th}-\bar{s}_{th}\big)+$$
$$+\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{Z}{N}$$
$$=\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}\left(a_s\big(s_{n,th}-\bar{s}_{th}\big)+\frac{\xi_\mu}{\xi_\sigma}\frac{Z}{N}\right).$$

The $\frac{Z}{N}$ will cancel out and the fractions simplify out to get

$$\frac{1}{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)$$
$$+\frac{e^{-rh}d(1+d)(1-q)}{\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+$$
$$=\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}a_s\big(s_{n,th}-\bar{s}_{th}\big).$$

This simplifies to

$$\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+$$
$$=\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}a_s(s_{n,th}-\bar{s}_{th}).$$

Matching coefficients yields

$$\frac{\xi_\mu}{\xi_\sigma}\frac{r(N\alpha-1)}{\lambda(N-1)}=a_s.$$

Therefore, the unique linear equilibrium, up to a constant shift, is

$$\mu_{n,th}=\frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)-z_{n,th}+\frac{Z}{N}\right),$$

which allocates

$$Y_{n,th}(\mu)=\frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)-z_{n,th}+\frac{Z}{N}\right).$$

To check the second-order condition, differentiate the FOC again. As every term has a Y_μ in it, the second-order condition can be written as $Y_{\mu\mu}$ times the FOC, which is zero in equilibrium, plus

$$\mathbb{E}\Bigg[-\frac{(1-e^{-rh})\lambda\xi_t^2Y_\mu^2}{r}-\frac{(1-e^{-rh})\lambda Y_\mu^2\xi_t^2}{r}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})^2\left(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\right)^2 \tag{100}$$
$$-\frac{(1-e^{-rh})\lambda Y_\mu^2\xi_t^2(1+d)^2}{r}\sum_{k=1}^{\infty}e^{-rhk}\mathbb{1}^c_{M_{t+k}}\left(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\right)^2\Bigg|H_{n,th}\Bigg]<0$$

This is always less than zero, and, therefore, the solved for optimal demand quantity to submit is a maximum of the continuation utility for trader n when a size-discovery session occurs.

Appendix B.1 Proof of the Impaired Size-Discovery Demand Quantities Optimality with Price Chosen by Operator

In this appendix section, I prove that the demand quantity (17) is the optimal demand quantity to submit for size-discovery sessions in the class of linear, symmetric, and stationary demand quantities for when size-discovery sessions occur. The proof is unaffected if the demand schedule played in price-discovery sessions accounts for the size-discovery session as impaired or not (the coefficients used during the price-discovery sessions differ but do not affect the solution). Define ξ_t to be a random variable on the unit interval with CDF F. ξ_t represents the fraction of the competitive allocation that a trader receives when there is a size-discovery trading session. In the standard discrete-time setting that is the main focus of this paper, I take $\xi_t = 1$ almost surely. The demand quantity conjectured will take the following form:

$$\mu_{n,th} = a_s s_{n,th} - b_s p + d_s z_{n,th} + f_s Z. \tag{101}$$

The exact functional form for the allocation mechanism, $Y_{n,th}(\mu)$, as a function of the quantities demanded, does not play a role in the equilibrium demanded quantity as long as it generates the competitive allocation when all traders submit (101). The cash transfer function will be (3). If $\xi_t \neq 1$ almost surely, then both $Y_{n,th}$ the cash transfer function will be scaled by ξ_t. I will use the one-stage deviation principle to verify the equilibrium strategy. As a reminder, let $\mathbb{1}_{M_k}$ be an indicator function equal to one if a size discovery session occurs in period k and zero otherwise (a price discovery session occurs in period k). The inventory of a trader evolves according to

$$z_{n,(t+1)h} = z_{n,th} + \mathbb{1}_{M_t}\xi_t\Big(a_s s_{n,th} - b_s p + d_s z_{n,th} + f_s Z\Big) + \mathbb{1}_{M_t^c}\Big(a s_{n,th} - b p^*_{th} + d z_{n,th} + f Z\Big). \tag{102}$$

Plugging in the equilibrium demand schedule solved for in Appendix A when price-discovery sessions occur gives the slightly simplified equation

$$z_{n,(t+1)h} = \mathbb{1}_{M_t}\xi_t\Big(a_s s_{n,th} - b_s p + f_s Z\Big) - \mathbb{1}^c_{M_t} d\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th} - \bar{s}_{th}\big) + \frac{Z}{N}\Big) + \big(1 + \mathbb{1}_{M_t}\xi_t d_s + \mathbb{1}^c_{M_t} d\big) z_{n,th}. \tag{103}$$

Therefore taking the first order condition of trader n's continuation utility with respect to trader n's demand quantity, the first order condition becomes

$$\frac{\partial}{\partial \mu_{n,th}} V_{n,th}\Big|_{\mu_{n,th}} = \tag{104}$$

$$\mathbb{E}\Bigg[\sum_{k=0}^{\infty} e^{-rhk}\Bigg(\mathbb{1}_{M_{t+k}}\Big(-\frac{\partial(p\xi_{t+k}Y_{n,(t+k)h})}{\partial\mu_{n,th}} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})}{\partial\mu_{n,th}} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(\xi_{t+k}Y_{n,(t+k)h} + z_{n,(t+k)h})^2}{\partial\mu_{n,th}}\Big) +$$
$$\mathbb{1}^c_{M_{t+k}}\Big(-\frac{\partial(p_{(t+k)h}X_{n,(t+k)h})}{\partial\mu_{n,th}} + (1-e^{-rh})v_{n,(t+k)h}\frac{\partial(X_{n,(t+k)h} + z_{n,(t+k)h})}{\partial\mu_{n,th}} - \frac{\lambda}{2r}(1-e^{-rh})\frac{\partial(X_{n,(t+k)h} + z_{n,(t+k)h})^2}{\partial\mu_{n,th}}\Big)\Bigg)\Bigg|H_{n,th}\Bigg].$$

To study the evolution of inventory, iterate equation (104) forward k times. Then the inventory of trader n in period $t+k$ can be written as

$$z_{n,(t+k)h} = \sum_{j=1}^{k-1}\Bigg(\prod_{i=1}^{j-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h} - \bar{s}_{(t+k-j)h}\big) + \frac{Z}{N}\Big) +$$
$$\sum_{j=1}^{k-1}\Bigg(\prod_{i=1}^{j-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s s_{n,(t+k-j)h} - b_s p + f_s Z\Big) + \tag{105}$$
$$\Bigg(\prod_{i=1}^{k-1}\big(1 + \mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s + \mathbb{1}^c_{M_{t+k-i}}d\big)\Bigg)\big(z_{n,th} + \xi_t Y_{n,th}\big).$$

Note that, without loss of generality, we can assume that $\mathbb{1}_{M_t} = 1$ ($\mathbb{1}^c_{M_t} = 0$) as otherwise the demand quantity submitted is irrelevant. Using (101), (39), (105) and the residual quantity trader n faces, we can derive the partial derivatives in (104). First, by the market-clearing condition, the equilibrium price is only a function of the aggregate signal and the total amount of inventory. Therefore, a change in quantity demanded in period t has no effect on prices in equilibrium.

$$\frac{\partial p^*_{(t+k)h}}{\partial \mu_{n,th}} = 0, \text{ for } k \geq 0 \tag{106}$$

Finally, the change in the allocated quantity with respect to the quantity demanded is the subderivative of $\frac{\partial Y_{n,th}}{\partial \mu_{n,th}} := Y_\mu$, as $Y_{n,th}$ may or may not be differentiable everywhere. If $Y_{n,th}$ is the proportional rationing mechanism, then it is not differentiable at $\mu_{n,th} = 0$, but the subdifferential is in a bounded range that does not include 0, but it is differentiable everywhere else. If $Y_{n,th}$ is the additive rationing mechanism, then the subderevative is the derivative. Using the above equations, we can derive the relationship of the change in future inventory as the demanded quantity in period t as

$$\frac{\partial z_{n,(t+k)h}}{\partial \mu_{n,th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t, & \text{if } k = 1 \\ Y_\mu \xi_t \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{107}$$

With the above partial, (107), we can see how future demand in price-discovery sessions changes as the demand in size-discovery sessions in period t changes.

$$\frac{\partial X_{n,(t+k)h}}{\partial \mu_{th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t d, & \text{if } k = 1 \\ Y_\mu \xi_t d \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{108}$$

From (107) and the residual demand quantity, we know that for $k \geq 1$

$$\frac{\partial Y_{n,(t+k)h}}{\partial \mu_{n,th}} = \begin{cases} Y_\mu \xi_t d_s, & \text{if } k = 1 \\ Y_\mu \xi_t d_s \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2. \end{cases} \tag{109}$$

Finally, from (107), (108), and (109) it is simple to see how the post-trade inventory for price and size-discovery sessions change with respect to quantity demanded in period t as

$$\frac{\partial (z_{n,(t+k)h} + X_{n,(t+k)h})}{\partial \mu_{n,th}} = \begin{cases} 0, & \text{if } k = 0 \\ Y_\mu \xi_t (1+d), & \text{if } k = 1 \\ Y_\mu \xi_t (1+d) \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2, \end{cases} \tag{110}$$

$$\frac{\partial (z_{n,(t+k)h} + \xi_{t+k} Y_{n,(t+k)h})}{\partial \mu_{n,th}} = \begin{cases} Y_\mu \xi_t, & \text{if } k = 0 \\ Y_\mu \xi_t (1 + d_s \xi_{t+k}), & \text{if } k = 1 \\ Y_\mu \xi_t (1 + d_s \xi_{t+k}) \Big(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Big), & \text{if } k \geq 2, \end{cases} \tag{111}$$

Now that the partials have been derived, equations (106), (107), (108), (109), (110), and (111) can be plugged into (104) which when set equal to zero will yield the optimal demand quantity for trader n to play in period t when a size-discovery session occurs.

$$\mathbb{E}\Bigg[- \xi_t p Y_\mu - d_s \xi_t Y_\mu p \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}_{M_{t+k}} \xi_{t+k} \Bigg(\prod_{i=1}^{k-1} \left(1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d\right) \Bigg) + (1 - e^{-rh}) \xi_t v_{n,th} Y_\mu$$

$$- \frac{Y_\mu (1 - e^{-rh}) \lambda \xi_t}{r} (z_{n,th} + \xi_t Y_{n,th}) + \xi_t Y_\mu (1 - e^{-rh}) \sum_{k=1}^{\infty} e^{-rhk} \mathbb{1}_{M_{t+k}} (1 + d_s \xi_{t+k}) v_{n,(t+k)h} \Bigg(\prod_{i=1}^{k-1} (1 + \mathbb{1}_{M_{t+k-i}} \xi_{t+k-i} d_s + \mathbb{1}^c_{M_{t+k-i}} d) \Bigg)$$

$$-\frac{(1-e^{-rh})\lambda Y_\mu \xi_t}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})(z_{n,(t+k)h}+\xi_{t+k}Y_{n,(t+k)h})\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$-dY_\mu\xi_t\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}p_{(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$+(1+d)Y_\mu\xi_t(1-e^{-rh})\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}v_{n,(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$-\frac{(1-e^{-rh})\lambda Y_\mu\xi_t(1+d)}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}(z_{n,(t+k)h}+X_{n,(t+k)h})\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big|H_{n,th}\Big]=0 \tag{112}$$

To start the simplification, factor out the common term Y_μ, which is never zero. Then, plug in the inventory process to get

$$\mathbb{E}\Big[-\xi_t p-d_s\xi_t p\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}\xi_{t+k}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)+(1-e^{-rh})\xi_t v_{n,th}$$

$$-\frac{(1-e^{-rh})\lambda\xi_t}{r}(z_{n,th}+\xi_t Y_{n,th})+\xi_t(1-e^{-rh})\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})v_{n,(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$-\frac{(1-e^{-rh})\lambda\xi_t}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big[\Big(a_s s_{n,(t+k-j)h}-b_s p+f_s Z\Big)$$

$$+(1+\xi_{t+k}d_s)\Big(\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)+\frac{Z}{N}\Big)+$$

$$\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s s_{n,(t+k-j)h}-b_s p+f_s Z\Big)+$$

$$\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\big(z_{n,th}+\xi_t Y_{n,th}\big)\Big)\Big]$$

$$-d\xi_t\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}p_{(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)$$

$$+(1+d)\xi_t(1-e^{-rh})\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}v_{n,(t+k)h}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big) \tag{113}$$

$$-\frac{(1-e^{-rh})\lambda(1+d)\xi_t}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\Big[(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)-\frac{Z}{N}\Big)$$

$$+(1+d)\Big(\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}^c_{M_{t+k-j}}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,(t+k-j)h}-\bar{s}_{(t+k-j)h}\big)+\frac{Z}{N}\Big)+$$

$$\sum_{j=1}^{k-1}\Big(\prod_{i=1}^{j-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\mathbb{1}_{M_{t+k-j}}\xi_{t+k-j}\Big(a_s s_{n,(t+k-j)h}-b_s p+f_s Z\Big)+$$

$$\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big)\big(z_{n,th}+\xi_t Y_{n,th}\big)\Big)\Big]\Big|H_{n,th}\Big]=0$$

Now, we're in a form where the expectation can be taken. Note that ξ_{t+k}, $\mathbb{1}_{M_{t+k}}$, and $v_{n,(t+k)h}$ are independent of each other, and ξ_{t+k} and $\mathbb{1}_{M_{t+k}}$ are also independent across time (k). Finally, define $\mathbb{E}(\xi_{t+k})=\xi_\mu$ and $\mathbb{E}(\xi^2_{t+k})=\xi_\sigma$. Finally, recall that the total signal process and, therefore, prices are martingales. Also, factor out a ξ_μ after taking the expectations. Using this, (113) can be written as

$$-p-q\xi_\mu d_s p\sum_{k=1}^{\infty} e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}+(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]$$

$$
\begin{aligned}
&-\frac{(1-e^{-rh})\lambda}{r}(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th})+q(1+\xi_\mu d_s)(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\\
&-\frac{(1-e^{-rh})\lambda}{r}\sum_{k=1}^{\infty}e^{-rhk}\Big[q(1+\xi_\mu d_s)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\Big(a_s s_{n,th}-b_s p+f_s Z\Big)\\
&+q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)\Bigg(\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{j=1}^{k-1}(-d)(1-q)(1+d)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&\Big(a_s s_{n,th}-b_s p+f_s Z\Big)\sum_{j=1}^{k-1}q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{k-1}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Bigg)\Big]\\
&-d(1-q)p_{(t-1)h}\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\\
&+(1+d)(1-q)\,\mathbb{E}[v_{n,th}|H_{n,th}](1-e^{-rh})\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1} \qquad (114)\\
&-\frac{(1-e^{-rh})\lambda(1+d)(1-q)(-d)}{r}\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{k=1}^{\infty}e^{-rhk}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-1}\\
&-\frac{(1-e^{-rh})\lambda(1+d)^2(1-q)}{r}\sum_{k=1}^{\infty}e^{-rhk}\Bigg((1-q)(1+d)(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\sum_{j=1}^{k-1}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)\Big(a_s s_{n,th}-b_s p+f_s Z\Big)\sum_{j=1}^{k-1}\big(1+q\xi_\mu d_s+(1-q)d\big)^{k-j-1}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{j-1}+\\
&\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)^{k-1}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Bigg)=0
\end{aligned}
$$

In the next step, I will evaluate the partial sums and then the infinite sums. Assume, which can be verified after solving the equilibrium, that $|e^{-rh}(q(1+\xi_\mu d_s)+(1-q)(1+d))|<1$ and $|e^{-rh}(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2)|<1$. Then (114) can be simplified and written as

$$
\begin{aligned}
&-p-\frac{q\xi_\mu e^{-rh}d_s}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}p+(1-e^{-rh})\,\mathbb{E}[v_{n,th}|H_{n,th}]\\
&-\frac{(1-e^{-rh})\lambda}{r}(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th})+\frac{q(1+\xi_\mu d_s)e^{-rh}(1-e^{-rh})}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\,\mathbb{E}[v_{n,th}|H_{n,th}]\\
&-\frac{(1-e^{-rh})\lambda}{r}\Big[\frac{q(1+\xi_\mu d_s)e^{-rh}}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\Big(a_s s_{n,th}-b_s p+f_s Z\Big)\\
&+q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)\Bigg(\frac{(1-q)(1+d)e^{-2rh}(-d)\big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)e^{-2rh}\big(a_s s_{n,th}-b_s p+f_s Z\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+\\
&\frac{e^{-rh}}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Bigg)\Big]\\
&-\frac{d(1-q)e^{-rh}}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}p_{(t-1)h}\\
&+\frac{(1+d)(1-q)e^{-rh}(1-e^{-rh})}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\,\mathbb{E}[v_{n,th}|H_{n,th}] \qquad (115)\\
&-\frac{(1-e^{-rh})\lambda(1+d)(1-q)e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}(-d)\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\Big)\\
&-\frac{(1-e^{-rh})\lambda(1+d)^2(1-q)}{r}\Big[\frac{(1-q)(1+d)e^{-2rh}(-d)\big(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+\frac{Z}{N}\big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+
\end{aligned}
$$

$$\frac{q\xi_\mu(1+\frac{\xi_\sigma}{\xi_\mu}d_s)e^{-2rh}\Big(a_s s_{n,th}-b_s p+f_s Z\Big)}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}+$$
$$\frac{e^{-rh}}{1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)\Big]=0$$

From here, it is all algebra to show that the $Y_{n,th}$ is linear in $s_{n,th}$ and $z_{n,th}$ in equilibrium. Then we will match the coefficients from the conjectured linear form to pin down the coefficients in equilibrium. Start by combining the like terms and simplify out a $1-e^{-rh}$ of (115) to get

$$\Big(-\frac{1-e^{-rh}\big(1+(1-q)d\big)}{(1-e^{-rh})\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}+\frac{\lambda e^{-rh}b(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\Big)p$$
$$-\frac{d(1-q)e^{-rh}}{(1-e^{-rh})\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}p_{(t-1)h}+\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\mathbb{E}[v_{n,th}|H_{n,th}]$$
$$-\frac{\lambda e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s s_{n,th}(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big(s_{n,th}-\bar{s}_{th}\big)\big)+$$
$$+\frac{\lambda e^{-rh}}{r\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(f_s Z(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu+d(1+d)(1-q)\frac{Z}{N}\big)+$$
$$=\frac{\lambda}{r\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)$$

Without loss of generality, we can assume there was no shock since the last trading period, as in reality a session wouldn't be offered. Multiply through by $\frac{r}{\lambda}$ Plug in the equilibrium price and expected value and combine to get:

$$\Big(-\frac{r}{\lambda}\frac{1-e^{-rh}\big(1+(1-q)d\big)}{(1-e^{-rh})\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}+\frac{e^{-rh}b_s(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\Big)p$$
$$-\frac{d(1-q)e^{-rh}}{(1-e^{-rh})\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)}\Big(\frac{r}{\lambda}\bar{s}_{th}-\frac{1}{N}Z\Big)+\frac{1}{1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)}\frac{r}{\lambda}\Big(\frac{N\alpha-1}{N-1}s_{n,th}+\frac{N(1-\alpha)}{N-1}\bar{s}_{th}\Big)$$
$$-\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(a_s s_{n,th}(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu-\frac{r(N\alpha-1)}{\lambda(N-1)}d(1+d)(1-q)\big(s_{n,th}-\bar{s}_{th}\big)\big)+$$
$$+\frac{e^{-rh}}{\big(1-e^{-rh}\big(1+q\xi_\mu d_s+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(f_s Z(1+\frac{\xi_\sigma}{\xi_\mu}d_s)q\xi_\mu+d(1+d)(1-q)\frac{Z}{N}\big)+$$
$$=\frac{1}{\big(1-e^{-rh}\big(q(1+2\xi_\mu d_s+\xi_\sigma d_s^2)+(1-q)(1+d)^2\big)\big)}\big(z_{n,th}+\frac{\xi_\sigma}{\xi_\mu}Y_{n,th}\big)$$

It is clear that $d_s=-\frac{\xi_\mu}{\xi_\sigma}$, and take the strategy of platform operator as given as $p=a_p\bar{s}_{th}+f_p Z$. Therefore, $Y_{n,th}=a_s s_{n,th}-b_s a_p\bar{s}_{th}+d_s z_{n,th}+(f_s-b_s f_p)Z$. The above then simplifies to

$$-\frac{r}{\lambda}\frac{1-e^{-rh}\big(1+(1-q)d\big)}{(1-e^{-rh})\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)}\Big(a_p\bar{s}_{th}+f_p Z\Big)$$
$$-\frac{d(1-q)e^{-rh}}{(1-e^{-rh})\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)}\Big(\frac{r}{\lambda}\bar{s}_{th}-\frac{1}{N}Z\Big)+\frac{1}{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}\frac{r}{\lambda}\Big(\frac{N\alpha-1}{N-1}s_{n,th}+\frac{N(1-\alpha)}{N-1}\bar{s}_{th}\Big)$$
$$+\frac{e^{-rh}d(1+d)(1-q)}{\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big)+$$
$$+\frac{e^{-rh}d(1+d)(1-q)}{\big(1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)\big)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\frac{Z}{N}+$$
$$=\frac{1}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}\Big(a_s s_{n,th}-b_s a_p\bar{s}_{th}+(f_s-b_s f_p)Z\Big)$$

Combine like terms again and simplify some

$$\Big(\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big)\frac{r(N\alpha-1)}{\lambda(N-1)}s_{n,th}$$

$$-\frac{r}{\lambda}\Big(\frac{(1-e^{-rh}\big(1+(1-q)d\big))a_p}{1-e^{-rh}}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}-\frac{N(1-\alpha)}{N-1}+\frac{(N\alpha-1)e^{-rh}d(1+d)(1-q)}{(N-1)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\Big)\bar{s}_{n,th}$$

$$\Big(-\frac{rN(1-e^{-rh}\big(1+(1-q)d\big))f_p}{\lambda(1-e^{-rh})}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}+\frac{e^{-rh}d(1+d)(1-q)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big)\frac{Z}{N}$$

$$=\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}\Big(a_s s_{n,th}-b_s a_p\bar{s}_{th}+(f_s-b_s f_p)Z\Big)$$

First, match coefficients on $s_{n,th}$, which is simply

$$a_s=\frac{\xi_\mu}{\xi_\sigma}\frac{r(N\alpha-1)}{\lambda(N-1)}.$$

This leaves

$$-\frac{r}{\lambda}\Big(\frac{(1-e^{-rh}\big(1+(1-q)d\big))a_p}{1-e^{-rh}}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}-\frac{N(1-\alpha)}{N-1}+\frac{(N\alpha-1)e^{-rh}d(1+d)(1-q)}{(N-1)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\Big)\bar{s}_{n,th}$$

$$\Big(-\frac{rN(1-e^{-rh}\big(1+(1-q)d\big))f_p}{\lambda(1-e^{-rh})}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}+\frac{e^{-rh}d(1+d)(1-q)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big)\frac{Z}{N}$$

$$=\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}\Big(-b_s a_p\bar{s}_{th}+(f_s-b_s f_p)Z\Big)$$

Now match coefficients on $\bar{s}_{th}$ and $\frac{Z}{N}$

$$\frac{r}{\lambda}\Big(\frac{(1-e^{-rh}\big(1+(1-q)d\big))a_p}{1-e^{-rh}}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}-\frac{N(1-\alpha)}{N-1}+\frac{(N\alpha-1)e^{-rh}d(1+d)(1-q)}{(N-1)\big(1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)\big)}\Big)$$

$$=\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}b_s a_p$$

$$\Big(-\frac{rN(1-e^{-rh}\big(1+(1-q)d\big))f_p}{\lambda(1-e^{-rh})}+\frac{d(1-q)e^{-rh}}{1-e^{-rh}}+\frac{e^{-rh}d(1+d)(1-q)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big)$$

$$=\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}(f_s-b_s f_p)N$$

Add a funny form of 0 to a_p and f_p to get

$$\frac{r}{\lambda}\Big(\frac{(1-e^{-rh}\big(1+(1-q)d\big))(a_p-1)}{1-e^{-rh}}+\frac{N\alpha-1}{N-1}\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big) \tag{116}$$

$$=\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}b_s a_p$$

$$\left(-\frac{(1-e^{-rh}\big(1+(1-q)d\big))(\frac{rNf_p}{\lambda}+1)}{1-e^{-rh}} + \frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)} \right) \tag{117}$$

$$= \frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)} \frac{\xi_\sigma}{\xi_\mu}(f_s-b_s f_p)N$$

Finally, see Appendix C to see that the platform operator offers the correct price. Therefore, $a_p = 1$ and $f_p = -\frac{\lambda}{rN}$. Therefore, the unique linear equilibrium is

$$\mu_{n,th} = \frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big) - z_{n,th} + \frac{Z}{N}\right),$$

which allocates

$$Y_{n,th}(\mu) = \frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th}-\bar{s}_{th}\big) - z_{n,th} + \frac{Z}{N}\right).$$

To check the second-order condition, differentiate the FOC again. As every term has a Y_μ in it, the second-order condition can be written as $Y_{\mu\mu}$ times the FOC, which is zero in equilibrium, plus

$$\mathbb{E}\Bigg[-\frac{(1-e^{-rh})\lambda\xi_t^2 Y_\mu^2}{r} - \frac{(1-e^{-rh})\lambda Y_\mu^2 \xi_t^2}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}_{M_{t+k}}(1+d_s\xi_{t+k})^2\left(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\right)^2 \tag{118}$$

$$-\frac{(1-e^{-rh})\lambda Y_\mu^2\xi_t^2(1+d)^2}{r}\sum_{k=1}^{\infty} e^{-rhk}\mathbb{1}^c_{M_{t+k}}\left(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}\xi_{t+k-i}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\right)^2 \Bigg| H_{n,th}\Bigg] < 0$$

This is always less than zero, and, therefore, the solved for optimal demand quantity to submit is a maximum of the continuation utility for trader n when a size-discovery session occurs.

Appendix C Proving the Platform Operator Offers the Correct Price

In this section, I will prove that the platform operator will set the correct price in equilibrium when a matching session occurs when the equilibrium is restricted to be an affine and stationary function of the total signal. First, conjecture that the platform operator will choose a price that is a stationary and affine function of the average total signal

$$p = a_p\bar{s}_{th} + f_p Z. \tag{119}$$

The platform operator chooses the price to maximize the expected total trading volume over the expected length of time before liquidation. Therefore, they solve

$$\frac{\partial}{\partial p}\sum_{n=1}^{N}\mathbb{E}\left[\sum_{k=0}^{\infty} e^{-rhk}\Big(\mathbb{1}_{M_k}|Y_{n,kh}| + \mathbb{1}^c_{M_k}|X_{n,kh}|\Big)\right] = 0. \tag{120}$$

Without loss of generality that the first trading session is a size-discovery trading session, the first-order condition is

$$\sum_{n=1}^{N}\mathbb{E}\left[(-1)^{\mathbb{1}(Y_{n,0}<0)}Y_\mu\frac{\partial\mu_{n,0}}{\partial p} + \sum_{k=1}^{\infty}e^{-rhk}\left(\mathbb{1}_{M_k}(-1)^{\mathbb{1}(Y_{n,kh}<0)}Y_\mu\frac{\partial\mu_{n,kh}}{\partial p} + \mathbb{1}^c_{M_k}(-1)^{\mathbb{1}(X_{n,kh}<0)}\frac{\partial X_{n,kh}}{\partial p}\right)\right] = 0. \tag{121}$$

From Appendix A and Appendix B, we know that the necessary derivatives are

$$\frac{\partial\mu_{n,kh}}{\partial p} = \begin{cases} -b_s, & \text{if } k=0 \\ -b_s d_s, & \text{if } k=1 \\ -b_s d_s\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big), & \text{if } k\geq 2. \end{cases} \tag{122}$$

$$\frac{\partial X_{n,kh}}{\partial p} = \begin{cases} -b_s d\frac{N-1}{N}, & \text{if } k=1 \\ -b_s d\frac{N-1}{N}\Big(\prod_{i=1}^{k-1}\big(1+\mathbb{1}_{M_{t+k-i}}d_s+\mathbb{1}^c_{M_{t+k-i}}d\big)\Big), & \text{if } k\geq 2. \end{cases} \tag{123}$$

Plugging in the above derivatives, the first-order condition becomes, after factoring out a $-b_s$ from every term and plugging in that $d_s = -1$

$$\sum_{n=1}^{N}\mathbb{E}\Big[(-1)^{\mathbb{1}(Y_{n,0}<0)}Y_\mu - e^{-rh}\mathbb{1}_{M_1}(-1)^{\mathbb{1}(Y_{n,h}<0)}Y_\mu - \sum_{k=2}^{\infty}e^{-rhk}\Big(\mathbb{1}_{M_k}(-1)^{\mathbb{1}(Y_{n,kh}<0)}Y_\mu\Big(\prod_{i=1}^{k-1}(1+d)\mathbb{1}^c_{M_{t+k-i}}\Big)\Big) \tag{124}$$

$$+e^{-rh}d\frac{N-1}{N}\mathbb{1}^c_{M_1}(-1)^{\mathbb{1}(X_{n,h}<0)} + d\frac{N-1}{N}\sum_{k=2}^{\infty}e^{-rhk}\Big(\mathbb{1}^c_{M_k}(-1)^{\mathbb{1}(X_{n,kh}<0)}\Big(\prod_{i=1}^{k-1}(1+d)\mathbb{1}^c_{M_{t+k-i}}\Big)\Big)\Big] = 0.$$

I will use the independence of the processes and that the expectation is unconditional from the platform operator's perspective. Define $\mathbb{E}\big[(-1)^{\mathbb{1}(Y_{n,kh}<0)Y_\mu}\big] = c_y$ and $\mathbb{E}\big[(-1)^{\mathbb{1}(X_{n,kh}<0)}\big] = c_x$. Therefore, the first-order condition can be rewritten as

$$\sum_{n=1}^{N}\Big[c_y + \Big(d(1-q)c_x - qc_y\Big)\sum_{k=1}^{\infty}e^{-rhk}(1-q)^{k-1}(1+d)^{k-1}\Big] = 0. \tag{125}$$

Therefore, the platform operator needs to choose the price such that $c_y = c_x = 0$. Intuitively, this means the unconditional probability that there are buyers and sellers is the same, which helps maximize volume due to the execution uncertainty in this trading protocol. Therefore, they choose the price such that there is no bias unconditionally toward buyers or sellers. Note that the absolute value function is even for proportional rationing. Therefore, the conditional expectation of Y_μ when you are on the heavy side of the market and having your order rationed is equal for both net buying or selling. Therefore, Y_μ does not affect the price offered by the law of iterated expectations. For additive rationing, Y_μ is constant. To not introduce bias, the platform operator sets p such that

$$P\Big(\frac{r(N\alpha-1)}{\lambda(N-1)}s_{n,kh} - b_s p - z_{n,kh} + f_s Z < 0\Big) = .5, \tag{126}$$

$$P\Big(a\big(s_{n,kh}-\bar{s}_{kh}\big) + d\big(z_{n,kh} - \frac{Z}{N}\big) < 0\Big) = .5. \tag{127}$$

The key is given everything is normally distributed, as long as the term inside the probability has mean zero, then the probability that demand is less than zero will be 50%. Therefore, it is sufficient to solve for the price to charge such that

$$\mathbb{E}\Big[\frac{r(N\alpha-1)}{\lambda(N-1)}s_{n,kh} - b_s p - z_{n,kh} + f_s Z\Big] = 0. \tag{128}$$

First, plug in the conjectured affine form of p and combine like terms to get

$$\mathbb{E}\Big[\frac{r(N\alpha-1)}{\lambda(N-1)}s_{n,kh} - b_s a_p \bar{s}_{kh} - z_{n,kh} + (f_s - b_s f_p)Z\Big] = 0. \tag{129}$$

Unconditionally, the expected total signal is equal to v. Therefore, it must be the case that $\frac{r(N\alpha-1)}{\lambda(N-1)} = b_s a_p$. Recall (116), b_s is defined by

$$\frac{r}{\lambda}\Big(\frac{(1-e^{-rh}\big(1+(1-q)d\big))(a_p-1)}{1-e^{-rh}} + \frac{N\alpha-1}{N-1}\frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\Big)$$

$$= \frac{1-e^{-rh}\big(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\big)}{1-e^{-rh}\big(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\big)}\frac{\xi_\sigma}{\xi_\mu}b_s a_p.$$

Therefore, subbing in for $b_s a_p$ and solving for a_p implies that $a_p = 1$, and then $b_s = \frac{r(N\alpha-1)}{\lambda(N-1)}$. Finally, we need to show

$$\mathbb{E}\left[z_{n,kh}\right] = (f_s - \frac{r(N\alpha-1)}{\lambda(N-1)} f_p)Z. \tag{130}$$

Using the fact that, unconditionally, as the asset is held in aggregate, each trader will have $\frac{Z}{N}$. Therefore,

$$\frac{1}{N} = (f_s - \frac{r(N\alpha-1)}{\lambda(N-1)} f_p). \tag{131}$$

Recall (117), f_s is defined by

$$\left(-\frac{(1-e^{-rh}\left(1+(1-q)d\right))(\frac{rNf_p}{\lambda}+1)}{1-e^{-rh}} + \frac{1-e^{-rh}\left(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\right)}{1-e^{-rh}\left(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\right)} \right)$$

$$= \frac{1-e^{-rh}\left(1-q\frac{\xi_\mu^2}{\xi_\sigma}+(1-q)d\right)}{1-e^{-rh}\left(q(1-\frac{\xi_\mu^2}{\xi_\sigma})+(1-q)(1+d)^2\right)} \frac{\xi_\sigma}{\xi_\mu}(f_s - b_s f_p)N$$

Therefore, subbing in for $(f_s - b_s f_p)$ and solving for f_p implies that $f_p = -\frac{\lambda}{rN}$. Therefore, $f_s = \frac{1-\alpha}{N-1}$. Putting it all together, the price offered, the quantity demanded as a function of the price offered, and the equilibrium quantity demanded is

$$p = \bar{s}_{kh} - \frac{\lambda}{rN} Z, \tag{132}$$

$$\mu_{n,tk} = \frac{r(N\alpha-1)}{\lambda(N-1)} s_{n,kh} - \frac{r(N\alpha-1)}{\lambda(N-1)} p - z_{n,kh} + \frac{1-\alpha}{N-1} Z, \tag{133}$$

$$\mu^*_{n,tk} = \frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{n,kh} - \bar{s}_{kh}\right) - z_{n,kh} + \frac{Z}{N}. \tag{134}$$

Appendix D Proof of Proposition 2 with Impairment

An important lemma needed for the welfare results is the rate at which a trader's inventory converges to the desired (competitive) inventory level given no new shocks (innovations) to the fundamental or private value of the asset. Given that total signals is a martingale, in expectation, the price a size-discovery session occurs at will be the same as the expected price that would occur in a price-discovery session. Taking the equilibrium demand schedules, (17), and quantities, (15), played when price and size-discovery sessions occur, respectively, and no new shocks to the fundamental or private values of the asset, one can write the inventory next period as

$$z_{i,(t+1)h} = z_{i,th} + \mathbb{1}_{M_t}\xi_t \frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{i,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) - z_{i,th} + \frac{Z}{N}\right) + \mathbb{1}^c_{M_t}\left(as_{i,th} - bp^*_{th} + dz_{i,th} + fZ\right). \tag{135}$$

Plugging in the equilibrium price, (18), for p^*_{th} and some simple rearranging of the terms by combing all $z_{i,th}$ terms, yields

$$= \mathbb{1}_{M_t}\xi_t \frac{\xi_\mu}{\xi_\sigma}\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{i,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) + \frac{Z}{N}\right) + \mathbb{1}^c_{M_t}\left(b\left(s_{i,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) - \frac{d}{N}Z\right) + \left(1 + \mathbb{1}^c_{M_t}d - \frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t\right)z_{i,th}. \tag{136}$$

Noting that $\frac{b}{-d} = \frac{r(N\alpha-1)}{\lambda(N-1)}$, you can factor out a $-d$ from the second term's parentheses and combine like terms to get

$$= (\frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t - \mathbb{1}^c_{M_t}d)\left(\frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{i,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) + \frac{Z}{N}\right) + \left(1 + \mathbb{1}^c_{M_t}d - \frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t\right)z_{i,th}. \tag{137}$$

Define $z^c_{i,(t+1)h} = \frac{r(N\alpha-1)}{\lambda(N-1)}\left(s_{i,th} - \frac{1}{N}\sum_{n=1}^{N} s_{n,th}\right) + \frac{Z}{N}$. This term is the desired (competitive) level of inventory a trader wishes to have given their information at time th. If the market was competitive, this is the inventory a trader would have at $(t+1)h$ after trading at time th. Subbing in this newly defined variable reduces the equation to

$$z_{i,(t+1)h} = (\frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t - \mathbb{1}^c_{M_t} d) z^c_{i,(t+1)h} + \big(1 + \mathbb{1}^c_{M_t} d - \frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t\big) z_{i,th}. \tag{138}$$

From (138), simple algebra yields a formula that tells the difference from the equilibrium inventory amount for trader i and the desired (competitive) amount as a function of the difference prior to trading times the fraction that the trader will have left after the trading session.

$$z_{i,(t+1)h} - z^c_{i,(t+1)h} = \big(1 + \mathbb{1}^c_{M_t} d - \frac{\xi_\mu}{\xi_\sigma}\mathbb{1}_{M_t}\xi_t\big)\big(z_{i,th} - z^c_{i,(t+1)h}\big) \tag{139}$$

Iterating (139) forward gives the lemma.

Appendix E Proof of Proposition 3 with Impairment

Define W^s as the total welfare of traders when there are price and size-discovery sessions, $q \neq 0$. W^s is defined to be the sum of all traders' ex-ante expected utilities over all traders (the sum of (15) over all traders). Note the first term of each type of trading session sums to zero by the market-clearing conditions, as it is just a wealth transfer. Therefore, the economic channel that drives the welfare results is simply how allocatively efficient the different market designs are. Setting $q = 0$ gives you the welfare of an economy with only price-discovery sessions. W^s is therefore

$$W^s = (1 - e^{-rh}) \sum_{n=1}^{N} \mathbb{E}\Bigg[\sum_{k=0}^{\infty} e^{-rhk}\Bigg(\mathbb{1}_{M_k}\Big(v_{n,kh}(z_{n,kh} + \xi_k Y_{n,kh}) - \frac{\lambda}{2r}(z_{n,kh} + \xi_k Y_{n,kh})^2\Big) + \mathbb{1}^c_{M_k}\Big(v_{n,kh}(X_{n,kh} + z_{n,kh}) - \frac{\lambda}{2r}(X_{n,kh} + z_{n,kh})^2\Big)\Bigg)\Bigg| H_{n,kh}\Bigg] \tag{140}$$

Recalling (40), which defines the next period's starting inventory, some simple algebra can be applied to get (140) to a form such that I can apply Lemma 2 from Du and Zhu (2017) to simplify the welfare expression.

$$W^s = (1 - e^{-rh}) \sum_{n=1}^{N} \mathbb{E}\Bigg[\sum_{k=0}^{\infty} e^{-rhk}\Big(v_{n,kh} z_{n,(k+1)h} - \frac{\lambda}{2r} z^2_{n,(k+1)h}\Big)\Bigg], \tag{141}$$

Recall the defined desired (competitive) inventory level $z^c_{n,(t+1)h} = \frac{r(N\alpha-1)}{\lambda(N-1)}\big(s_{n,th} - \frac{1}{N}\sum_{j=1}^{N} s_{j,th}\big) + \frac{Z}{N}$. Note that d_s is not the same d_s as in Appendix B, but the d from appendix A, where there are both size-discovery trading sessions. This d_s is a function of q and reduces to the d in Du and Zhu (2017) when $q = 0$. Using lemma 2 of Du and Zhu (2017), (141) can be written as

$$W^s = (1 - e^{-rh}) \sum_{n=1}^{N} \mathbb{E}\Bigg[\sum_{k=0}^{\infty} e^{-rhk}\Big(v_{n,kh} z^c_{n,(k+1)h} - \frac{\lambda}{2r}\big((z^c_{n,(k+1)h})^2 + (z_{n,(k+1)h} - z^c_{n,(k+1)h})^2\big)\Big)\Bigg]. \tag{142}$$

Note that from the lemma proved in appendix C, one can write the expected squared difference between the desired and actual inventory at clock time $(k+1)h$ as a function of expected initial excess inventory and the changes in the desired inventory positions as

$$\mathbb{E}\Big[\big(z_{n,(k+1)h} - z^c_{n,(k+1)h}\big)^2\Big] = \big((1-q)(1+d_s)^2 + q(1 - \frac{\xi^2_\mu}{\xi_\sigma})\big)\,\mathbb{E}\Big[\big(z_{n,kh} - z^c_{n,(k+1)h}\big)^2\Big]. \tag{143}$$

$$= \big((1-q)(1+d_s)^2 + q(1 - \frac{\xi^2_\mu}{\xi_\sigma})\big)\,\mathbb{E}\Big[\big(z_{n,kh} - z^c_{n,kh}\big)^2\Big] + \big((1-q)(1+d_s)^2 + q(1 - \frac{\xi^2_\mu}{\xi_\sigma})\big)\,\mathbb{E}\Big[\big(z^c_{n,kh} - z^c_{n,(k+1)h}\big)^2\Big]. \tag{144}$$

$$= \big((1-q)(1+d_s)^2 + q(1 - \frac{\xi^2_\mu}{\xi_\sigma})\big)^{k+1}\,\mathbb{E}\Big[\big(z_{n,0} - z^c_{n,h}\big)^2\Big] + \sum_{k'=1}^{k}\big((1-q)(1+d_s)^2 + q(1 - \frac{\xi^2_\mu}{\xi_\sigma})\big)^{k-k'+1}\,\mathbb{E}\Big[\big(z^c_{n,(k'+1)h} - z^c_{n,k'h}\big)^2\Big]. \tag{145}$$

A key part of (144) is that the competitive inventory process is a martingale and measurable with respect to prior information and therefore $\mathbb{E}\Big[(z_{n,kh} - z^c_{n,kh})(z^c_{n,(k+1)h} - z^c_{n,kh})\Big] = 0$ by the law of iterated expectations. The rest of the formula follows by induction. This is also shown in the proof of Lemma 3 in Du and Zhu (2017). It is important to note that $z^c_{n,th}$, the desired (competitive) inventory position is not a function of q and therefore can be "ignored" when studying the effect of q. Most of the following work will focus only on the terms that involve q, "welfare" (up to some additive constant that is not a function of q). Therefore welfare can be written as

$$W^s = (1-e^{-rh})\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(v_{n,kh}z^c_{n,(k+1)h}-\frac{\lambda}{2r}((z^c_{n,(k+1)h})^2\Big)\Big] \tag{146}$$

$$-\frac{\lambda(1-e^{-rh})}{2r}\sum_{n=1}^{N}\sum_{k=0}^{\infty}e^{-rhk}\Big(((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))^{k+1}\,\mathbb{E}\Big[(z_{n,0}-z^c_{n,h})^2\Big]+$$

$$\sum_{k'=1}^{k}((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))^{k-k'+1}\,\mathbb{E}\Big[(z^c_{n,(k'+1)h}-z^c_{n,k'h})^2\Big]\Big).$$

Evaluating the infinite sums in (146), W^s can be written as

$$W^s = (1-e^{-rh})\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(v_{n,kh}z^c_{n,(k+1)h}-\frac{\lambda}{2r}((z^c_{n,(k+1)h})^2\Big)\Big] \tag{147}$$

$$-\frac{\lambda(1-e^{-rh})}{2r}\Big[\frac{(1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{(1-e^{-rh}((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\sum_{n=1}^{N}\mathbb{E}[(z_{n,0}-z^c_{n,h})^2]+$$

$$\frac{(1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\sum_{n=1}^{N}\sum_{k'=1}^{\infty}\mathbb{E}[(z^c_{n,(k'+1)h}-z^c_{n,k'h})^2]e^{-rhk'}\Big].$$

The terms inside the brackets in second line of (147) are positive as are λ, r and $(1-e^{-rh})$. Therefore W^s can be written as

$$W^s = (1-e^{-rh})\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(v_{n,kh}z^c_{n,(k+1)h}-\frac{\lambda}{2r}(z^c_{n,(k+1)h})^2\Big)\Big] \tag{148}$$

$$-\frac{\lambda(1-e^{-rh})}{2r}\frac{(1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\Big[\sum_{n=1}^{N}\mathbb{E}[(z_{n,0}-z^c_{n,h})^2]+\sum_{n=1}^{N}\sum_{k'=1}^{\infty}\mathbb{E}[(z^c_{n,(k'+1)h}-z^c_{n,k'h})^2]e^{-rhk'}\Big].$$

Appendix E.1 Quantifying Welfare for Scheduled Arrivals of New Information

To quantify the strategic costs induced by the market design, it is useful to benchmark the allocative efficiency of the different marker designs to that of a competitive market. Therefore, the ex-ante welfare of traders under this efficient benchmark design is

$$W^c = (1-e^{-rh})\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(v_{n,kh}z^c_{n,(k+1)h}-\frac{\lambda}{2r}(z^c_{n,(k+1)h})^2\Big)\Big] \tag{149}$$

In this section, we assume that information arrival is scheduled and known in advance. For simplicity, it is assumed that $T_k = k\gamma$. Following Du and Zhu (2017), I will focus on the natural case where $z_{n,0} = Z/N$, which is a way to look at a "steady state" version of the model. It is also shown that the optimal trading frequency for this steady state is $h = \gamma$. Finally, define

$$\sigma_z^2 = \sum_{n=1}^{N}\mathbb{E}[(z^c_{i,T_k}-z^c_{i,T_{k-1}})^2] = \Big(\frac{r(N\alpha-1)}{\lambda(N-1)}\Big)^2\frac{(N-1)(\chi^2(\sigma_D^2+\sigma_\epsilon^2)+\sigma_w^2)}{\alpha^2}. \tag{150}$$

Therefore, we combine (149) and (148) to get

$$W^c - W^s = \sum_{n=1}^{N}\Big(\frac{\lambda(1-e^{-rh})}{2r}\frac{(1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}((1-q)(1+d_s)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\Big[\mathbb{E}[(z_{n,0}-z^c_{n,h})^2]+\sum_{k'=1}^{\infty}\mathbb{E}[(z^c_{n,(k'+1)h}-z^c_{n,k'h})^2]e^{-rhk'}\Big]\Big) \tag{151}$$

Applying (150), propositions 5 and 6 from Du and Zhu (2017), we get that

$$W^c - W^s = \frac{\lambda\sigma_z^2}{2r}\frac{(1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}. \tag{152}$$

Appendix E.2 Quantifying Welfare for Stochastic Arrivals of New Information

In this section, we assume that information arrival is stochastic. For analytical tractability, the information arrival process will follow a Poisson process. Specifically, the shocks will arrive at times $\{T_k\}_{k\geq 1}$ where this process is a homogeneous Poisson process with arrival intensity of $\mu > 0$. Therefore, the expected time between shocks will be $1/\mu$, and the expected number of shocks per interval h is $h\mu$. Given this process, the expected squared difference in desired inventory positions are

$$\sum_{n=1}^{N}\mathbb{E}[(z^c_{i,(t+1)h} - z^c_{i,th})^2] = h\mu\sigma_z^2, \tag{153}$$

$$\sum_{n=1}^{N}\mathbb{E}[(z^e_{i,\tau} - z^c_{i,\tau})^2] = (\tau - th)\mu\sigma_z^2, \tag{154}$$

where σ_z^2 is defined according to (150). Taking equation (151) and applying Lemma 2 from Du and Zhu (2017), the difference from the perfectly efficient welfare and that due to trading frequency and strategic actions by the traders can be written as

$$W^c - W^s = \frac{\lambda(1-e^{-rh})}{2r}\frac{(1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})}{1-e^{-rh}((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\sum_{n=1}^{N}\left[\mathbb{E}[(z_{n,0} - z^c_{n,h})^2] + \sum_{k'=1}^{\infty}\mathbb{E}[(z^c_{n,(k'+1)h} - z^c_{n,k'h})^2]e^{-rhk'}\right] \tag{155}$$

Substituting in the defined total expected squared deviations and focusing on the steady-state case, (155) simplifies down to

$$W^c - W^s = \frac{\lambda\sigma_z^2}{2r}\left(\frac{(1-e^{-rh})\big((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)}{1-e^{-rh}((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}\left[1 + \frac{he^{-rh}\mu}{1-e^{-rh}}\right]\right) \tag{156}$$

Appendix E.3 Comparative Static

In this subsection, I will prove that $\frac{\partial d}{\partial h} < 0$, when the equilibrium of interest exists. Using implicit differentiation of (67), the partial of d with respect to h can be expressed as

$$\frac{\partial d}{\partial h} = -\frac{(1+d)re^{-hr}\left(\frac{d(N\alpha-1)\big(1-((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))\big)}{\big(1-e^{-rh}((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))\big)^2} - \frac{q\xi_\mu^2}{\xi_\sigma}\right)}{\frac{2d(1+d)^2(1-q)e^{-rh}(1-e^{-rh})(N\alpha-1)}{\big(1-e^{-rh}((1-q)(1+d)^2+q(1-\frac{\xi_\mu^2}{\xi_\sigma}))\big)^2} + \frac{d}{1+d} - \frac{q\xi_\mu^2 e^{-hr}}{\xi_\sigma}\frac{1+d}{d}} < 0 \tag{157}$$

if $\left(\frac{|d|}{1+d}\right)^2 < \frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma}$. It is simple to see that $\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma} < 1$, as $0 \leq q < 1$, $r, h > 0$, and $0 \leq \xi_\mu^2 \leq \xi_\sigma \leq 1$. Finally, factor the right-hand side of the equation that defines d to get

$$\frac{d(N\alpha-1)(1-e^{-rh})}{1-e^{-rh}\big((1-q)(1+d)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma})\big)} = \left(\sqrt{\frac{|d|}{1+d}} - \sqrt{\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma}}\right)\left(\sqrt{\frac{|d|}{1+d}} + \sqrt{\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma}}\right) < 0 \tag{158}$$

Therefore, $\sqrt{\frac{|d|}{1+d}} - \sqrt{\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma}} < 0$, and since $\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma} < 1$, we have that

$$\frac{d}{1+d} < \frac{|d|}{1+d} < \frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma} < \sqrt{\frac{e^{-rh}q\xi_\mu^2}{\xi_\sigma}}. \tag{159}$$

Therefore, $\frac{\partial d}{\partial h} < 0$, when the equilibrium of interest exists.

Appendix E.4 Welfare Loss as Trading Frequency Changes

We know that $\frac{\partial d_s}{\partial h} < 0$ and assume for simplicity $z_{n,0} = \frac{Z}{N}$, and $\sum_{n=1}^{N} \mathbb{E}[(z^c_{n,(k'+1)h} - z^c_{n,k'h})^2] = \sigma_z^2 \mu h$, for all $k' \in \{0, 1, \ldots\}$. Then

$$W^c - W^s = \frac{\lambda \sigma_z^2}{2r} \left[\frac{(1-e^{-rh})((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma}))}{1-e^{-rh}((1-q)(1+d_s)^2 + q(1-\frac{\xi_\mu^2}{\xi_\sigma}))} \right] \left[1 + \frac{he^{-rh}\mu}{1-e^{-rh}} \right] \tag{160}$$

Using the equation that defines d_s, (67), the first term in brackets can be written as

$$\frac{(1-e^{-rh})(1-q)(1+d_s)^2}{1-e^{-rh}(1-q)(1+d_s)^2} = \frac{(1+d_s)\big((1+d_s)(1-qe^{-rh})-1\big)}{d_s(N\alpha-1)} \tag{161}$$

Differentiating (161) with respect to h directly is unwieldy. Instead, to assign the sign of the comparative static, notice that all terms inside of a bracket are always positive. Therefore, showing each term decreases in h, which then implies that the overall derivative would also decrease in h from the product rule. Differentiate the first bracketed term with respect to h to get

$$\frac{\partial}{\partial h}\left(\frac{(1+d_s)\big((1+d_s)(1-qe^{-rh})-1\big)}{d_s(N\alpha-1)} \right) = \frac{1-q}{N\alpha-1}\left(\frac{\partial d_s}{\partial h} + \frac{qe^{-rh}(1+d_s)\big(rd(1+d_s)+(1-d_s)\frac{\partial d_s}{\partial h}\big)}{d_s^2} \right) < 0. \tag{162}$$

Now, differentiate the second bracketed term

$$\frac{\partial}{\partial h} 1 + \frac{he^{-rh}\mu}{1-e^{-rh}} = -\frac{e^{-rh}u\big(rh+e^{-rh}-1\big)}{(1-e^{-rh})^2} < 0. \tag{163}$$

when $h > 0$ and $0 < h < \frac{1}{r}$, which is always true. Therefore, we know that

$$\frac{\partial W^c - W^s}{\partial h} < 0. \tag{164}$$

Therefore, the welfare loss due to strategic trading, relative to a competitive benchmark, is strictly monotonically increasing in h. This implies that the welfare loss due to the strategic actions of traders is maximized when trading is continuous.

Appendix E.5 Existence of a Welfare-Improving Trading Frequency

Work-Up Sessions

The goal of this subsection is to show that there exists a $h \in (0, \infty)$ such that

$$W^s - W^p > 0. \tag{165}$$

First, plug in the definitions of $W^c - W^p$ and $W^c - W^s$ to get

$$\sum_{n=1}^{N} \left(\frac{\lambda(1-e^{-rh})}{2r} \left(\frac{(1+d_p)^2}{1-e^{-rh}(1+d_p)^2} - \frac{(1-q)(1+d_s)^2}{1-e^{-rh}(1-q)(1+d_s)^2} \right) \left[\mathbb{E}[(z_{n,0} - z^c_{n,h})^2] + \sum_{k'=1}^{\infty} \mathbb{E}[(z^c_{n,(k'+1)h} - z^c_{n,k'h})^2] e^{-rhk'} \right] \right). \tag{166}$$

Given the goal is to prove there exists a trading frequency such that (166) is positive, it is equivalent to show that

$$\frac{(1+d_p)^2}{1-e^{-rh}(1+d_p)^2} - \frac{(1-q)(1+d_s)^2}{1-e^{-rh}(1-q)(1+d_s)^2} > 0, \tag{167}$$

for some h. Given the denominators are positive and combining the fractions simplifies the problem to

$$(1+d_p)^2 - (1-q)(1+d_s)^2 > 0. \tag{168}$$

The goal is to apply the Intermediate Value Theorem. Given the equilibrium exists, the solution to (67) is continuous. The first case is an arbitrarily large h. Letting $h \to \infty$, the solutions for d, in a market with and without a size-discovery mechanism, are

$$d_p = d_s = -\frac{N\alpha - 2}{N\alpha - 1}. \tag{169}$$

Plugging (169) into (168), it simplifies to

$$\frac{q}{(N\alpha - 1)^2} > 0. \tag{170}$$

For the second part, following Antill and Duffie (2020), when h is arbitrarily close to zero, and let $q = \gamma h$ for $\gamma > 0$, and $\gamma < \bar{\gamma}$, where $\bar{\gamma}$ is the unique positive solution of $\sqrt{8\bar{\gamma}(\bar{\gamma} + r)} + 3\bar{\gamma} = r(N\alpha - 2)$ so that the equilibrium exists. Note from (166) that the difference in welfare is equal to

$$\frac{(1 - e^{-rh})(1 + d_p)^2}{1 - e^{-rh}(1 + d_p)^2} - \frac{(1 - e^{-rh})(1 - q)(1 + d_s)^2}{1 - e^{-rh}(1 - q)(1 + d_s)^2}. \tag{171}$$

Using the equations that define d, (171) can be rewritten as

$$\frac{1 + d_p}{N\alpha - 1} - \frac{(1 + d_s)(1 - q)\big(d_s - e^{-rh}q(1 + d_s)\big)}{d_s(N\alpha - 1)}. \tag{172}$$

Note that as $h \to 0^+$ both solutions for d equal 0. Therefore, the second term needs L'Hospital's rule. After taking the limits, (172) simplifies to

$$\frac{1}{N\alpha - 1} \frac{\gamma}{\lim\limits_{h \to 0^+} \frac{\partial d_s}{\partial h}} < 0. \tag{173}$$

The inequality is true as $\lim\limits_{h \to 0^+} \frac{\partial d_s}{\partial h} < 0$. Therefore, by the Intermediate Value Theorem, there exists $h^* \in (0, \infty)$ such that $W^s - W^p > 0$.

Matching Sessions

The goal of this subsection is to show that

$$W^s - W^p > 0 \tag{174}$$

is always true, and only equal to zero in the limit as $h \to 0^+$. (174) is equivalent to showing that (168). Recall the solutions for d, in a market with and without a matching session, are

$$d_s = -\frac{(N\alpha - 1)(1 - e^{-rh}) + 2e^{-rh}(1 - q) - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}(1 - q)}}{2e^{-rh}(1 - q)}, \tag{175}$$

$$d_p = -\frac{(N\alpha - 1)(1 - e^{-rh}) + 2e^{-rh} - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}}}{2e^{-rh}}. \tag{176}$$

Plugging in (175) and (176) into (168), which we want to show is greater than 0, yields

$$\left(\frac{(N\alpha - 1)(1 - e^{-rh}) - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}}}{2e^{-rh}}\right)^2 - (1-q)\left(\frac{(N\alpha - 1)(1 - e^{-rh}) - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}(1 - q)}}{2e^{-rh}(1 - q)}\right)^2 > 0. \tag{177}$$

Multiply by $(1 - q)$ to both sides and this simplifies to

$$(1-q)\left((N\alpha-1)(1-e^{-rh}) - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}}\right)^2 - \left((N\alpha-1)(1-e^{-rh}) - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}(1 - q)}\right)^2 > 0. \tag{178}$$

Expand the equation, combine like terms and simplify to get

$$-q(N\alpha - 1)(1 - e^{-rh}) - \left((1 - q)\sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}} - \sqrt{(N\alpha - 1)^2(1 - e^{-rh})^2 + 4e^{-rh}(1 - q)}\right) > 0. \tag{179}$$

The, divide both sides by the term $\frac{1}{(N\alpha-1)(1-e^{-rh})}$ and move the $1-q$ inside the square root to get the equation

$$-q-\sqrt{(1-q)\Big(-q+1+\frac{4e^{-rh}(1-q)}{(N\alpha-1)^2(1-e^{-rh})^2}\Big)}+\sqrt{1+\frac{4e^{-rh}(1-q)}{(N\alpha-1)^2(1-e^{-rh})^2}}>0 \tag{180}$$

Define the variable $x=1+\frac{4e^{-rh}(1-q)}{(N\alpha-1)^2(1-e^{-rh})^2}$, which by inspection is always greater than 1. Therefore, it is left to show that for $x>1$ that the below equation is true

$$\sqrt{x}-q-\sqrt{(1-q)(x-q)}>0. \tag{181}$$

Note, there is equality when $x=1$. Therefore, if we can show that the left hand side increases in x for $x>1$, then the inequality must be true. The derivative of the left hand side of (181) with respect to x is

$$\frac{1}{\sqrt{x}}-\frac{1-q}{\sqrt{(1-q)(x-q)}}>0. \tag{182}$$

To show (182) is greater than 0 for $x>1$, add the negative term to the other side, cross multiply, square both sides, and simplify out a $1-q$ which yields the inequality

$$x-q>1-q. \tag{183}$$

(183) is always true for $x>1$. Therefore, $W^s-W^p>0$ is always true when the size-discovery session is modeled as a matching session.

In the second part of this subsection, I will show that in the limit $(h\to 0^+)$, as trading becomes continuous as in Antill and Duffie (2020), that $W^s-W^p=0$. Let $q=\gamma h$. Note that the solutions for d limit to zero as $h\to 0^+$. Note from (166) that the difference in welfare is equal to

$$\frac{(1-e^{-rh})(1+d_p)^2}{1-e^{-rh}(1+d_p)^2}-\frac{(1-e^{-rh})(1-q)(1+d_s)^2}{1-e^{-rh}(1-q)(1+d_s)^2}. \tag{184}$$

Using the equations that define d, (184) can be rewritten as

$$\frac{1+d_p}{N\alpha-1}-(1-\gamma h)\frac{1+d_s}{N\alpha-1}. \tag{185}$$

Taking the limit as $h\to 0^+$, (185) equals 0. Therefore, as trading becomes continuous, size-discovery sessions occur at a rate, and size-discovery sessions are modeled as matching sessions, the welfare of the two market designs is always equal.

Appendix F Quantifying Volume

The platform operator makes the decision to offer a size-discovery trading protocol by maximizing the time-discounted expected future volume traded. This is written as

$$\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(\mathbb{1}_{M_k}|Y_{n,kh}|+\mathbb{1}_{M_k}^c|X_{n,kh}|\Big)\Big]. \tag{186}$$

In equilibrium, this can be rewritten as

$$\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\big(\mathbb{1}_{M_k}-d\mathbb{1}_{M_k}^c\big)\big|Y_{n,kh}\big|\Big]. \tag{187}$$

Note that $Y_{n,kh}=z^e_{n,kh}-z_{n,kh}$, the desired inventory position minus the current inventory position. Finally, the current inventory position is a function of what happened in the past and evolves in equilibrium according to

$$\begin{aligned} z_{n,(k+1)h} &= z_{n,kh}+\mathbb{1}_{M_k}\big(z^e_{n,kh}-z_{n,kh}\big)-d\mathbb{1}_{M_k}^c\big(z^e_{n,kh}-z_{n,kh}\big) \\ &= (1+d)\mathbb{1}_{M_k}^c z_{n,kh}+\big(1-(1+d)\mathbb{1}_{M_k}^c\big)z^e_{n,kh}. \end{aligned} \tag{188}$$

Putting this all together, let us first focus on a single trader and the k^{th} trading period. By the law of iterated expectations, we can iterate to

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\mathbb{1}_{M_k} - d\mathbb{1}^c_{M_k}\big)\big|z^e_{n,kh} - z_{n,kh}\big|\Big] \qquad (189)\\
&= \big(q - d(1-q)\big)\,\mathbb{E}\Big[\big|z^e_{n,kh} - z_{n,kh}\big|\Big]\\
&= \big(q - d(1-q)\big)\,\mathbb{E}\Big[\big|z^e_{n,kh} - (1+d)\mathbb{1}^c_{M_{k-1}} z_{n,(k-1)h} - \big(1-(1+d)\mathbb{1}^c_{M_{k-1}}\big)z^e_{n,(k-1)h}\big|\Big]\\
&= \big(q - d(1-q)\big)\Big(q\,\mathbb{E}\Big[\big|z^e_{n,kh} - z^e_{n,(k-1)h}\big|\Big] + (1-q)\,\mathbb{E}\Big[\big|z^e_{n,kh} - z^e_{n,(k-1)h} + (1+d)\big(z^e_{n,(k-1)h} - z_{n,(k-1)h}\big)\big|\Big]\Big).
\end{aligned}
$$

This process can be iterated until the initial inventory position is reached. For simplicity, we will assume a "steady state", $z^e_{n,0} = z_{n,0}$. Therefore, you get

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\mathbb{1}_{M_k} - d\mathbb{1}^c_{M_k}\big)\big|z^e_{n,kh} - z_{n,kh}\big|\Big] \qquad (190)\\
&= \big(q-d(1-q)\big)\Big(\sum_{j=0}^{k-1} q(1-q)^j\,\mathbb{E}\Big[\big|\sum_{l=0}^{j}(1+d)^l(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h})\big|\Big] + (1-q)^k\,\mathbb{E}\Big[\big|\sum_{l=0}^{k}(1+d)^l(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h})\big|\Big]\Big).
\end{aligned}
$$

The platform operator does not have any specific information about the desired inventory positions other than knowing which distribution they are drawn from. All shocks are normally distributed and linear. Therefore, $z^e_{n,kh}$ will be normally distributed and the absolute value will then be a folded normal. The expectation of a folded normal, given the underlying random variable, is normally distributed with mean μ and standard deviation σ, is $\sigma\sqrt{\frac{2}{\pi}}e^{-\mu^2/2\sigma^2} + \mu(1-2\Phi(-\mu/\sigma))$. To write out volume explicitly, the mean and variance of the change in shocks are needed. Assume for tractability a time-homogeneous Poisson process of arrival of shocks with intensity $\mu > 0$. The expected value and variance are

$$
\mathbb{E}\Big[\sum_{l=0}^{j}(1+d)^l(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h})\Big] = 0, \qquad (191)
$$

$$
\begin{aligned}
\mathbb{E}\Big[\big(\sum_{l=0}^{j}(1+d)^l(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h})\big)^2\Big] &= \sum_{l=0}^{j}(1+d)^{2l}\,\mathbb{E}\Big[\big(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h}\big)^2\Big] \qquad (192)\\
&= h\mu\frac{\sigma_z^2}{N}\frac{1-(1+d)^{2+2j}}{1-(1+d)^2}
\end{aligned}
$$

Note that the inventory process are martingales and, therefore, $\mathbb{E}\big[(z^e_{n,(k-l)h} - z^e_{n,(k-l-1)h})(z^e_{n,(k-j)h} - z^e_{n,(k-j-1)h})\big] = 0$ for $j \neq l$. Therefore, using the expected value of a folded normal gives that (190) is equal to

$$
= \sqrt{h\mu}\frac{\sigma_z}{\sqrt{N}}\sqrt{\frac{2}{\pi}}\big(q-d(1-q)\big)\Big(q\sum_{j=0}^{k-1}(1-q)^j\sqrt{\frac{1-(1+d)^{2+2j}}{1-(1+d)^2}} + (1-q)^k\sqrt{\frac{1-(1+d)^{2+2k}}{1-(1+d)^2}}\Big). \qquad (193)
$$

Finally, putting (193) into (187), the expected total volume is equal to

$$
\begin{aligned}
&\sum_{n=1}^{N}\mathbb{E}\Big[\sum_{k=0}^{\infty}e^{-rhk}\Big(\mathbb{1}_{M_k}\big|Y_{n,kh}\big| + \mathbb{1}^c_{M_k}\big|X_{n,kh}\big|\Big)\Big] = \qquad (194)\\
&\sigma_z\sqrt{Nh\mu}\sqrt{\frac{2}{\pi}}\big(q-d(1-q)\big)\Big(1 + \sum_{k=1}^{\infty}e^{-rhk}\Big(q\sum_{j=0}^{k-1}(1-q)^j\sqrt{\frac{1-(1+d)^{2+2j}}{1-(1+d)^2}} + (1-q)^k\sqrt{\frac{1-(1+d)^{2+2k}}{1-(1+d)^2}}\Big)\Big).
\end{aligned}
$$