A p Theory of Government Debt and Taxes*

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Abstract

An optimal tax and borrowing plan determines the marginal cost of servicing government debt, p', and makes the government's debt risk-free. An option to default restricts debt capacity. Optimal debt-GDP ratio dynamics are driven by 1) a primary deficit, 2) interest payments, 3) GDP growth, and 4) hedging costs. Hedging influences debt capacity and debt transition dynamics. For plausible parameter values, we make comparative dynamic quantitative statements about debt-GDP ratio transition dynamics, debt capacity, and how long it would take our example economy to attain that calibrated equilibrium debt capacity.

Keywords: sovereign debt; default; limited commitment; Ricardian equivalence; debt capacity; debt sustainability

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1 Introduction

What is a maximum sustainable government debt-to-GDP ratio? Under a good policy, how long should it take to attain that limit? How costly is it for a government to service its debt and how do costs depend on today's debt-GDP ratio? Should a government plan to borrow more when, as in the US today, interest rates on government debt are lower than prospective GDP growth rates? Under an optimal policy, how much would US tax rates eventually have to rise in order to finance the U.S. 123% debt-GDP in Q4 2021?[1]

To answer such questions, we construct a tractable stochastic continuous-time model of government debt and taxes. To highlight model components that we nickname A, B, C, and D, we call it an **ABCD** model. The country's exogenous (potential) output/GDP process follows a geometric Brownian motion (GBM) process[2] driven by idiosyncratic and systematic shocks. At each point in time there is a complete set of competitively priced one-time-increment-ahead **A**rrow securities that fully span all one-time-increment-ahead state contingencies, represented in the manner of (Merton, 1971, Black and Scholes, 1973).

We capture distortionary taxes via a convex deadweight loss function as in **B**arro's deterministic model. As in Eaton and Gersovitz (1981), the government can default on its debt. Upon default, the government's debt balance drops to zero, output decreases, and the government permanently loses access to the debt market: thereafter it must always set its primary budget surplus to zero. It might also face a more adverse tax distortion function.[3] As in Thomas and Worrall (1988), Worrall (1990), Kehoe and Levine (1993), and Ray (2002), adverse continuation values after default deter a borrower from reneging on its debt and bounds its sustainable debt from above.[4] **C**redit constraints that shape the government's debt capacity emerge endogenously as a consequence of our no-commitment-to-repay assumption.[5] The final component of our **ABCD** structure is a **D**iscount rate of the representative household that is larger than that of sovereign creditors. This assumption puts a non-zero drift into optimal debt-GDP ratio dynamics.

[1]This number is from Fred at https://fred.stlouisfed.org.

[2]The GBM process is the continuous-time counterpart of the endowment process used in the classic equilibrium asset-pricing and cost-of-business-cycle models: (Lucas, 1978, 1987).

[3]Our main qualitative results are robust to the detailed specification of punishments for default. The key is that default is costly and hence the government faces a consequence from default. Costly default supports a debt capacity. Otherwise, optimal debt capacity would be zero as shown by Bulow and Rogoff (1989).

[4]Our model shares some of the structure of the simple villager-money-lender model that Ljungqvist and Sargent (2023, ch. 22) use to introduce some of the ideas in the closed economy model of Kocherlakota (1996b) that builds on and reinterprets Thomas and Worrall (1988).

[5]Our model emphasizes effects of financial constraints on sovereign finance, in the same spirit as Bolton (2016), Bolton and Huang (2018), and Rebelo, Wang, and Yang (2021).

Starting from the **B** environment from Barro (1979), our **C**, **A**, and **D** components, respectively, add an option like Eaton and Gersovitz (1981) for the government to default on its debt, complete markets and the ability to hedge government risks by paying appropriate risk premia, and a representative household that is more impatient than creditors, as in Aguiar and Amador (2021).[6]

A household's optimal value function is as $P(B, Y) = p(b)Y$, where Y is GDP, B is government debt, and $b = B/Y$. The marginal cost $-P_B(B, Y) = -p'(b)$ of servicing debt, which we refer to as marginal p, motivates the title of this paper. A government optimally smooths a representative household's tax burdens over time by equating the marginal cost of taxing it with the marginal benefit of using tax proceeds to service government debt. The government in Barro (1979) solves a discounted deadweight loss minimization problem. But the structure of our model instead directs a government to maximize a risk-adjusted present value of total cashflow payouts to the household.[7]

Working in continuous time facilitates sharp explicit characterizations of debt limits and debt dynamics.[8] Two conditions determine a maximum sustainable risk-free debt-to-GDP ratio $\overline{b}$: 1.) the government's indifference condition between defaulting and servicing its debt induced by its limited commitment, and 2.) a zero-drift condition for the debt-GDP ratio b at debt capacity $\overline{b}$ that boils down to a Gordon growth valuation formula at a steady state $\overline{b}$.[9] If we withhold our component **C**: the limited-commitment debt-market participation constraint, we retrieve a stochastic version of Barro's that shares his commitment-to-repay assumption. That model predicts debt capacities that we think are implausibly high, 10-15 times GDP.

Another difference from Barro (1979) is that, while in our model net government debt

[6]While there is no default in equilibrium in our model, the default option induces a limited-commitment constraint. Outcomes in our model differ from sovereign debt models with limited commitment constructed by Eaton and Gersovitz (1981). Aguiar and Amador (2021).

[7]In Barro (1979), the household's value maximization problem is equivalent to the tax distortion cost minimization problem because the government fully commits to repay its debt and output is exogenous. Therefore, the solution in Barro (1979) is indeed welfare maximizing. However, in our model, we have to work with the value maximization problem because the government's limited commitment to repay its debt makes output and debt capacity be endogenous. We cannot simply follow Barro (1979) to solve the distortion cost minimization problem.

[8]DeMarzo, He, and Tourre (2021) construct a continuous-time sovereign-debt model that generates equilibrium debt ratcheting. Rebelo, Wang, and Yang (2021) construct a continuous-time sovereign-debt model in which a country's degree of financial development, defined as how easily it can issue debt denominated in domestic currency in international capital markets, generates "debt intolerance" in the sense of Rogoff, Reinhart, and Savastano (2003).

[9]The zero-drift condition at $\overline{b}$ is an equilibrium argument based on local changes. The Gordon growth model at the steady state is a forward-looking present value calculation argument for the determination of $\overline{b}$. They are equivalent. A non-zero drift of b at $\overline{b}$ would be inconsistent with the notion of debt capacity.

continues to be risk free as it is in Barro's model, it nevertheless bears a risk premium because the revenue stream that ultimately funds it is stochastic and must be insured. For trading a macro-security proposed by Shiller (1994) as well as other state-contingent securities, the government pays an insurance bill to insure itself against risk in GDP growth rates. That insurance bill appears in the debt transition equation and leads to an adjustment of an "$r - g$" term featured by Blanchard (2019) and Mehrotra and Sergeyev (2021). We adopt a "small open economy" assumption that there is an exogenous stochastic discount factor (SDF) process that is not affected by the government's tax and borrowing policy.[10] How government debt is evaluated in complete markets settings is studied empirically by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).

Thus, important features of our complete-market formulation are that 1.) the optimal debt-GDP ratio b process evolves deterministically and 2.) a risk premium from the government's hedging has a first-order effect on the dynamics of the risk-free net debt-GDP ratio b. Both features come from the government's incentive to reduce the household's tax burdens. Applying Jensen's inequality to the first-order condition for the tax rate that to smooth taxes over time and across states implies that it is optimal to make the b process deterministic. By trading state-contingent securities the government can set to zero all contributions to the volatility of b coming from both systematic and idiosyncratic risks. Optimal risk management policies do set them to zero. It is costless to hedge idiosyncratic risk, but the government has to pay a risk premium to hedge the systematic risk component of its GDP shock by trading the security whose payouts are indexed to the GDP process that Shiller (1994) described.[11]

Optimal debt-GDP ratio dynamics are driven by four forces: 1) the primary deficit, 2) interest payments, and 3) GDP growth, and 4) hedging costs. We can summarize these dynamics as follows:

$$\text{change of } b = \text{primary deficit} + \text{interest rate } (r) \times b \; - \text{growth } (g) \times b + \text{hedging cost } (\lambda) \times b. \tag{1}$$

The first term on the right side of (1) is the scaled primary deficit, the difference between government spending and tax revenues, divided by contemporaneous GDP. The second term

[10] We use a geometric Brownian motion process to model the SDF process. It resembles an endogenous SDF that emerges from the equilibrium asset-pricing model of Lucas (1978) as well as the SDF processes that appear in the portfolio-choice model of Merton (1971) and the option pricing model of Black and Scholes (1973).

[11] States in which the return on the Shiller macro security is high are also ones in which investors' marginal utility (equivalently the SDF) is low. That is why the SDF and the return on the Shiller security are negatively correlated.

is a (scaled) interest payment that equals b times the risk-free rate r. The third term is debt reduction due to GDP growth. These three terms are discussed, for example, by Blanchard (2019) and Mehrotra and Sergeyev (2021). In addition to these three terms, (1) contains a fourth term that emerges because the government optimally chooses to hedge its GDP process in a way that ends up making b evolve deterministically. Hedging costs equal b times the risk premium of a risky asset whose cash flow process is the same as the GDP process. Combining the second and fourth terms in (1), we obtain:

$$\text{change of } b = \text{primary deficit} + \text{Shiller's macro security return } (r+\lambda) \times b \; - \text{growth } (g) \times b \,. \tag{2}$$

Evidently, the appropriate expected rate of return to multiply b in the b dynamics (2) is not the risk-free rate that Blanchard (2019) used but instead the expected return on the macro security of Shiller (1994). Because GDP is positively correlated with the aggregate shock, the government buys GDP-indexed hedging contracts that require it to pay a continuous premium payment of λ for each unit of debt outstanding. By making that trade, the government eliminates its exposure to the aggregate shock and makes its net debt risk-free.

Relative to Barro (1979), a third difference is that our government is impatient as in Aguiar and Amador (2021) and DeMarzo, He, and Tourre (2021). This implies that the government's discount rate exceeds the interest rate. This outcome is consistent with US Treasury bonds bringing a convenience yield that lowers the US cost of borrowing below a risk-free rate (Krishnamurthy and Vissing-Jorgensen, 2012).[12] The government's impatience generates a backloaded tax schedule in which the tax rate increases over time. Fiscal deficits scaled by GDP decrease over time and eventually become surpluses. The debt-GDP ratio approaches a steady state that attains a maximum sustainable level $\bar{b}$. If a government that is impatient enough starts from a sufficiently small debt, it immediately increases b to an optimal target level $\underline{b} > 0$ at which its marginal cost of servicing debt equals one. This is a Blanchard (2019) "debt is cheap" response on steroids.[13] Thus, optimal debt-GDP dynamics reside in three disjoint regions:[14] 1.) a lump-sum debt issuance and payout region in which

[12] Since the US borrowing cost is lower than the risk-free rate, various investment projects and welfare transfer programs, e.g., infrastructure, seem to become more attractive. One would still need to evaluate net payoff streams for such projects, but with a lower cost of capital than the risk-free rate. Van Binsbergen, Diamond, and Grotteria (2022) estimate the convenience yield to be about 40 basis points per annum.

[13] To construct an optimal fiscal plan, our government uses both singular control (lump-sum debt issuance and payout to the household) and convex control (tax smoothing). The US government's 2020 and 2021 covid stimulus checks and related transfers might be interpreted as examples of such payouts financed by lump-sum debt issuances.

[14] Only at time 0 is only possible for the government to be in either the lump-sum debt issuance and payout region or the default region. If starting in the lump-sum debt issuance and payout region where $b < \underline{b}$, the

$b < \underline{b}$; 2.) a default region in which debt is unsustainable ($b > \overline{b}$); and 3.) an interior region in which $b \in [\underline{b}, \overline{b}]$. Because debt is inexpensive here it is optimal for the government to jump its debt to $\underline{b}$ in the $b < \underline{b}$ region. In honor of Blanchard (2019), we refer to this as the Blanchard region. In the $b \in [\underline{b}, \overline{b}]$ interval, Barro tax-smoothing prevails. When $b \geqslant \overline{b}$, the government balances its budget period by period.

An optimal policy is described by 1) a nonlinear first-order ordinary differential equation (ODE) for the government's (scaled) value $p(b)$; 2) a first-order condition for the optimal tax rate $\tau(b)$; 3) a zero-drift condition and an indifference condition between defaulting and not defaulting that characterize a steady state in which debt is at the maximum sustainable level $\overline{b}$; 4) value-matching and smooth-pasting conditions that characterize the lump-sum debt issuance and payout boundary $\underline{b}$. The upper debt-capacity boundary $\overline{b}$ is an absorbing state and the lower lump-sum debt issuance boundary $\underline{b}$ is a reflecting barrier. These two boundaries embody economic forces on the government's maximum sustainable debt and its policy for an initial lumpy payout to the household.[15] To the best of our knowledge, we are the first to derive a zero-drift condition that pins down an endogenous debt capacity. We characterize the two boundaries in ways that show case how the continuous-time setting allows us to represent underlying economic forces concisely.

The government's marginal cost of servicing debt $-P_B(B, Y) = -p'(b)$ measures how much the household's value decreases when government debt increases by one unit. Tax distortions and limited commitment make $-p'(b)$ exceed one; $-p'(b)$ appears in both the first-order condition for an optimal tax rate and in an equation that restricts the government's optimal value function.

A calibrated version of our **ABCD** model provides a back-of-the-envelope estimate of how long it will take for the US to attain its maximum sustainable debt. Such calculations can help us sort through current debates about debt sustainability. We tell how the time to reach debt capacity critically depends on the prevailing interest rate and on a government's impatience.[16] For fixed government impatience, the lower is the interest rate, the higher is a government's debt capacity. So in an economy in which the interest rate on government debt is low, a government initially taxes less and borrows more, thereby making the debt-GDP

government increases its debt so that its b instantly equals $\underline{b}$ after time 0 and then the b process is dictated by the law of motion in the interior region. If starting in the default region where $b > \overline{b}$, the government immediately defaults and sets taxes to its expenditure so that its primary deficit is zero at all time.

[15] Our baseline model is amenable to extensions that will allow additional sources of randomness not included in the baseline model – e.g., a Markov process for the government expenditures/GDP ratio rather than the fixed ratio in the baseline model.

[16] Bohn (1998) described measures that the US took in response to the accumulation of debt during the 1970s and 1980s that are broadly consistent with dynamics prescribed by our model.

ratio increase faster. In this situation, a direct debt-capacity effect dominates an indirect (debt-GDP ratio) drift effect in shaping how long it takes to exhaust its debt capacity. This logic underlies an argument that a government should borrow more when debt is cheap, e.g., Blanchard (2019). A reader wanting a quick preview of how the **ABCD** components combine to shape outcomes can skip ahead to subsection 6.2 and the all-in-one Figure 1.

By adopting a continuous-time contracting framework used by DeMarzo and Sannikov (2006) and Sannikov (2008), Internet Appendix I.A formulates the dual to the government's optimal debt management problem. It takes the form of present-value-of-revenue maximizing government that confronts a household that has the option of forcing the government to balance primary budget always.[17] In the dual, a key state variable confronting the revenue-maximizing government is a continuation value that the government has promised the household. A well-diversified planner maximizes the present value of cash flows subject to a stream of incentive constraints on those promised values.

Related Literature. Our model assembles building blocks from Lucas and Stokey (1983) (complete state-contingent debt) and Barro (1979) (tax distortion costs) in a tractable continuous-time framework with an exogenously specified SDF along lines of Black and Scholes (1973), Merton (1973), and Harrison and Kreps (1979). Our assuming an exogenous stochastic discount factor process distinguishes our model markedly from Lucas and Stokey (1983). In their model, a government's tax and borrowing strategy affects the stochastic discount factor process. That motivates their government to manipulate equilibrium debt prices by altering distorting taxes. Like Lucas and Stokey (1983), we assume complete financial markets that allow the government to issue fully state contingent debt.[18] By staying within the Barro tradition of an exogenous SDF process, we remove dynamic inconsistencies that arise from bond-price-manipulation motives central to models in the Lucas-Stokey tradition.[19] We focus on implications of limited commitment for debt capacity and debt dynamics.

Bohn (1995) valued government debt with an SDF like that of Lucas (1978). Bohn (1990) studied how hedging with financial instruments shapes optimal fiscal policy of a risk-neutral government in a stochastic reformulation of Barro (1979). A difference between our paper and Bohn (1990) is that hedging costs play a key role in debt-GDP dynamics in our model.

[17] Ai and Li (2015) and Bolton, Wang, and Yang (2019) construct recursive contracts to cope with limited-commitment problems in corporate finance.

[18] Our complete financial spanning setting eases analysis and exposition. We leave important extensions to incomplete markets settings along the line of Aiyagari, Marcet, Sargent, and Seppälä (2002) for subsequent research.

[19] Because the Barro (1979) model is deterministic, his SDF is an exponential function that decays at the risk-free rate per unit of time.

We extend Bohn's insights by incorporating effects of default opportunities on debt dynamics and sustainability. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) analyze how the covariance between an intertemporal marginal rate of substitution and a primary government surplus ought to affect the value of government debt.

Brunnermeier, Merkel, and Sannikov (2020, 2022) incorporate a bubble term within a fiscal theory of the price level, develop a model of safe assets with a negative beta in an incomplete-markets setting, and analyze implications for debt sustainability. Kocherlakota (2021) develops a model of government debt bubbles associated with tail risk in a heterogeneous-agent incomplete-markets Aiyagari-Bewley-Huggett style model. Reis (2021) studies debt capacity in a related model with a bubble on government debt. D'Erasmo, Mendoza, and Zhang (2016) review the literature on government debt sustainability. Abel, Mankiw, Summers, and Zeckhauser (1989) and Abel and Panageas (2022) analyze maximum budget-feasible government debt in overlapping generations models with perpetually zero primary budget surpluses. Elenev, Landvoigt, Shultz, and Van Nieuwerburgh (2021) construct a New Keynesian model that includes financial intermediation, risk premia, production, fiscal policies, and conventional and unconventional monetary policies.[20]

In calling our model a p theory of taxes and government debt, we provoke an analogy with a q theory of investment. A convex tax distortion cost that we take from Barro (1979) serves as a counterpart to the convex capital adjustment cost in a q theory of investment, e.g., Hayashi (1982). In q theory, an optimum condition sets marginal q, the marginal value of capital, equal to the marginal cost of investing. In our p theory, an optimal fiscal policy sets the marginal cost of taxing equal to the marginal cost of servicing government debt, $-p'(b)$. In a q theory, a firm's asset is productive capital that generates a cash flow. Government debt is both "backward" and "forward looking": while it cumulates *past* primary government deficits, it has to be serviced from *prospective* primary surpluses. Because it is costly to adjust productive capital, marginal q exceeds one in q theory ,while the marginal cost of servicing debt, $-p'(b)$, exceeds one in our p theory because the prospective taxes that will service government are distortionary. It is enlightening to watch our model unleash forces that resemble ones that appear in the q-theories of costly capital stock adjustment of Lucas and Prescott (1971), Hayashi (1982), and Abel and Eberly (1994). Tax distortions in our model affect asset valuations and act in ways similar to the costs of capital adjustment in the q theories.

[20]For other discussions of '$r - g$' and debt sustainability, please see Barro (2020), Van Wijnbergen, Olijslagers, and de Vette (2020), Aguiar, Amador, and Arellano (2021), Mian, Straub, and Sufi (2021), Reis (2021), and Liu, Schmid, and Yaron (2021).

2 The Setting

Time $t \in [0, +\infty)$. Subject to a sequence of limited-commitment constraints, the government wants an optimal taxation and financing plan. The government trades a complete set of history-contingent securities. We extend Barro (1979) along three lines. First, we introduce both idiosyncratic and systematic shocks that allow us to analyze how risks affect taxation and debt management. Second, in the spirit of Thomas and Worrall (1988), Worrall (1990), Kehoe and Levine (1993), Kocherlakota (1996b), Alvarez and Jermann (2000, 2001), and Chien and Lustig (2010), at each instant the government is free to default, an option that limits its ability to borrow. Third, we assume that the household is impatient. Unlike Lucas and Stokey (1983), debt at time 0 is endogenous and the SDF is exogenous.[21]

There are two coupled "regimes." In the "no-default" regime, the government trades state-contingent securities and services its debts and chooses how much to tax. If the government ever defaults on its debt, it permanently enters a "balanced-budget" regime.

2.1 Output, Government Spending, and Taxation

After describing GDP, government spending, and taxation in the no-default regime, we'll describe them in the balanced-budget regime.

Output, Government Spending, and Taxation in the no-default Regime.

Output process. GDP $\{Y_t; t \geqslant 0\}$ is exogenous and follows a geometric Brownian motion (GBM) process

$$\frac{dY_t}{Y_t} = gdt + \sigma_Y d\mathcal{Z}_t^Y \,, \tag{3}$$

where, under the physical measure $\mathbb{P}$, $\mathcal{Z}_t^Y$ is a standard Brownian motion, g is the expected GDP growth rate, $\sigma_Y > 0$ is the growth volatility, and $Y_0 > 0$ is the known initial value of Y_t.

GDP Y_t is subject to idiosyncratic shocks that warrant no risk premium and to systematic shocks that warrant a risk premium. Let the standard Brownian motion $\mathcal{Z}_t^h$ represent the idiosyncratic shock and the standard Brownian motion $\mathcal{Z}_t^m$ represent the systemic shock under a physical measure $\mathbb{P}$, respectively. We refer to the systematic shock $d\mathcal{Z}_t^m$ as the

[21] We represent the household's risk aversion indirectly via the SDF. In a sequel, we extend our model to describe risk aversion via an intertemporal utility functional. We can solve this more general model in closed form up to an ODE with economically interpretable boundary conditions. While some results reported here are altered under those preferences, key qualitative results about equilibrium debt capacity and optimal tax smoothing remain unaltered.

Shiller macro security shock.[22] We can decompose the output shock $d\mathcal{Z}_t^Y$ over dt under the physical measure $\mathbb{P}$ as

$$d\mathcal{Z}_t^Y = \sqrt{1-\rho^2}\, d\mathcal{Z}_t^h + \rho\, d\mathcal{Z}_t^m \,, \tag{4}$$

where ρ is the constant correlation coefficient between the output shock $d\mathcal{Z}_t^Y$ and the shock $d\mathcal{Z}_t^m$. For convenience, we can also write the output process $\{Y_t; t \geqslant 0\}$ in (3) as

$$\frac{dY_t}{Y_t} = gdt + \left(\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m\right) \,, \tag{5}$$

where ψ_m and ψ_h are systematic and idiosyncratic volatility parameters given by

$$\psi_m = \rho\sigma_Y \qquad \text{and} \qquad \psi_h = \sqrt{1-\rho^2}\,\sigma_Y \,, \tag{6}$$

respectively. Expressions (5)-(6) for $\{Y_t; t \geqslant 0\}$ help us isolate distinct roles of systematic and idiosyncratic shocks.

Government spending and debt. Let $\{\Gamma_t; t \geqslant 0\}$ denote an exogenous government spending process that brings no utility to the household. We assume that in the no-default regime Γ_t varies with contemporaneous output Y_t according to

$$\Gamma_t = \gamma_t Y_t \,, \tag{7}$$

where γ_t is exogenous. We set $\gamma_t = \gamma \in [0, 1]$ so that government spending is proportional to GDP in the no-default regime. The government finances its spending Γ_t with taxes and debts.

Debt and taxes. Let $\{B_t; t \geqslant 0\}$ denote the government's debt balance and $\{\mathcal{T}_t; t \geqslant 0\}$ denote the tax revenue process. As in Barro (1979), we assume that taxes are distortionary. Let $C_t = C(\mathcal{T}_t, Y_t)$ denote deadweight loss in units of consumption goods when the government collects tax revenue $\mathcal{T}_t$ and GDP is Y_t in the no-default regime. Following Barro (1979), we assume that the deadweight loss function, $C(\mathcal{T}_t, Y_t)$, is homogeneous of degree one in output Y_t and tax revenue $\mathcal{T}_t$:

$$C_t = C(\mathcal{T}_t, Y_t) = c(\tau_t) Y_t \,, \tag{8}$$

[22] For mnemonic convenience, we use superscript m to refer to the Shiller *macro* security shock and the superscript h to refer to the *hedgeable* idiosyncratic shock.

where $\tau_t = \mathcal{T}_t / Y_t$ is the average tax rate on output. Again following Barro (1979), we assume that the scaled deadweight loss, $c(\tau)$, is increasing, convex, and smooth.

Since tax revenue at time t cannot exceed net output $Y_t - \Gamma_t$, we require $\mathcal{T}_t \leqslant \overline{\tau}\, Y_t$, which is equivalent to the following constraint on the tax rate τ_t:

$$\tau_t \leqslant \overline{\tau}\,, \tag{9}$$

where $\overline{\tau}$ is a maximal politically feasible tax rate on GDP Y_t in the no-default regime. Keynes (1923, pp.56–62) and Keynes (1931) inferred limits on a country's debt-GDP ratio partly from an upper bound $\overline{\tau}$ based on political considerations.

Output, Government Spending, and Taxation in the Balanced-budget Regime. Defaulting brings disruptions to economic activities that cause an output loss. Let $\widehat{Y}_t$ denote GDP in the balanced-budget regime and let $T^{\mathcal{D}}$ denote an endogenous time when the government defaults. Following Aguiar and Gopinath (2006) and Rebelo, Wang, and Yang (2021), we assume that when the government defaults it repudiates all of its debt, that GDP immediately drops from $Y_{T^{\mathcal{D}}-} = \lim_{s \uparrow T^{\mathcal{D}}-} Y_s$, the pre-default GDP level, to $\widehat{Y}_{T^{\mathcal{D}}} = \alpha Y_{T^{\mathcal{D}}-}$, and that the government permanently resides in the balanced-budget regime.[23]

In the balanced-budget regime ($t \geqslant T^{\mathcal{D}}$), the government can issue no debt ($B_t = 0$) and output $\widehat{Y}_t$ follows a downward scaled version of the GBM process (5). Therefore,

$$\widehat{Y}_t = \alpha Y_t\,, \quad t \geqslant T^{\mathcal{D}}\,, \tag{10}$$

where $\alpha \in (0,1)$ is a constant.[24] So output in the balanced-budget regime equals an α fraction of Y_t given in (3), where $\{Y_t; t \geqslant 0\}$ would have been GDP had the economy permanently stayed in the no-default regime.

Let $\widehat{\mathcal{T}}_t$ denote tax revenue in the balanced-budget regime. Since the government can issue no debt in the balanced-budget regime, it has to finance its spending period by period according to

$$\widehat{\mathcal{T}}_t = \Gamma_t = \gamma_t Y_t\,, \quad t \geqslant T^{\mathcal{D}}\,. \tag{11}$$

Note that government spending $\{\Gamma_t; t \geqslant 0\}$ is exogenous and independent of its default decision.

Taxation continues to be distortionary in the balanced-budget regime. Let $\widehat{C}_t = \widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t)$ denote deadweight loss when the government collects tax revenue $\widehat{\mathcal{T}}_t$ and output is $\widehat{Y}_t$ in the

[23] To ease exposition, we assume no exit from the balanced-budget regime.

[24] Hébert and Schreger (2017) provide supporting empirical evidence.

balanced-budget regime. We assume that $\widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t)$ is homogeneous of degree one in tax revenue $\widehat{\mathcal{T}}_t$ and output $\widehat{Y}_t$:

$$\widehat{C}_t = \widehat{C}(\widehat{\mathcal{T}}_t, \widehat{Y}_t) = \widehat{c}(\widehat{\tau}_t)\widehat{Y}_t \,, \tag{12}$$

where $\widehat{\tau}_t = \widehat{\mathcal{T}}_t/\widehat{Y}_t$ is the tax rate in the balanced-budget regime. We assume that $\widehat{c}(\widehat{\tau})$ is increasing, convex, and smooth.

Deadweight loss functions in the two regimes are connected:

$$\widehat{c}(\,\cdot\,) = \kappa\, c(\,\cdot\,)\,. \tag{13}$$

The parameter $\kappa \geqslant 1$ measures how much more costly taxation is in the balanced-budget regime than in the no-default regime.

As in the no-default regime, we require $\widehat{\mathcal{T}}_t \leqslant \overline{\tau}\,\widehat{Y}_t$, which is equivalent to the following constraint on the tax rate $\widehat{\tau}_t$ in the balanced-budget regime:

$$\widehat{\tau}_t \leqslant \overline{\tau}\,, \quad t \geqslant T^{\mathcal{D}}\,, \tag{14}$$

where $\overline{\tau}$ is the same maximum politically feasible tax rate described above.

Thus, default brings three costs: 1) a loss of output (as $\widehat{Y}_t = \alpha Y_t < Y_t$); 2) possibly a worse deadweight loss function than it faced in the no-default regime ($\kappa \geqslant 1$); and 3) period-by-period primary budget balance.

2.2 Financial Markets

A government in the no-default regime has the following investment and financing opportunities: (1) it can insure its idiosyncratic risk through actuarially fairly priced hedging contracts; (2) it can invest in a Shiller macro security portfolio; and (3) it can issue risk-free debt that matures instantaneously and is continuously rolled over. Outcomes would not change if we were to include longer term government debt too. Markets are dynamically complete.[25]

Idiosyncratic risk hedging asset. There is a competitive market in a financial asset that is perfectly correlated with the idiosyncratic shock $\mathcal{Z}_t^h$. Because no risk premium is

[25]Subject to quantity limits coming from its inability to commit to repayment, the government can dynamically trade a complete set of Arrow securities. Our analysis builds on a dynamic replicating portfolio argument used in Black and Scholes (1973) and Harrison and Kreps (1979) under complete markets with full commitment.

awarded for bearing idiosyncratic risk, an investor who holds one unit of this asset at time t receives no up-front payment but receives a gain or loss equal to $d\mathcal{Z}_t^h = \left(\mathcal{Z}_{t+dt}^h - \mathcal{Z}_t^h\right)$ at time $t + dt$. We normalize the volatility parameter of this hedging contract to be one. We denote the government's holdings of this idiosyncratic risk hedging asset at time t by $-\Xi_t^h$, so the government's idiosyncratic risk exposure in levels is $-\Xi_t^h d\mathcal{Z}_t^h$ over dt.

Shiller's macro security and equivalent futures contract. The government can manage its exposure to risks in GDP growth by trading an asset whose payouts are proportional to the aggregate shock, a type of security described by Shiller (1994). In the spirit of Merton (1971) and Black and Scholes (1973), we assume that under the physical measure $\mathbb{P}$ the Shiller macro security return dR_t over dt is independently and identically distributed (i.i.d) with the drift parameter μ_m and the volatility parameter σ_m:[26]

$$dR_t = \mu_m dt + \sigma_m d\mathcal{Z}_t^m \,, \tag{15}$$

where $\mathcal{Z}_t^m$ is a standard Brownian motion under the physical measure $\mathbb{P}$.

We can rewrite the return process (15) as $dR_t = rdt + \sigma_m d\widetilde{\mathcal{Z}}_t^m$, where η is the Sharpe ratio of the Shiller macro security

$$\eta = \frac{\mu_m - r}{\sigma_m} \tag{16}$$

and $\widetilde{\mathcal{Z}}_t^m$ represents the risk-adjusted aggregate shock[27]

$$d\widetilde{\mathcal{Z}}_t^m = \eta dt + d\mathcal{Z}_t^m \,. \tag{17}$$

We interpret $d\widetilde{\mathcal{Z}}_t^m = \eta dt + d\mathcal{Z}_t^m$ as the payoff on a unit of the futures contract on the Shiller macro security (an example of a dt-step-ahead Arrow security.) The value of this futures contract with payoff (17) is zero (Cox, Ingersoll, and Ross, 1981). Thus, a risk-averse investor requires a payment of ηdt to bear a unit of the aggregate shock $d\mathcal{Z}_t^m$. Once we add the drift payoff ηdt with the aggregate shock exposure $d\mathcal{Z}_t^m$, the investor is indifferent between investing and not investing in this futures contract, implying that the value of the futures contract is zero.

As for the idiosyncratic risk hedging position, we denote the government's holdings of this Shiller macro security futures contract at time t by $-\Xi_t^m$, which implies that in levels

[26] This widely used geometric Brownian motion process for stock price is fully consistent with the asset pricing model of Lucas (1978). If we generalized our model to allow for disasters/jumps as in Barro (2006), all of our insights would remain valid.

[27] In Appendix B, we show that $\widetilde{\mathcal{Z}}_t^m$ is a standard Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. The drift of the price of the stock futures contract is zero under $\widetilde{\mathbb{P}}$ (Duffie, 2001).

the government's systematic risk exposure is $-\Xi_t^m(\eta dt + d\mathcal{Z}_t^m)$ over dt. Because financial market risk spanning is complete, the government can also use the Shiller macro security rather than the futures contract to manage aggregate shocks. We use the futures contract in order to preserve the expositional symmetry in our treatment of idiosyncratic risk and systematic risk management.

Stochastic discount factor. We assume that a single aggregate shock $\mathcal{Z}_t^m$ drives the Shiller macro security payout. Using the standard no-arbitrage argument for complete-markets economies, we obtain a unique stochastic discount factor process (SDF), Λ_t:

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta d\mathcal{Z}_t^m\,, \quad \Lambda_0 = 1\,. \tag{18}$$

No arbitrage requires that the drift of $d\Lambda_t/\Lambda_t$ equals $-r$. In our one-factor model, the volatility of $d\Lambda_t/\Lambda_t$ equals $-\eta$, where $\eta = (\mu_m - r)/\sigma_m$ is the market price of risk; this is also the Sharpe ratio for the Shiller macro security (Duffie, 2001).

2.3 Government Budget and Objective

Budget constraints. Given an initial debt level (B_0), the government at $t = 0$ has intertemporal budget constraint:

$$B_0 \leqslant \mathbb{E}_0 \int_0^{T^{\mathcal{D}}} \Lambda_t \left[(\mathcal{T}_t - \Gamma_t)\, dt - dU_t \right]\,, \tag{19}$$

where $\{U_t; t \geqslant 0\}$ is the undiscounted cumulative lump-sum transfer to households process so that dU_t is the incremental non-negative lump-sum transfer to households over dt.[28] The right side of (19) is the present value of the government's primary surplus $[(\mathcal{T}_t - \Gamma_t)\, dt - dU_t]$. The left side of (19) is the initial debt level B_0. Inequality (19) states that the value of debt B_0 cannot exceed the time-0 value of the government's primary surpluses until it defaults at $T^{\mathcal{D}}$. After the government defaults, creditors recover nothing.

Flow payoffs to the household. Let $\mathbf{1}_t^{\mathcal{D}}$ be an indicator function that equals one in the balanced-budget regime when $t \geqslant T^{\mathcal{D}}$ and zero in the no-default regime when $t < T^{\mathcal{D}}$. In the balanced-budget regime ($\mathbf{1}_t^{\mathcal{D}} = 1$), the government has no debt and the household continuously receives payments at rate $(\widehat{Y}_t - (\Gamma_t + \widehat{C}_t))$ because $\widehat{\mathcal{T}}_t = \Gamma_t$. In the no-default regime ($\mathbf{1}_t^{\mathcal{D}} = 0$), the household continuously receives payments at the rate $(Y_t - (\mathcal{T}_t + C_t))$,

[28] Technically, $\{U_t; t \geqslant 0\}$ is a singular control process. We shall show that at an optimum U_t is non-decreasing.

which equals the difference between GDP Y_t and total costs of taxes $(\mathcal{T}_t + C_t)$. The household also receives a lump-sum transfer dU_t when the government issues debt dU_t and distributes the proceeds. When the benevolent government, acting in the interests of the representative household, is impatient, a lump-sum transfer can occur under an optimal government plan.

Thus, the household receives flow payments from three sources: 1) lump-sum transfer to the household financed by debt issuance dU_t in the no-default regime; 2) recurrent payments in the no-default regime $(Y_t - (\mathcal{T}_t + C_t))$; and 3) recurrent payments in the balanced-budget regime $(\widehat{Y}_t - (\widehat{\mathcal{T}}_t + \widehat{C}_t))$.

Intertemporal discounting and risk premium specifications. Let $(\zeta + r)$ denote the rate at which the household discounts future payoffs. We assume that the household values risk in the same way as investors and hence uses the same market price η for aggregate risk.[29] As a result, when $\zeta = 0$, the benevolent government and the market are equally patient. In this case, the household and investors use the same SDF Λ_t to value payouts. However, when the household is impatient ($\zeta > 0$), a common assumption in the sovereign debt literature (e.g., Aguiar and Gopinath, 2006), the benevolent government front loads consumption and postpones debt repayments. This leads us to use $e^{-\zeta t}\Lambda_t$ as the effective SDF for the household to value their risky payoffs instead of the SDF Λ_t that investors use to price payoffs. Appendix B provides technical details.

Government objective. Combining our assumptions about flow payoffs and the household's effective SDF, we obtain the the household's value:

$$\mathbb{E}_0 \int_0^\infty e^{-\zeta t}\Lambda_t \left[\left(1 - \mathbf{1}_t^{\mathcal{D}}\right)\left(dU_t + \left(Y_t - (\mathcal{T}_t + C_t)\right)dt\right) + \mathbf{1}_t^{\mathcal{D}}\left(\widehat{Y}_t - (\widehat{\mathcal{T}}_t + \widehat{C}_t)\right)dt\right], \tag{20}$$

where $\zeta \geqslant 0$ measures the household's impatience. The government chooses lump-sum transfers (dU_t), tax rates (τ_t and $\widehat{\tau}_t$), and idiosyncratic and systematic risk hedging demands (Π_t^h and Π_t^m) to maximize (20) subject to budget constraint (19), constraint (9) on the tax rate τ in the no-default regime, and constraint (14) on $\widehat{\tau}$ in the balanced-budget regime. The government's access to complete markets and the inefficiency of default induce the government not to default and to make its net debt be risk free. Risk-free debt capacity $\overline{B}_t$ is part of an optimal plan. To economize on free parameters, we use the same SDF Λ_t to value flow payoffs to the household in both no-default and balanced-budget regimes.[30]

Let $P_t = P(B_t, Y_t)$ denote the household's continuation value at time t. Let $S_t = S(B_t, Y_t)$

[29]The household and investors use the same Radon-Nikodym derivative that links physical measure $\mathbb{P}$ to risk-neutral measure $\widetilde{\mathbb{P}}$ (Duffie, 2001). With complete markets, this Radon-Nikodym derivative is unique.

[30]To capture additional adverse affects from defaulting, we could modify the SDF in the balanced-budget regime by using a different risk-free rate and market price of risk.

denote the sum of debt value B_t and the household's value $P(B_t, Y_t)$:

$$S_t = S(B_t, Y_t) = P(B_t, Y_t) + B_t \,. \tag{21}$$

The household's value function after a default affects its value and optimal decisions before it has defaulted. Since government debt is always zero in the balanced-budget regime, the household's value function in the balanced-budget regime depends on only contemporaneous GDP $\widehat{Y}_t = \alpha Y_t$; so we denote the value function in the balanced-budget regime by $\widehat{P}(\widehat{Y}_t)$. Because default is costly, the government wants to manage its state-contingent debt dynamics to avoid default. That gives rise to the following participation constraint:

$$P(B_t, Y_t) \geqslant \widehat{P}(\widehat{Y}_t) \,. \tag{22}$$

3 Model A: Ricardian Equivalence

Before deducing an optimal government plan in our ABCD model, we revisit the Ricardian equivalence logic of Barro (1974). A Ricardian equivalence version of our model features only a complete set of **A**rrow's one-period ahead securities. We call this special case Model **A**. Taxes are not distorting all budget-feasible tax policies are equivalent. The household's value (20) becomes

$$P_0 = \mathbb{E}_0 \int_0^\infty \Lambda_t \left[dU_t + (Y_t - \mathcal{T}_t)\, dt \right] \,, \tag{23}$$

and its present value budget constraint becomes

$$B_0 \leqslant \mathbb{E}_0 \int_0^\infty \Lambda_t \left[(\mathcal{T}_t - \Gamma_t)\, dt - dU_t \right] \,. \tag{24}$$

Combining (23) and (24) at equality yields

$$P_0 + B_0 = \mathbb{E}_0 \int_0^\infty \Lambda_t \left(Y_t - \Gamma_t \right) dt \,. \tag{25}$$

Expression (25) states that the total value $S_0^{FB} = P_0 + B_0$ is independent of policies $\{U_t, \mathcal{T}_t; t \geqslant 0\}$, a version of "Ricardian equivalence". We use superscript FB to denote the value attained when our three extensions to Barro (1979) have been deactivated.

In the spirit of Shiller (1994), consider a financial asset whose cash flow almost surely equals net output $\{Y_t - \Gamma_t = (1-\gamma)Y_t; t \geqslant 0\}$ process. Its value equals the right side of (25). The SDF (18) implies that the risk premium of this macro asset λ equals $\beta(\mu_m - r)$, where $\beta = \rho\sigma_Y/\sigma_m$ is the CAPM regression coefficient of this asset's return on the Shiller macro security portfolio return. We can equivalently write this asset's risk premium as

$$\lambda = \psi_m \eta = \rho \sigma_Y \eta \,. \tag{26}$$

Since tax and debt policies are irrelevant here, total value $S_t^{FB} = P_t + B_t$ equals the value of this Shiller-like financial asset:

$$S_0^{FB} = \mathbb{E}_0 \int_0^\infty \Lambda_t \left(Y_t - \Gamma_t\right) dt = \frac{1-\gamma}{r+\lambda-g} Y_0 \,. \tag{27}$$

To assure convergence of the integral on right side of the above equation, we require the expected return $r + \lambda$ to be larger than the GDP growth rate g:

$$r + \lambda > g \,. \tag{28}$$

4 Model AB: Stochastic Version of Barro (1979)

We now briefly describe an **AB** version of our general **ABCD** model. This model includes **B**arro's tax distortions and a complete set of **A**rrow's one-period-ahead securities, but it excludes the **CD** features of the general model. The government chooses a policy to maximize

$$\mathbb{E}_0 \int_0^\infty \Lambda_t \left[dU_t + \left(Y_t - \left(\mathcal{T}_t + C_t\right)\right) dt \right] \,, \tag{29}$$

subject to budget constraint (24) and the Keynes constraint (9) on the tax rate. Substituting budget constraint (24) at equality into objective function (29), we obtain that the value being maximized by the government is

$$\mathbb{E}_0 \int_0^\infty \Lambda_t \left(Y_t - \Gamma_t - C_t\right) dt - B_0 \,. \tag{30}$$

Choosing $\{\mathcal{T}_t; t \geqslant 0\}$ to maximize (30) is equivalent to minimizing the present value of deadweight losses $\mathbb{E}_0 \int_0^\infty \Lambda_t C_t dt$ subject to the constraint of honoring an initial debt B_0 that satisfies (24) with equality. This was Barro's justification for recasting the government's value maximization problem as a deadweight loss minimization problem. Such an equivalence does not prevail in our **ABCD** model because the government's option to default contributes endogenous distortion costs. So we must work with a value-maximization problem rather than a cost-minimization problem.

It is useful to scale variables by contemporaneous GDP. Let b_t denote a debt-GDP ratio

$$b_t = \frac{B_t}{Y_t}. \tag{31}$$

Similarly, let

$$p(b_t) = \frac{P(B_t, Y_t)}{Y_t} \quad \text{and} \quad s(b_t) = \frac{S(B_t, Y_t)}{Y_t} = p(b_t) + b_t \,. \tag{32}$$

We have

Proposition 4.1. Stochastic Barro (1979) Model. *Assuming $\zeta = 0$ and a government commited to service its debt, the optimal debt-GDP ratio $b_t = b_0$ for all t; the optimal tax rate τ_t is constant over time and depends only on b_0:*

$$\tau(b_t) = \tau(b_0) = (r + \lambda - g)b_0 + \gamma \,. \tag{33}$$

The government's scaled value function, $f(b_t)$, is also constant over time and given by

$$p(b_t) = p(b_0) = \frac{1 - \tau(b_0) - c(\tau(b_0))}{r + \lambda - g} \,. \tag{34}$$

Any initial debt-GDP level b_0 satisfying $b_0 \leqslant \overline{b}$ is sustainable, where

$$\overline{b} = \frac{\overline{\tau} - \gamma}{r + \lambda - g} \,. \tag{35}$$

We relegate a proof of Proposition 4.1 to Appendix A. In our stochastic Barro economy, any initial condition is a steady state, since $b_t = b_0$ and $p(b_t) = p(b_0)$. Therefore, the present value of the (scaled) primary surplus $\tau(b_t) - \gamma$ equals the (scaled) debt b_t at all t:

$$\frac{\tau(b_t) - \gamma}{r + \lambda - g} = b_t = b_0 \,. \tag{36}$$

Notice that discount rate $r + \lambda$ appears in present value equation (36), not the risk-free rate r. An optimal tax rate $\tau(b_t)$ satisfies the following first-order condition:

$$1 + c'(\tau(b_t)) = -p'(b_t) \,. \tag{37}$$

The government optimally equates the marginal cost $1 + c'(\tau(b_t))$ of taxing the household with the marginal benefit $-p'(b_t) > 0$ of reducing debt, a version of Barro's tax smoothing recommendation.

Were it to be offered an option to choose its initial debt level, a government would set $b_0 = 0$ because that maximizes $s(b_0) = p(b_0) + b_0$. Using (37), we obtain $s'(b_0) = p'(b_0) + 1 = -c'(\tau(b_0)) \leqslant 0$. Our assertion that an optimal $b_0 = 0$ follows from the assumption that $c(\cdot)$ is increasing and convex: issuing lump-sum debt yields no benefits but induces distorting debt servicing costs.

Next we show that when the government has the option to default, equivalence between the government's value maximization and cost minimization problem no longer holds .

5 Tax and Debt Management in Environment ABCD

After posing our section 2 government's dynamic debt and risk management problem as a dynamic program, we characterize policies in both interior and lump-sum transfer regions of the state space.

5.1 No-default Regime

Dynamic State-Contingent Debt Management. The government manages idiosyncratic and systematic risk by choosing Ξ_t^h and Ξ_t^m. Government debt B_t evolves as

$$dB_t = (rB_t + (\Gamma_t - \mathcal{T}_t))\,dt + dU_t - \Xi_t^h d\mathcal{Z}_t^h - \Xi_t^m\,(\eta dt + d\mathcal{Z}_t^m)\ . \tag{38}$$

Since $\Gamma_t - \mathcal{T}_t$ is the primary deficit and rB_t is an interest payment, the first term on the right side of (38) is government saving. The second term dU_t is the government's lump-sum transfer to households. The third and fourth terms record gains and losses from government holdings of the idiosyncratic and systematic risk-hedging assets. That the household is strictly better offer if the government makes the evolution of $\{B_t\}$ state contingent is captured by the last two terms in (38).

Let $\overline{B}_t$ denote the government's maximum sustainable debt, to be determined in Section 5.2. We shall show that the government's optimal lump-sum transfer policy $\{dU_t\}$ is characterized by an endogenous debt threshold level, $\underline{B}_t$, below which it issues and makes a payout $dU_t > 0$ to the household.

Interior Region ($\underline{B}_t \leqslant B \leqslant \overline{B}_t$). Next, we characterize the optimal policies and value function for the interior region ($B_t \in [\underline{B}_t, \overline{B}_t]$). In this region of the state space, the government sets $dU_t = 0$ and relies exclusively on risk hedging strategies and taxation to shape its state-contingent debt dynamics.

Dynamic programming. The government chooses tax revenue $\mathcal{T}$, idiosyncratic-risk hedging demand Ξ^h, and the systematic risk hedging demand Ξ^m. The optimal value function $P(B, Y)$ solves Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}(\zeta + r)P(B, Y) = \max_{\mathcal{T}, \Xi^h, \Xi^m} \; & (Y - \mathcal{T} - C(\mathcal{T}, Y)) + [rB + \Gamma - \mathcal{T}\,]\, P_B(B, Y) \\ & + \frac{(\Xi^h)^2 + (\Xi^m)^2}{2} P_{BB}(B, Y) + (g - \rho\eta\sigma_Y) Y P_Y(B, Y) \\ & + \frac{\sigma_Y^2 Y^2}{2} P_{YY}(B, Y) - \left(\psi_h \Xi^h + \psi_m \Xi^m\right) Y P_{BY}(B, Y)\,. \end{aligned} \tag{39}$$

The first term on the right side of (39), $(Y - \mathcal{T} - C(\mathcal{T}, Y))$, is the net payment flow to the household. The second and third terms are drift and diffusion volatility effects of increasing debt B on $P(B, Y)$. The fourth and fifth terms express effects of drift and volatility of GDP on $P(B, Y)$. The sixth term captures effects of the intertemporal idiosyncratic and systematic risk hedging demands on $P(B, Y)$.

First-Order conditions. Tax revenue $\mathcal{T}$ satisfies the FOC:

$$1 + C_{\mathcal{T}}(\mathcal{T}, Y) = -P_B(B, Y)\,. \tag{40}$$

It equates the marginal cost of taxing the household, $1 + C_{\mathcal{T}}(\mathcal{T}, Y)$, with the marginal benefit of using taxes to reduce debt, $-P_B(B, Y) > 0$.

As in Merton (1971), systematic risk intertemporal hedging demand Ξ^m satisfies:

$$\Xi^m = \psi_m \frac{Y P_{BY}(B, Y)}{P_{BB}(B, Y)}. \tag{41}$$

Similarly, the FOC for the intertemporal diffusion risk hedging demand is

$$\Xi^h = \psi_h \frac{Y P_{BY}(B, Y)}{P_{BB}(B, Y)}\,. \tag{42}$$

The cross partial derivative P_{BY} that appears in equations (41) and (42) shapes the government's idiosyncratic and systematic risk intertemporal hedging demands. Note the symmetry between (41) and (42).

We can use FOCs (40), (41), and (42) to represent the HJB equation (39) as

$$(\zeta + r)P(B,Y) = \max_{\mathcal{T} \leqslant \overline{\tau} Y} Y - \mathcal{T} - C(\mathcal{T}, Y) + [rB + \Gamma - \mathcal{T}\ ] P_B(B,Y) \\ + \widetilde{g} Y P_Y(B,Y) + \frac{\sigma_Y^2 Y^2}{2} P_{YY}(B,Y) - \frac{\sigma^2 Y^2}{2} \frac{P_{BY}^2(B,Y)}{P_{BB}(B,Y)}, \tag{43}$$

where $\widetilde{g} = g - \rho\eta\sigma_Y$ is a risk-adjusted growth rate.[31] The household's value function $P(B,Y)$ is homogeneous of degree one in B and Y. Consequently the following expression holds:[32]

$$P_{YY}(B,Y) = \frac{P_{BY}^2(B,Y)}{P_{BB}(B,Y)}. \tag{44}$$

Using (44) to simplify (43), we obtain the following first-order partial differential equation:

$$(\zeta + r)P(B,Y) = \max_{\mathcal{T} \leqslant \overline{\tau} Y} (Y - \mathcal{T} - C(\mathcal{T}, Y)) + (rB + \Gamma - \mathcal{T})\, P_B + (g - \rho\eta\sigma_Y) Y P_Y. \tag{45}$$

The first term on the right side of (45) is the flow payoff to the household. The second term captures the effect of fiscal deficit $(rB + \Gamma - \mathcal{T})$ on its value function $P(B,Y)$ and the last term describes the risk-adjusted growth effect of Y on the household's value. Optimality implies that the sum of these three terms equals $(\zeta + r)P(B,Y)$. Access to complete markets lets the government optimally hedge and make its debt be risk free; consequently no diffusion terms associated with P_{BB}, P_{YY}, or P_{BY} appear in (45). Systematic volatility ψ_m of output growth appears in the last term because it influences the household's value via the standard discount rate channel present in the CAPM.

Lump-sum Debt Issuance and Payout Region ($0 \leqslant B_t < \underline{B}_t$).

Next, we turn to a region $0 \leqslant B_t < \underline{B}_t$ where the government issues a lump-sum amount of debt to finance a one-time pay out to the household. In this region, the debt-output ratio $b_t = B_t/Y_t$ is so low that it is optimal for the government immediately to issue debt and pay out the proceeds to the household. The optimal lump-sum transfer policy for a given B_t is

$$dU_t = \max\{\underline{B}_t - B_t, 0\}. \tag{46}$$

[31] Technically, it is the growth rate under the risk-neutral measure $\widetilde{\mathbb{P}}$.

[32] Using the homogeneity property $P(B,Y) = p(b)Y$, we obtain $P_B = f'(b)$, $P_{BB} = p''(b)/Y$, $P_Y = p(b) - p'(b)b$, $P_{YY} = p''(b)bB/Y^2 = p''(b)b^2/Y$, and $P_{BY} = -p''(b)b/Y$. Therefore, we can verify $P_{BB}P_{YY} = (p''(b)b/Y)^2 = P_{BY}^2$.

Equation (46) implies the following value-matching condition when $B_t < \underline{B}_t$:

$$P(B_t, Y_t) = P(\underline{B}_t, Y_t) + (\underline{B}_t - B_t)\,. \tag{47}$$

Rewriting (47) and using the definitions $S(B_t, Y_t) = P(B_t, Y_t) + B_t$ and $S(\underline{B}_t, Y_t) = P(\underline{B}_t, Y_t) + \underline{B}_t$, we find that $S(B_t, Y_t) = S(\underline{B}_t, Y_t)$, so that sums of the household's value and debt value are equated before and after new debt issuances.

By appropriately setting dU_t, the government optimally sets a new debt level $\underline{B}_t \geqslant 0$ that attains

$$\max_{\underline{B} \geqslant 0} \quad S(\underline{B}_t, Y_t) = P(\underline{B}_t, Y_t) + \underline{B}_t\,. \tag{48}$$

If an optimal $\underline{B}_t$ is interior (i.e., if $\underline{B}_t > 0$), it satisfies the FOC:

$$P_B(\underline{B}_t, Y_t) = -1 \quad \text{or equivalently} \;\; S_B(\underline{B}_t, Y_t) = 0\,. \tag{49}$$

Otherwise, the government issues no lump-sum new debt and $\underline{B}_t = 0$.

5.2 Debt Capacity $\overline{B}_t$ and Balanced-budget Regime ($B_t > \overline{B}_t$)

Balanced-budget Regime ($B_t > \overline{B}_t$). When government debt B_t exceeds debt capacity $\overline{B}_t$, the government defaults and permanently enters the balanced-budget regime.[33] The household's value function $P(B_t, Y_t)$ at $B_t > \overline{B}_t$ satisfies

$$P(B_t, Y_t) = \widehat{P}(\widehat{Y}_t)\,, \tag{50}$$

where $\widehat{Y}_t = \alpha Y_t$ and the household's value in the balanced-budget regime $\widehat{P}(\widehat{Y})$ satisfies the differential equation

$$(\zeta + r)\widehat{P}(\widehat{Y}) = \left(\widehat{Y} - \Gamma - \widehat{C}(\Gamma, \widehat{Y})\right) + (g - \rho\eta\sigma_Y)\widehat{Y}\widehat{P}'(\widehat{Y}) + \frac{\sigma_Y^2 \widehat{Y}^2}{2}\widehat{P}''(\widehat{Y})\,. \tag{51}$$

The first term on the right side of (51) is the net payment flow received by the household in the balanced-budget regime. Since the government can neither borrow nor lend in the default regime, tax revenues $\mathcal{T}_t$ must equal government spending Γ_t. The second and third terms capture impacts of the risk-adjusted drift and volatility of output on the household's value function $\widehat{P}(\widehat{Y})$. The balanced-budget regime is absorbing. Here for $t \geqslant T^{\mathcal{D}}$, output

[33] We can generalize our model to allow for the possibility where the government has a probability to exit the balanced-budget regime and return to the no-default regime.

equals $\widehat{Y}_t = \alpha Y_t$ and there is no debt ($B_t = 0$). Let $\widehat{p}_t = \widehat{P}(\widehat{Y}_t)/\widehat{Y}_t$. Later we'll show that $\widehat{p}_t = \widehat{p}$, a constant. To ensure that the value function in the balanced-budget regime is non-negative, we impose:[34]

$$1 - \gamma/\alpha - \kappa c(\gamma/\alpha) \geqslant 0\,. \tag{52}$$

Debt Capacity $\overline{B}$. The government's debt capacity $\overline{B}_t$ is shaped by 1) the government's incentive to renege on its debt, which gives rise to a limited-commitment constraint; and 2) the "Keynesian" tax constraint $\tau \leqslant \overline{\tau}$, where $\overline{\tau}$ is the maximal rate at which the government can tax the household (again see Keynes (1923, pp.56–62) and Keynes (1931).) If the government's default incentive is strong, the limited-commitment constraint binds at its debt capacity. If the government has limited ability to tax output (i.e., when the maximum feasible tax rate on output, $\overline{\tau}$, is relatively low), the tax constraint $\tau \leqslant \overline{\tau}$ binds at debt capacity.

When limited-commitment constraint binds at $\overline{B}_t$. When the government is indifferent between servicing its debt and defaulting, it has reached its debt capacity, $\overline{B}_t$, and the following value-matching condition prevails:

$$P(\overline{B}_t, Y_t) = \widehat{P}(\widehat{Y}_t)\,, \tag{53}$$

where $\widehat{Y}_t = \alpha Y_{t-}$ and $\widehat{P}(\widehat{Y}_t)$ satisfies (51). Counterparts of this condition play key roles in models of Worrall (1990), Kehoe and Levine (1993), and Kocherlakota (1996b).[35]

When tax constraint $\mathcal{T}(B,Y) \leqslant \overline{\tau} Y$ binds at $\overline{B}_t$. When the government's tax constraint $\tau_t \leqslant \overline{\tau}$ binds at debt capacity:

$$\mathcal{T}(\overline{B}_t, Y_t) = \overline{\tau} Y_t\,. \tag{54}$$

Either (53) or (54) holds at debt capacity $\overline{B}_t$. Because $\overline{B}_t$ is a free boundary, we require one more condition to pin it down. After describing some simplifications, we supply this condition in the next subsection.

[34] The value function in the balanced-budget regime is non-negative if and only if the condition $\widehat{Y} - \Gamma - \widehat{C}(\Gamma, \widehat{Y}) \geqslant 0$ holds, which is equivalent to the condition given in (52) after we use the homogeneity property and $\widehat{Y}_t = \alpha Y_t$.

[35] Our approach is related to Bolton, Wang, and Yang (2019) and Rebelo, Wang, and Yang (2021) who incorporate the limited-commitment constraints into corporate finance and international finance in continuous-time models.

5.3 Exploiting Homogeneity

The debt-output ratio b is the state variable. Let $du_t = dU_t/Y_t$ be scaled lump-sum transfer and $\overline{b}_t = \overline{B}_t/Y_t$ be the maximum feasible debt-GDP ratio.

Optimal tax rate $\tau(b)$. Substituting $P(B,Y) = p(b)Y$ into FOC (40) for tax revenue $\mathcal{T}$, we obtain the following simplified FOC for the tax rate $\tau(b)$:[36]

$$1 + c'(\tau(b)) = -p'(b)\,. \tag{55}$$

Since $c''(\,\cdot\,) > 0$, we can invert the marginal tax distortion cost function $c'(\,\cdot\,)$ to obtain the unique tax rate $\tau(b)$ for a given b.

Debt-GDP (b_t) dynamics in the interior region: $b \in [\underline{b}, \overline{b}]$. When the debt-GDP ratio is not too low, i.e., $b \geqslant \underline{b}$, the government presents no lump-sum payments to the household: $du_t = 0$, because the marginal benefit of financing an immediate payout to the household is smaller than the marginal cost of financing debt, including deadweight losses. Using Ito's Lemma, we can show that in this interior region b_t evolves deterministically according to

$$\dot{b}_t \equiv \mu_t^b = \mu^b(b_t) = \underbrace{\gamma - \tau(b_t)}_{\text{primary deficit}} + \underbrace{r \times b_t}_{\text{interest payment}} - \underbrace{g \times b_t}_{\text{growth}} + \underbrace{\lambda \times b_t}_{\text{hedging cost}}\,. \tag{56}$$

The first term on the right side of (56) is the scaled "primary" or net-of-interest fiscal deficit $\gamma - \tau(b)$. The second term is the interest cost of servicing debt. The sum of these two terms is the scaled fiscal deficit, gross of interest payments. The third term is a debt-GDP ratio reduction contributed by output growth. The last term captures the hedging cost due to the risk premium payment, a term that Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) also included in a related but different setting. Although payouts on the government's net debt B_t are risk-free, the λb_t term appears because the source of funds for these payouts is a future primary surplus stochastic process that must be discounted at $r + \lambda$ to account for the costs of hedging transactions that the government undertakes in order to purchase the claims that allow it to make B be risk free.

Debt-GDP ratio limit $\overline{b}$. Drift of the debt-GDP ratio b_t is zero at $\overline{b}$. To see this, note that *ipso facto* b cannot exceed $\overline{b}$, which implies $\mu^b(\overline{b}) \leqslant 0$. Furthermore, with $\zeta \geqslant r$, the government weakly has incentives at the margin to postpone tax burdens, which implies that $\mu^b(\overline{b}) \geqslant 0$. These two inequalities jointly imply that the drift of b at debt capacity is zero so

[36] This condition holds regardless of whether the tax constraint (9) binds or not. The reason is that the tax constraint may bind only at $\overline{b}$. Tax smoothing implies that the FOC (55) holds also at the boundary $\overline{b}$.

that $\mu^b(\bar{b}) = 0$.

Substituting $\mu^b(\bar{b}) = 0$ into (56) yields

$$\bar{b} = \frac{\tau(\bar{b}) - \gamma}{r + \lambda - g}\,. \tag{57}$$

Equation (57) asserts that at the maximum sustainable debt-GDP ratio $\bar{b}$ equals the present value of the primary deficit $(\tau(\bar{b}) - \gamma)$ evaluated at the appropriate discount rate $r+\lambda$, because the optimal primary deficit is risky and bears an insurance premium of λ. Condition (57) fulfills our Section 5.2 promise to pin down the endogenous debt-GDP capacity $\bar{b}$.

Scaled lump-sum debt issuance boundary $\underline{b}$ and payout policy du_t. We can use homogeneity to simplify (48) and verify that the lump-sum debt issuance boundary $\underline{b}$ solves

$$\max_{\underline{b} \geqslant 0} \quad s(\underline{b}) = p(\underline{b}) + \underline{b}\,. \tag{58}$$

If the optimal $\underline{b}$ is interior (i.e., $\underline{b} > 0$), the marginal cost of debt issuance must be zero at $\underline{b}$ so that $s'(\underline{b}) = 0$. Otherwise, the government issues no lumpy debt and $\underline{b} = 0$, since $s'(\underline{b}) < 0$. Thus, an optimal lump-sum transfer policy satisfies

$$du_t = \max\{\underline{b} - b_t, 0\}. \tag{59}$$

Distinct economic forces shape the optimal upper and lower boundaries. The lower boundary $\underline{b}$ is about an optimal lump-sum transfer to households financed by a lump-sum debt issue; it is characterized by smooth-pasting and super-contact conditions. The upper boundary $\bar{b}$ is absorbing and can be approached only from the left. That certifies it as the maximum sustainable level of debt per unit of GDP.

If at $t = 0$ initial government debt were zero and if an optimal $\underline{b} > 0$, a government would immediately issue debt and uses the proceeds to finance a lump-sum payment $dU_0 = \underline{b}Y_0$ to the household, thereby resetting b to equal $\underline{b}$; thereafter b_t stays inside $[\underline{b}, \bar{b}]$ until it reaches the maximum sustainable debt capacity $\bar{b}$.

When the optimal $\underline{b}$ is strictly positive ($\underline{b} > 0$), there is no deadweight cost of debt and the marginal cost of servicing debt, $-p'(\underline{b})$, equals one. This outcome differs from the zero fiscal cost of debt asserted in Blanchard (2019) and Sims (2022). "Debt is cheap" statements like theirs apply when $b < \underline{b}$. Here the government has not borrowed enough and should increase its debt-GDP ratio to $\underline{b} > 0$.

5.4 Optimal Fiscal Plan

Theorem 5.1. *Under restriction (28) that $r + \lambda > g$ as well as the additional restrictions $\kappa \geqslant 1$, $\alpha \leqslant 1$, and the condition $1-\gamma/\alpha-\kappa c(\gamma/\alpha) \geqslant 0$ given in (52), the scaled value function in the no-default regime, $p(b)$, satisfies the first-order nonlinear differential equation:*

$$[\zeta + (r + \lambda - g)]\, p(b) = 1 - \tau(b) - c(\tau(b)) + [(r + \lambda - g)b + \gamma - \tau(b)]\, p'(b)\,, \tag{60}$$

subject to the debt-sustainability condition (57) and one of the following two conditions for the scaled debt capacity $\overline{b}$:

$$p(\overline{b}) \quad = \quad \alpha\widehat{p}\,, \quad \textit{when the tax rate constraint (9) does not bind}\,; \tag{61}$$

$$\tau(\overline{b}) \quad = \quad \overline{\tau}\,, \quad \textit{when the tax rate constraint (9) binds}\,. \tag{62}$$

The scaled value $\widehat{p}$ in the balanced-budget regime is

$$\widehat{p} \quad = \quad \frac{1 - \gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + (r + \lambda - g)}\,. \tag{63}$$

The lump-sum debt issue boundary $\underline{b}$ is described by (58), and the optimal lump-sum transfer policy, du_t, is given by (59). The optimal tax rate policy $\tau(b)$ is given by (55) and the debt-output ratio $\{b_t\}$ evolves deterministically at rate of $\dot{b}_t$ described by (56).

Unlike Mehrotra and Sergeyev (2021) who study debt limits for exogenous tax and debt paths, the debt capacity in our model depends on optimal tax and debt paths. Because we dropped the commitment-to-repay assumption of Barro (1979), our model contains an endogenous debt capacity that turns out to be quantitatively important. Our section 6 calibration shows that debt capacity is much smaller in our model than it would be without the default option. Because our model contains shocks to GDP growth rates, debt-GDP ratio dynamics and debt capacity both depend on a risk premium. The following proposition asserts that the equilibrium debt capacity exists and is unique.

Proposition 5.2. *Under the $r + \lambda > g$ condition given in (28), $\kappa \geqslant 1, \alpha \leqslant 1$, and the condition $1 - \gamma/\alpha - \kappa c(\gamma/\alpha) \geqslant 0$ given in (52), the equilibrium debt capacity $\overline{b}$ is unique and given by*

$$\overline{b} = \min\left\{b^*, \frac{\overline{\tau} - \gamma}{r + \lambda - g}\right\}\,, \tag{64}$$

where b^* is the unique positive root of the following equation

$$1-(r+\lambda-g)b-c((r+\lambda-g)b+\gamma)=\alpha-\alpha\kappa c(\gamma/\alpha)\,. \tag{65}$$

We now report a closed-form optimal plan in for the special **ABC** version of our model that has no extra household impatience: $\zeta=0$. This version of the model includes a complete set of **A**rrow's one-period-ahead securities, **B**arro's tax distortions, and equilibrium **C**redit Constraints.

Lemma 5.3. *When $\zeta=0$, $b_t=b_0$ and the optimal tax rate $\tau(b_t)$ is affine for all t: $\tau(b_t)=\tau(b_0)=(r+\lambda-g)b_0+\gamma$. The scaled value function in the no-default regime, $p(b)$, is constant and given by*

$$p(b_t)=p(b_0)=\frac{1-\tau(b_0)-c(\tau(b_0))}{r+\lambda-g}\,. \tag{66}$$

The scaled value $\widehat{p}$ under autarky is $\widehat{p}=\frac{1-\gamma/\alpha-\kappa c(\gamma/\alpha)}{r+\lambda-g}>0$. There is no lumpy debt issue and hence $\underline{b}=0$. Scaled debt capacity is $\overline{b}=p^{-1}(\alpha\widehat{f})$ when tax rate constraint (9) does not bind. Otherwise, $\overline{b}=\frac{\overline{\tau}-\gamma}{r+\lambda-g}$. We thus obtain

$$\overline{b}=\min\left\{p^{-1}(\alpha\widehat{p}),\frac{\overline{\tau}-\gamma}{r+\lambda-g}\right\}\,. \tag{67}$$

Note that with no extra impatience ($\zeta=0$), the debt-GDP ratio remains constant: $b_t=b_0$ for all t. The optimal plan entails tax smoothing and features constant tax rate over time as in Barro (1979). Moreover, the debt balance, B_t, is volatile and non-stationary: because $B_t=b_0Y_t$, it follows a geometric Brownian motion process with drift μ and volatility σ.

6 Quantitative Illustration

To prepare the way for quantitative illustrations of our model's salient properties, we first describe how we set key parameters.

6.1 Parameters

We set the mean of output growth to $g=2\%$ per annum in line with the estimates in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020). We set the annual risk-free rate r to 1%, the risk premium λ to 3%, and the government spending/output ratio to $\gamma=20\%$, in line with the estimates in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).[37] Our choice of a 3% annual risk premium aligns with an equilibrium consumption CAPM analysis.[38] Consider a Lucas (1978) equilibrium asset pricing model in which the source of aggregate risk is the world stock market and the β of a financial claim on the US aggregate output proposed by Shiller (1994) is between 1/2 and one, which seems plausible in light of sizes of the US stock market and the US economy relative to the world's. With a 6% annual world stock market risk premium and a β of 1/2 for the financial claim on the US output, we obtain a risk premium of $\lambda = 3\%$ for a financial claim on US output.

We set the upper bound for the maximum politically feasible tax rate $\overline{\tau}$ at 50%.[39] As benchmarks, Denmark has the highest average tax-output ratio: 46.3% and the average tax rate in OECD countries is 33.8%. We calibrate $\Omega = \{\zeta, \alpha, \varphi_\tau\}$ from the US debt data[40] from 2000 to 2020 (see Appendix C). The impatience parameter is $\zeta = 0.1\%$ per annum. The output recovery fraction in the balanced-budget regime is $\alpha = 0.94$.

We follow Barro (1979) in using a quadratic deadweight loss function:

$$c(\tau) = \frac{\varphi_\tau}{2}\tau^2\,, \tag{68}$$

where the parameter $\varphi_\tau > 0$ measures the deadweight cost caused by distortionary taxes. With this specification, we obtain a closed form for debt capacity.

Lemma 6.1. *Under the conditions given in Theorem 5.1 and when the deadweight loss function is quadratic so that $c(\tau) = \frac{\varphi_\tau}{2}\tau^2$ as given in (68), debt capacity $\overline{b}$ is*

$$\overline{b} = \min\left\{\frac{\left(\sqrt{1+2\varphi_\tau\left(1-\alpha+\gamma+\varphi_\tau\kappa\gamma^2/\alpha/2\right)}-1\right)/\varphi_\tau-\gamma}{r+\lambda-g}, \frac{\overline{\tau}-\gamma}{r+\lambda-g}\right\}. \tag{69}$$

When tax constraint (9) does not bind, the debt capacity $\overline{b}$ equals the first term in (69). Debt capacity $\overline{b}$ increases with increases in the expected growth rate g, default cost κ, and output loss α; it decreases with increases in tax distortion costs φ_τ, the expected risky asset return $r + \lambda$, the risk free rate r, and the risk premium λ.

We calibrate tax distortion parameter at $\varphi_\tau = 2.8$. To avoid freely choosing another

[37] We do not need to choose the value for output growth volatility σ_Y once we calibrate risk premium λ.

[38] See Kocherlakota (1996a) for a critical review of the early literature on the equity risk premium.

[39] Keynes (1931) guessed .25 for this parameter for France in 1926.

[40] We use our calibration principally to illustrate our model's mechanism via a serious back-of-the-envelope calculation.

Table 1: **Parameter Values.** This table summarizes the parameter values for our baseline quantitative analysis. Whenever applicable, parameter values are continuously compounded and annualized.

Parameter	Symbol	Value
A. Calibration inputs		
risk-free rate	r	1%
risk premium	λ	3%
average output growth rate	g	2%
government spending to output ratio	γ	20%
default deadweight loss	κ	1
B. Calibration outputs		
(relative) impatience	ζ	0.1%
output recovery in the balanced-budget regime	α	0.94
tax deadweight loss	$\varphi_{\mathcal{T}}$	2.8

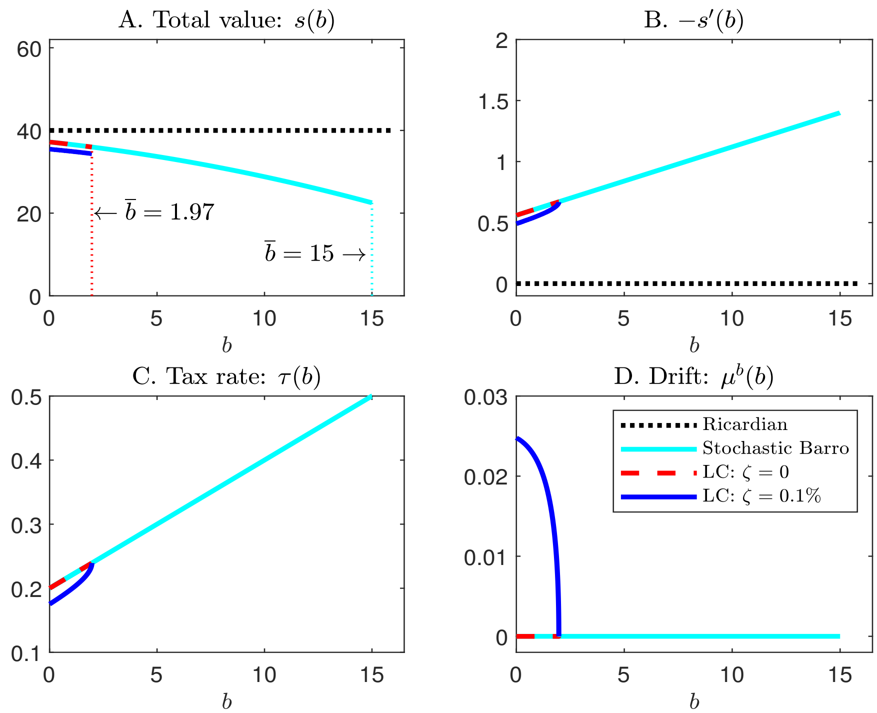
parameter, we set $\kappa = 1$ so that the dead deadweight loss function is the same in the two regions: $c(\,\cdot\,) = \widehat{c}(\,\cdot\,)$. Table 1 summarizes parameter values for our baseline analysis.

6.2 All in One Figure

Figure 1 portrays how outcomes in the interior region of the state space vary as we include or withhold components of our **ABCD** model. The dotted black lines show Ricardian outcomes that prevail in our component-**A**-only model. In this model, the value function is constant and independent of b, the tax rate is indeterminate (and so absent from panel C on the lower left). When we add the **B** distorting taxes component to get a Stochastic Barro model, the drift of debt is constant at zero and the tax rate is constant over time at the value determined by $\tau(b)$ at the initial debt level. When we add limited commitment component **C** but not component **D** by keeping $\zeta = 0$, the only consequence is that maximum sustainable debt $\bar{b}$ drops from its higher value under a Keynes (1931) guess about a maximum tax rate to a much lower value. Thus, notice that for all levels of b up to $\bar{b} = 1.97$, the optimal government plan in our limited-commitment model coincides with that for the Stochastic Barro model that had assumed commitment and $\zeta = 0$; here $\bar{b} = 1.97$ is debt capacity in our limited-commitment model. A notable result from this figure is that the government's debt capacity is reduced by 87% from $\bar{b} = 15$ in the stochastic Barro model to 1.97 in our model.[41] This 87% reduction of debt capacity is attributable solely to the government having the option to default in our model. Note that the drift of debt continues to be zero for all debt levels. It is only when we add the **D** component of our **ABCD** model, namely extra discounting to reflect

[41]The government's debt capacity for the stochastic Barro model equals $\bar{b} = \frac{\overline{\tau}-\gamma}{r+\lambda-g} = \frac{0.5-0.2}{4\%-2\%} = 15$.

Figure 1: **Four Models in One Graph.** Outcomes for the pure **A** model (denoted Ricardian), the stochastic Barro model, a limited commitment model without added impatience ($\zeta = 0$), and a limited commitment model with added impatience ($\zeta = .01$). All parameter values other than ζ are those reported in Table 1. In the stochastic Barro (full-commitment) model, debt capacity is $\bar{b} = 15$ with $\bar{\tau} = 0.5$. In our limited-commitment model, debt capacity is $\bar{b} = 1.97$. Under Ricardian equivalence, an outcome prevails at which $s(b) = s^{FB} = 40$ and $s'(b) = 0$.



extra impatience of the representative household relative to the investors who bequeath the discount factor process Λ_t to the model, that the drift now becomes a decreasing function of b, starting high at $b = 0$ and declining monotonically to 0 at $\bar{b}$.

Figure 1 shows how taxes distort. An undistorted outcome is attained under the special section 3 version of our model that we used to retrieve a Ricardian equivalence outcome. In our model, the total scaled value in this case is $s^{FB} = (1-\gamma)/(r+\lambda-g) = 40$. Under Ricardian equivalence, tax and debt policies are irrelevant and therefore the marginal deadweight cost of debt, $-s'(b) = 0$, is zero for all admissible levels of b (panel B). The gap between the solid blue line (the $s(b)$ solution for the stochastic Barro model) and the horizontal Ricardian (dotted black) line $s(b) = s^{FB} = 40$ increases with b. In the special section 4 stochastic

Barro (1979) version of our model, the marginal deadweight cost of debt increases with b and reaches $-s'(\bar{b}) = 1.40$ at its debt limit $\bar{b} = 15$. To sustain such a high level of debt, the government has to tax output at 50%: $\tau(\bar{b}) = 0.5$.

6.3 Time to Reach Debt Capacity

Our model asserts that a government's debt-output ratio b_t evolves deterministically at rate $\dot{b}_t = \mu^b(b_t)$ described by (56). For a given initial b_0, the time it takes for the government to reach its debt capacity $\bar{b}$ is

$$\int_{b_0}^{\bar{b}} \frac{db_t}{\dot{b}_t} = \int_{b_0}^{\bar{b}} \frac{1}{(r + \lambda - g)b_t + \gamma - \tau(b_t)} db_t. \tag{70}$$

Figure 2: **Time to Reach Debt Capacity as a Function of Impatience ζ.** All other parameter values are reported in Table 1. The initial the debt-GDP ratio is $b_0 = 108.1\%$ and debt capacity is 197%.

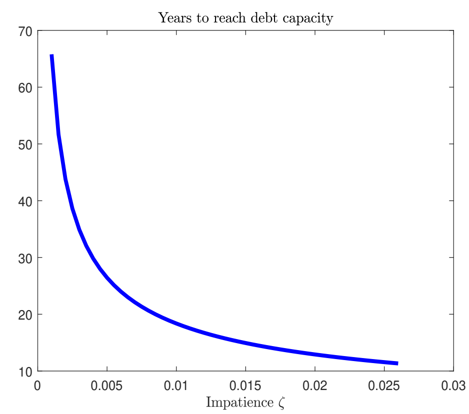
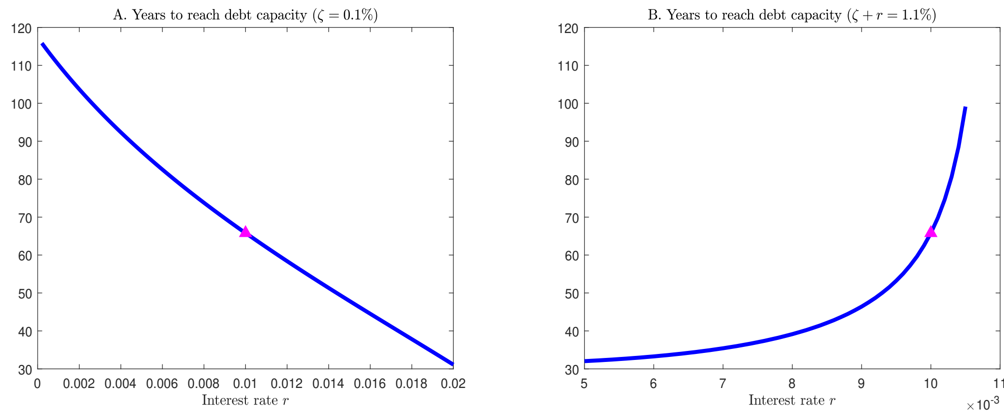


Figure 2 shows that as governments become more impatient across economies (i.e., as ζ increases), the time it takes for the government to exhaust its debt capacity decreases. Even for a seemingly small increase of impatience, effects of impatience are large. In our calculation, starting from the current US debt level of $b = 108\%$, it will take about 66 years to reach the debt limit in 2086 if $\zeta = 0.1\%$, but it would take less than 20 years to reach the debt limit in 2038 if impatience were to increases to $\zeta = 1\%$. If we interpret populism as impatience, these comparative dynamics are consistent with a commonly held view that debt capacity is smaller for a populist government.

Figure 3 plots time it takes for the government to reach its debt capacity as a function of interest rate r. First recall that when facing a lower interest rate, a forward-looking govern-

Figure 3: **Time to Reach Debt Capacity as a Function of Interest Rate r.** For both panels, the initial b is $b_0 = 108.1\%$. In panel A, the impatience parameter is fixed at $\zeta = 0.1\%$. In panel B, the discount rate is fixed at $\zeta + r = 1.1\%$. All other parameter values are reported in Table 1.



ment can finance its debt repayment with a lower tax rate $\tau(b)$, which is less distortionary (a lower marginal cost of debt, $-p'(b)$). As a result, debt is more sustainable, which means a larger debt capacity $\bar{b}$, but the debt-GDP ratio also drifts upward at a faster rate $\dot{b}_t$, *ceteris paribus*. Holding impatience ζ fixed, we see that it takes longer to reach the steady state and exhaust its debt capacity if the interest rate is lower (panel A). This is because the debt capacity force is stronger than the drift effect. Across economies, the level of the interest rate has big consequences. At our parameter settings, starting from the current US debt level of $b = 108\%$, it takes about 87 years to reach the debt limit in 2107 if $r = 0.5\%$, but takes about 66 years to reach the debt limit in 2086 if $r = 1\%$. This pattern is in line with reasoning of Blanchard (2019) and Furman and Summers (2020).

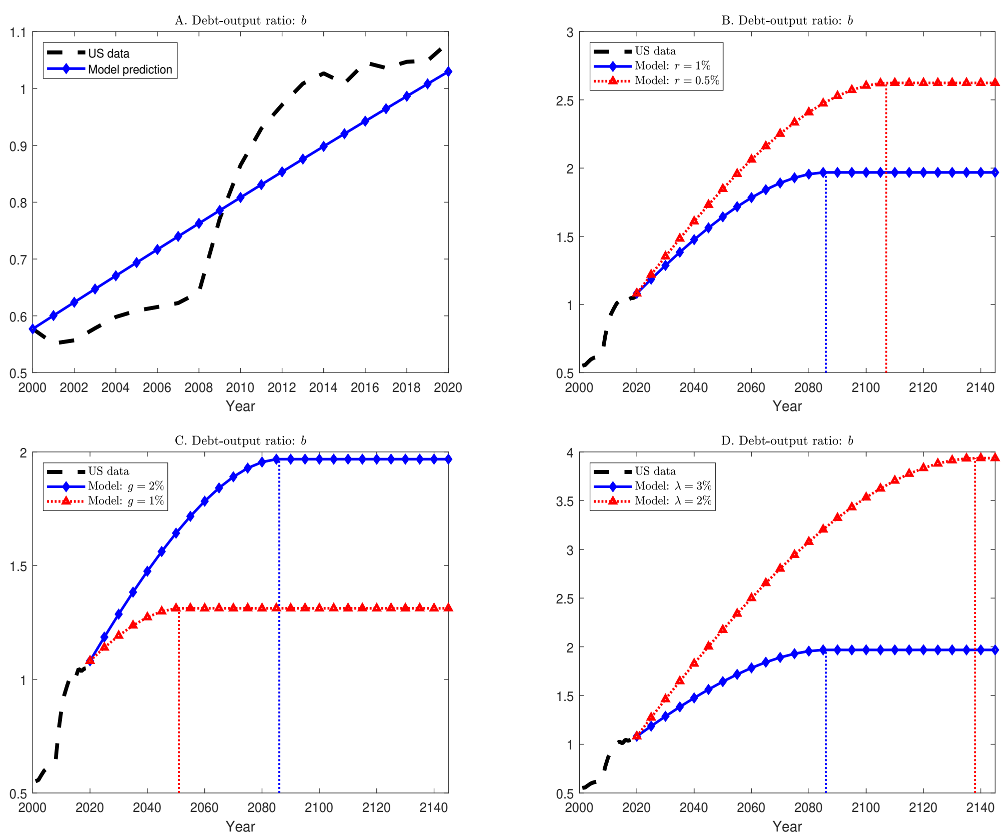
We now perform a distinct calculation that holds a government's discount rate fixed even though we alter the interest rate. We hold a government's discount rate $(\zeta + r)$ fixed and plot time to reach debt capacity as a function of r in panel B of Figure 3. Evidently, it takes less time to reach steady-state debt capacity if interest rate is lower. This is because the drift effect (due to a corresponding increase in impatience ζ) becomes much stronger than the debt capacity effect. For a fixed value of $\zeta + r = 1.1\%$, starting from the current US debt level of $b = 108\%$, it would take about 32 years to reach the debt limit in 2052 if $r = 0.5\%$; but if $r = 1\%$, it would take about 66 years to reach the debt limit in 2086.

A key takeaway from the two panels of Figure 3 is that time to reach the steady-state debt capacity crucially depends on both how impatient the government is and the level of interest rate.

6.4 Quantitative Debt-GDP Ratio Dynamics

Next, we analyze prospective debt-GDP ratio dynamics using our calibrated parameter values. Since we are interested in both the maximum sustainable debt $\bar{b}$ at the optimal steady state and transition dynamics towards $\bar{b}$, we assume that a government can completely hedge its exposures to risks, with the consequence that dynamics of the debt-GDP ratio are deterministic. We have designed our model parsimoniously in a way that can capture a long-run trend and the steady state of debt dynamics.

Figure 4: **Prospective Debt-GDP Ratio Dynamics for Scenarios.** The US debt-output ratios in 2000 and 2020 are 57.5% and 108.1%, respectively. For all model-predicted b processes in panels B, C, and D, the left-end points of the horizontal lines are the corresponding levels of debt capacity $\bar{b}$.



Panel A of Figure 4 plots the implied debt-GDP ratio dynamics from 2000 to 2020 using parameters from our baseline calibration.[42] Our model (the blue solid line) does a good job

[42]Recall that our calibration procedure did not target the debt-GDP ratio dynamics that we plot, which only conditions on the initial condition. Our calibration procedure minimizes the sum of the squared of the difference between one-step-ahead model-predicted b_t and the realized b_t.

of approximating the trend of debt-GDP ratio dynamics $\{b_t\}$ over this 20-year period in the US (the black dashed line). Panels B, C, and D of Figure 4 plot the predicted debt-output ratio $\{b_t\}$ processes starting from 2021 until the government exhausts its debt capacity and reaches the steady state for various scenarios where we change interest rate r, growth rate g, and risk premium λ.

Panel B shows that the government can be expected to reach its debt capacity ($\overline{b} = 1.97$) in 2086 if $r = 1\%$ as we noted earlier. The debt-GDP ratio gradually builds up until reaching the steady state where $\overline{b} = 1.97$ (the solid blue line.) But if the interest rate were unexpectedly and permanently decreases to $r = 0.5\%$, the debt-GDP ratio would increase at a much faster rate, so that a steady state $\overline{b} = 2.62$ (the dotted red line) would be reached in 2107.

Panel C shows that if a government's growth rate permanently drops to 1% from 2%, the government will reach its reduced debt capacity ($\overline{b} = 1.31$ from 1.97) in 2051. This result confirms the intuition that economic growth is a key source of servicing debt.

Panel D shows that if the risk premium λ were unexpectedly and permanently to drop to 2% from 3%, the government's debt capacity would then increase to $\overline{b} = 3.94$ from 1.97; it would take almost 118 years to exhaust its debt limit around 2138. This result shows that the risk premium λ has a very large quantitative effect on both debt capacity and on transition dynamics to a steady state.

Additional quantitative experiments appear in Appendix D.

7 Concluding Remarks and Extensions

To construct streamlined formulas that allow us to isolate salient forces that determine optimal fiscal policy, debt capacity, and debt dynamics, we purposefully chose to work with a complete-markets limited-commitment model with only one aggregate shock. We have neglected other sources of aggregate risks that governments face including stochastic interest rates, a stochastic government spending-GDP ratio γ, and market prices of risk (Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2019). We can extend our model to include such risks by making γ, the risk-free rate r, or GDP growth g an n-state Markov process. These extended models remain tractable and generate richer dynamics of debt, debt capacity, and taxes. They can be used to study various long-run risks that can confront a government. In a sequel, we extend our model to include a quantity theory of money and an inflation tax as an additional source of government revenues. That model allows us to extend an analysis of the interdependence of fiscal and monetary policies provided by Sargent and Wallace (1981).

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Appendices

A Stochastic Barro Model

In this appendix, we compute an optimal fiscal policy for the Section 4 model, which is a stochastic formulation of Barro (1979). This model is a special case of our general model formulation with full commitment and no impatience ($\zeta = 0$). We characterize the household's value function and show that the government's tax policies are time consistent.

To solve the government's optimization problem given by (29) subject to the budget constraint (19), we introduce the following Lagrangian $\mathcal{L}$

$$\begin{aligned}\mathcal{L} \quad = \quad & \max_{\mathcal{T}_t, U_t; t \geqslant 0} \quad \mathbb{E}_0 \int_0^\infty \Lambda_t \left[dU_t + \left(Y_t - \left(\mathcal{T}_t + C_t \right) \right) dt \right] \\ & + \vartheta \left[\mathbb{E}_0 \int_0^\infty \Lambda_t \left(\mathcal{T}_t - \Gamma_t \right) dt - \mathbb{E}_0 \int_0^\infty \Lambda_t dU_t - B_{0-} \right] , \end{aligned} \tag{I-1}$$

where ϑ is the Lagrangian multiplier for the government's budget constraint (19).

The first order condition for the optimal tax rate at time t is given by

$$1 + C_{\mathcal{T}}(\mathcal{T}, Y) = \vartheta \, . \tag{I-2}$$

Using the homogeneity property of the tax deadweight cost function (8) to simplify the FOC (I-2), we obtain $c'(\tau_t^*) = \vartheta - 1$ for the optimal tax rate τ_t^* at any time t. Since ϑ is a constant, the optimal tax rate τ_t^* is constant at all t: $\tau_t^* = \tau^*$ for all t, where τ^* satisfies:

$$c'(\tau^*) = \vartheta - 1 \, . \tag{I-3}$$

The (strict) convexity of the deadweight loss function $c(\tau)$ implies that the Lagrangian multiplier for the government budget constraint is (strict) larger than one: $\vartheta > 1$. Because tax is distortionary and there is no incentive for the government to front load consumption (as $\zeta = 0$), there is no lump-sum transfer at any time t: $dU_t = 0$. (Moreover, the optimal debt target should be zero: $\underline{b} = 0$, if the government were given the option to chooses its initial debt b_0.) We obtain ϑ by using (I-3): $\vartheta = 1 + c'(\tau^*)$. Next, we determine τ^*.

Because the government's budget constraint (19) holds with equality (as $\vartheta > 1$), the present value of primary surplus $\{(\tau^* - \gamma) Y_t; t \geqslant 0\}$, discounted at the rate of $r + \lambda$, the sum of the risk-free rate r and risk premium λ, equals the outstanding debt balance, B_0. This calculation yields the following explicit equation:[43]

$$\tau^* = b_0(r + \lambda - g) + \gamma\,. \tag{I-5}$$

Substituting (I-3) and $dU_t = 0$ into the Lagrangian (I-1) and using the homogeneity property, we obtain the following expression for the value function (also the Lagrangian) under optimal policies:

$$p(b_0) = \mathcal{L} = \frac{1 - \tau^* - c(\tau^*)}{r + \lambda - g}\,, \tag{I-6}$$

where τ^* is given in (I-5). As the budget constraint (19) binds, we only need to calculate the first term in (I-1) under the optimal policies.

Using the tax policy given by (I-5), the government optimally adjusts its debt balance B_t in each step with output Y_t so that the debt-GDP ratio is constant at all $t \geqslant 0$: $b_t = b_0$. The government in the future will follow the same strategy chosen by the time-0 government. Therefore, the government's optimization problem is time consistent (Lucas and Stokey 1983).

Finally, we discuss the maximum sustainable debt under commitment. Suppose that the maximal tax burden that the household is willing to tolerate without triggering a revolution, denoted by $\overline{\mathcal{T}}^*_t$, is the level at which the household's value function is zero. Given the stationarity of our perpetual growth model, the household's net cash flow payoff in each period is zero:

$$Y_t - \overline{\mathcal{T}}^*_t - C(\overline{\mathcal{T}}^*_t, Y_t) = 0\,. \tag{I-7}$$

Let $\overline{B}$ denote the corresponding largest sustainable debt that the government can credibly honor. Then, $\overline{B}$ satisfies the following equation:

$$\overline{B} = \mathbb{E}_0 \int_0^\infty \Lambda_t \left(\overline{\mathcal{T}}^*_t - \Gamma_t\right) dt. \tag{I-8}$$

The maximum sustainable debt-GDP ratio $\overline{b}$ is then given by $\overline{b} = (\overline{\tau}^* - \gamma)\,/\,(r + \lambda - g)$, where $\overline{\tau}^* = \overline{\mathcal{T}}^*_t / Y_t$.

[43]The present value formula is

$$\frac{\tau^* - \gamma}{r + \lambda - g} = \frac{B_0}{Y_0} \equiv b_0 \tag{I-4}$$

under the condition that the tax policy τ^* is feasible.

B Optimal Fiscal Plan for ABCD model

In this appendix, we describe the optimal plan that appeared in Section 5 for the primal dynamic debt management problem defined in Section 2.

HJB equation for $P(B,Y)$. Using Ito's formula, we obtain the following SDF-adjusted dynamics for the household's value function $P(B_t, Y_t)$:

$$d(\Lambda_t P(B_t, Y_t)) = \Lambda_t dP(B_t, Y_t) + P(B_t, Y_t) d\Lambda_t + < d\Lambda_t, dP(B_t, Y_t) >, \tag{I-9}$$

where the SDF $\{\Lambda_t; t \geqslant 0\}$ is given in (18) and

$$\begin{aligned} dP(B_t, Y_t) &= P_B dB_t + \frac{P_{BB}}{2} < dB_t, dB_t > + P_Y dY_t + \frac{P_{YY}}{2} < dY_t, dY_t > + P_{BY} < dB_t, dY_t > \\ &= \left[(rB + (\Gamma - \mathcal{T}) - \Xi^m \eta) P_B + gYP_Y + \frac{\sigma_Y Y^2 P_{YY}}{2} \right] dt \\ &\quad + \left[\frac{\left((\Xi^h)^2 + (\Xi^m)^2\right) P_{BB}}{2} - (\Xi^h \psi_h + \Xi^m \psi_m) Y P_{BY} \right] dt \\ &\quad - P_B(\Xi^h d\mathcal{Z}_t^h + \Xi^m d\mathcal{Z}_t^m) + YP_Y(\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m) \,. \end{aligned} \tag{I-10}$$

Note that the process defined by

$$\int_0^t \left(e^{-\zeta s} \Lambda_s \left(Y_s - \mathcal{T}_s - C(\mathcal{T}_s, Y_s) \right) ds \right) + e^{-\zeta s} \Lambda_s dU_s + e^{-\zeta t} \Lambda_t P(B_t, Y_t)$$

is a martingale under the physical measure $\mathbb{P}$. Therefore, its drift under $\mathbb{P}$ is zero:

$$\mathbb{E}_t \left[d\left(e^{-\zeta t} \Lambda_t P(B_t, Y_t) \right) \right] + e^{-\zeta t} \Lambda_t \left(Y_t - \mathcal{T}_t - C(\mathcal{T}_t, Y_t) \right) dt = 0. \tag{I-11}$$

Note that we have used the result that $dU_t = 0$ in the interior region. Simplifying (I-11) gives the HJB equation (39) for the household's value function $P(B_t, Y_t)$.

We do not repeat the first-order condition (FOC) for the tax rate and other derivations contained in the main body. Below we provide the details for risk management policies.

Shiller macro security allocation ξ^m. Let $\xi_t^m = \Xi_t^m / Y_t$ denote the scaled Shiller macro security allocation. Using the homogeneity property, we show that ξ_t^m is a function of b_t, which we denote by $\xi^m(b_t)$. Simplifying the FOC given in (41) for Ξ^m, we obtain the following expression for $\xi^m(b)$:

$$\xi^m(b) = -\psi_m b \,. \tag{I-12}$$

Idiosyncratic hedging demand ξ^h. Let $\xi_t^h = \Pi_t^h/Y_t$ denote the scaled idiosyncratic risk hedging demand. Similarly, using the homogeneity property, we show that ξ_t^h is a function of b_t, which we denote by $\xi^h(b_t)$. Simplifying the FOC given in (42) for Ξ^h, we obtain the following expression for $\xi_t^h = \xi^h(b_t)$:

$$\xi^h(b) = -\psi_h b \,. \tag{I-13}$$

Debt-GDP ratio b_t dynamics. Applying Ito's lemma to $b_t = B_t/Y_t$, where B_t is given in (38) and Y_t is given in (3), we obtain

$$db_t \;=\; \mu_t^b \, dt + du_t + \sigma_t^{b,h} \, d\mathcal{Z}_t^h + \sigma_t^{b,m} \, d\mathcal{Z}_t^m \,, \tag{I-14}$$

where

$$\mu_t^b \;=\; (r-g)b_t + \gamma - \tau_t - \eta\xi_t^m + \left(\psi_h\xi_t^h + \psi_m\xi_t^m + b_t\sigma_Y^2\right) \tag{I-15}$$

$$\sigma_t^{b,h} \;=\; -\left(\xi_t^h + \psi_h b_t\right) \tag{I-16}$$

$$\sigma_t^{b,m} \;=\; -\left(\xi_t^m + \psi_m b_t\right) \,. \tag{I-17}$$

Substituting hedging policies (I-12) and (I-13) into (I-15), we show that the debt-output ratio, $\{b_t\}$, evolves deterministically at the rate given by:

$$\dot{b}_t = \mu_t^b = \mu^b(b_t) = (r+\lambda-g)b_t + \gamma - \tau(b_t) \tag{I-18}$$

where $\tau(b_t)$ is given by (55).

Equivalent formulation of optimization problem under risk-neutral measure $\widetilde{\mathbb{P}}$. As is standard in macro research, we have formulated the government's optimization problem in Section 2 and provided the solution in Section 5 under the physical measure $\mathbb{P}$. We can equivalently formulate the problem and solve it under the risk-neutral measure $\widetilde{\mathbb{P}}$. Recall that under the physical measure $\mathbb{P}$, the Brownian motions for idiosyncratic shock and systemic shock are given by $\mathcal{Z}_t^h$ and $d\mathcal{Z}_t^m$, respectively. Because the shock to the market portfolio is systematic with a constant Sharpe ratio of η, using the standard Black-Merton-Scholes dynamic replication argument, we can show that the Brownian motion for systemic shock under the risk-neutral measure $\widetilde{\mathbb{P}}$, denoted by $\widetilde{\mathcal{Z}}_t^m$, is given by

$$d\widetilde{\mathcal{Z}}_t^m = d\mathcal{Z}_t^m + \eta dt \,. \tag{I-19}$$

This equation is also the reason why a well-diversified investor who holds a long position in the market futures contract demands a positive payment at the rate of ηdt to break even. This explains the last term in the law of motion (38) for B_t. The Brownian motion for the idiosyncratic shock under the risk-neutral measure $\widetilde{\mathbb{P}}$ is the same as that under the physical measure $\mathbb{P}$:

$$d\widetilde{\mathcal{Z}}_t^h = d\mathcal{Z}_t^h\,, \tag{I-20}$$

as there is no risk premium.

Using (I-19) amd (I-20) under the risk-neutral measure, we may express the output process (3) under the risk-neutral measure $\widetilde{\mathbb{P}}$ as follows:

$$\frac{dY_t}{Y_t} \quad = \quad \widetilde{g}dt + \sigma_Y\left(\sqrt{1-\rho^2}d\widetilde{\mathcal{Z}}_t^h + \rho d\widetilde{\mathcal{Z}}_t^m\right)\,, \tag{I-21}$$

where $\widetilde{g}$ is the average output growth rate under the risk-neutral measure $\widetilde{\mathbb{P}}$:

$$\widetilde{g} = g - \rho\sigma_Y\eta\,. \tag{I-22}$$

In the interior region where $dU_t = 0$, we may equivalently express the government's optimization problem under the risk-neutral measure $\widetilde{\mathbb{P}}$ as follows:

$$\max_{\mathcal{T}_t\leqslant\overline{\tau}Y_t,\Xi_t^h,\Xi_t^m} \widetilde{\mathbb{E}}_0\left[\int_0^\infty e^{-(\zeta+r)t}\left(\left(Y_t-\mathcal{T}_t-C(\mathcal{T}_t,Y_t)\right)\left(1-\mathbf{1}_t^{\mathcal{D}}\right)+\left(\widehat{Y}_t-\widehat{\mathcal{T}}_t-\widehat{C}(\widehat{\mathcal{T}}_t,\widehat{Y}_t)\right)\mathbf{1}_t^{\mathcal{D}}\right)dt\right]\,, \tag{I-23}$$

subject to the government's tax constraint $\mathcal{T}_t \leqslant \overline{\tau}Y_t$ and the budget constraint:

$$B_t = \widetilde{\mathbb{E}}_t\left[\int_t^{T^D} e^{-r(s-t)}\left(\mathcal{T}_s - \Gamma_s\right)ds\right]\,. \tag{I-24}$$

Note that the budget constraint (I-24) is under the risk-neutral measure $\widetilde{\mathbb{P}}$.

Equation (I-24) implies that $e^{-rt}B_t + \int_0^t e^{-rs}\left(\mathcal{T}_s - \Gamma_s\right)ds$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$. Using the marginal representation theorem, we can equivalently express debt dynamics under the risk-neutral measure $\widetilde{\mathbb{P}}$ as:

$$dB_t = \left(rB_t + \left(\Gamma_t - \mathcal{T}_t\right)\right)dt - \Xi_t^h d\widetilde{\mathcal{Z}}_t^h - \Xi_t^m d\widetilde{\mathcal{Z}}_t^m\,. \tag{I-25}$$

Using (I-23), (I-25), and (I-21) in the interior region, we use the following HJB equation to solve the household's value function$P(B,Y)$:

$$
\begin{aligned}
(\zeta+r)P(B,Y) = \max_{\mathcal{T}\leqslant\overline{\tau}Y,\,\Xi^h,\,\Xi^m} \quad & Y-\mathcal{T}-C(\mathcal{T},Y)+\left[rB+\Gamma-\mathcal{T}\,\right]P_B(B,Y) \\
& +(g-\rho\eta\sigma_Y)YP_Y(B,Y)+\frac{(\Xi^h)^2+(\Xi^m)^2}{2}P_{BB}(B,Y) \\
& +\frac{\sigma_Y^2Y^2}{2}P_{YY}(B,Y)-\left(\psi_h\Xi^h+\psi_m\Xi^m\right)YP_{BY}(B,Y)\,.
\end{aligned} \tag{I-26}
$$

Existence and uniqueness of equilibrium debt capacity. In Proposition 5.2, we show that under the $\kappa\geqslant1$ and $\alpha\leqslant1$ conditions, there exists a unique positive debt capacity $\overline{b}>0$. Furthermore, when the tax constraint (9) does not bind, there exists a unique $\overline{b}>0$ where $p(\overline{b})=\alpha\widehat{p}$. When taxes are more distortionary ($\kappa\geqslant1$) under the balanced-budget regime or when default causes output losses ($\alpha\leqslant1$), the government is always better off not defaulting and instead prudently managing risk exposures and debt dynamics to avoid default.

Proof of proposition 5.2. Equations (60) and (57) imply

$$
p(\overline{b})=\frac{1-\tau(\overline{b})-c(\tau(\overline{b}))}{\zeta+r+\lambda-g}\,, \tag{I-27}
$$

where $\tau(\overline{b})=(r+\lambda-g)\overline{b}+\gamma$. The debt capacity $\overline{b}$ solves one of the following two equations

$$
p(\overline{b}) = \alpha\widehat{p}\,,\quad \text{when the tax rate constraint (9) does not bind}\,; \tag{I-28}
$$

$$
\tau(\overline{b})=\overline{\tau}\,,\quad \text{when the tax rate constraint (9) binds}\,. \tag{I-29}
$$

If tax constraint (9) binds, the equilibrium debt capacity $\overline{b}$ is the unique solution of (I-29): $\overline{b}=\frac{\overline{\tau}-\gamma}{r+\lambda-g}$. If tax constraint (9) does not bind, we can show that the equilibrium debt capacity, which is the solution of (I-28), exists and is also unique. First, (I-27) implies that the left side of (I-28) is decreasing $\overline{b}$. Second, the left side of (I-28) when $\overline{b}=0$ equals $1-\gamma-c(\gamma)$, which is strictly larger than the right side of (I-28), given that the deadweight loss function $c(\,\cdot\,)$ is increasing and convex (in addition to the $\kappa\geqslant1$ and $\alpha\leqslant1$ conditions). Third, the left side of (I-28) approaches negative infinity as $\overline{b}\to\infty$. Therefore, there exists a unique value of $\overline{b}>0$ where (I-28) holds with equality. This unique value of $\overline{b}>0$ solves (65). Thus, an equilibrium debt capacity exists and is uniquely determined by

$$
\overline{b}=\min\left\{b^*,\frac{\overline{\tau}-\gamma}{r+\lambda-g}\right\}\,.
$$

$\square$

Extension: finite balanced-budget regime duration. Our baseline model assumes that the government stays in the balanced-budget regime forever after reneging on its liability. As typical in the sovereign-debt literature, we generalize our baseline model by allowing the government to regain access to international capital markets with probability χ per unit of time. Let T^ϵ denote the government's stochastic exogenous exit time from the balanced-budget regime. Upon exiting it at T^ϵ and returning to the no-default regime, the household's value function is $P(0, Y_{T^\epsilon})$, where output is continuous at T^ϵ, which means $Y_{T^\epsilon} = \widehat{Y}_{T^\epsilon}$. The household's value function in the balanced-budget regime $\widehat{P}(\widehat{Y})$ therefore satisfies

$$(\zeta + r + \chi)\widehat{P}(\widehat{Y}) = \widehat{Y} - \Gamma - \widehat{C}(\Gamma, \widehat{Y}) + (g - \rho\eta\sigma_Y)\widehat{Y}\widehat{P}'(\widehat{Y}) + \frac{\sigma_Y^2 \widehat{Y}^2}{2}\widehat{P}''(\widehat{Y}) + \chi P(0, \widehat{Y}) \,. \quad \text{(I-30)}$$

The scaled value in the balanced-budget regime, $\widehat{p}$, is then given by

$$\widehat{p} \quad = \quad \frac{1 - \gamma/\alpha - \kappa c(\gamma/\alpha) + \chi p(0)}{\zeta + (r + \lambda - g) + \chi} \,. \quad \text{(I-31)}$$

C Calibration

We use the US annual debt-output ratio from 2000 to 2020 to estimate our model. US debt and GDP data are from FRED provided by St. Louis fed: https://fred.stlouisfed.org.

Let $\Omega = \{\varphi, \zeta, \kappa\}$. Our model asserts that the government debt-GDP ratio b_t grows deterministically at rate $\dot{b}_t \equiv \mu^b(b_t)$ given in (56). Let $\mu^b(b_t; \Omega)$ denote the drift of b given Ω. To account for measurement errors, we introduce a noise term into the law of motion (56) for b_t and discretize the b_t process as follows:

$$b_{t_{i+1}} = b_{t_i} + \mu^b(b_{t_i}; \Omega)(t_{i+1} - t_i) + \varepsilon_{i+1} \,, \quad i = 1, 2, \cdots \,, \quad \text{(I-32)}$$

where ε_{i+1} is a random variable that captures the effect of measurement errors. Let $h(\varepsilon_{i+1})$ denote the density function of ε_{i+1}:

$$h\left(b_{t_{i+1}} - b_{t_i} + \mu^b(b_{t_i}; \Omega)(t_{i+1} - t_i)\right) \,. \quad \text{(I-33)}$$

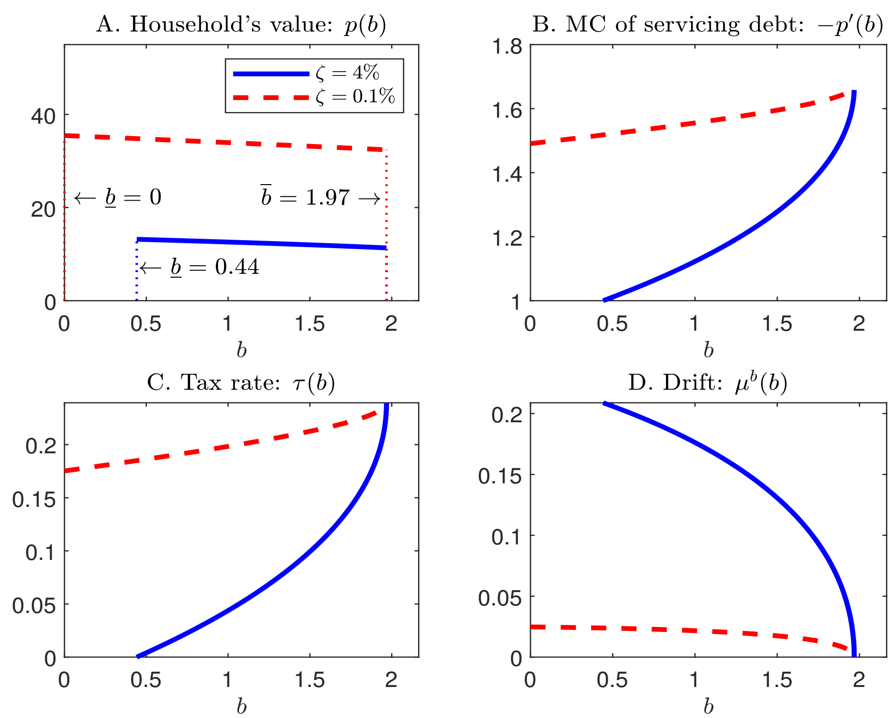
Let $\{\widehat{b}_{t_i}, i = 1, \cdots, 21\}$, where $t_i = 1999 + i$, denote the annual US debt-to-GDP ratio from

2000 to 2020. Our estimate of Ω is

$$\widehat{\Omega} = \arg\max_{\Omega} \quad \sum_{i=1}^{20} \ln h\left(\widehat{b}_{t_{i+1}} - \widehat{b}_{t_i} + \mu^b(\widehat{b}_{t_i};\Omega)\right) . \tag{I-34}$$

D Quantitative Comparative Dynamics

Figure 5: **Effects of Impatience ζ.** All parameter values other than ζ are reported in Table 1.



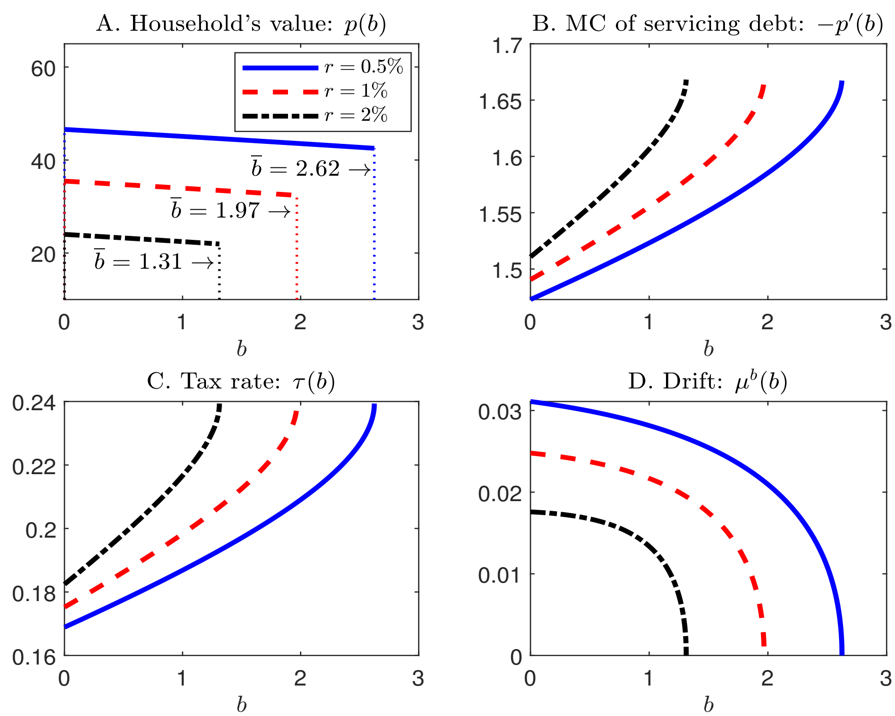
In this appendix we perturb some parameters one at a time and display effects on outcomes.

Effects of impatience ζ. A larger parameter ζ indicates more primal government's impatience. It introduces a wedge in first-order conditions that has quantitatively important effects on taxes and value functions. Figure 5 compares outcomes in our baseline ($\zeta = 0.1\%$) case with those from a $\zeta = 4\%$ case in which the government is much more impatient.

As ζ increases from 0.1% to 4%, the total value $p(b)$ decreases by about two thirds at all admissible levels of b (panel A.) This outcome emerges mostly from a typical discounting channel. More interesting to us is that the marginal cost of debt ($-p'(b)$) and the optimal tax rate ($\tau(b)$) both decrease substantially for most values of b (panels B and C). This happens because it is much less costly for the government to defer taxation. As a result, the marginal cost of debt ($-p'(b)$) at $b = 0.44$ is one when $\zeta = 4\%$ but equals 1.52 dollars in our baseline $\zeta = 0.1\%$ case. The optimal tax rate ($\tau(b)$) at $b = 0.44$ is zero when $\zeta = 4\%$ but equals 18% in our baseline $\zeta = 0.1\%$ case.

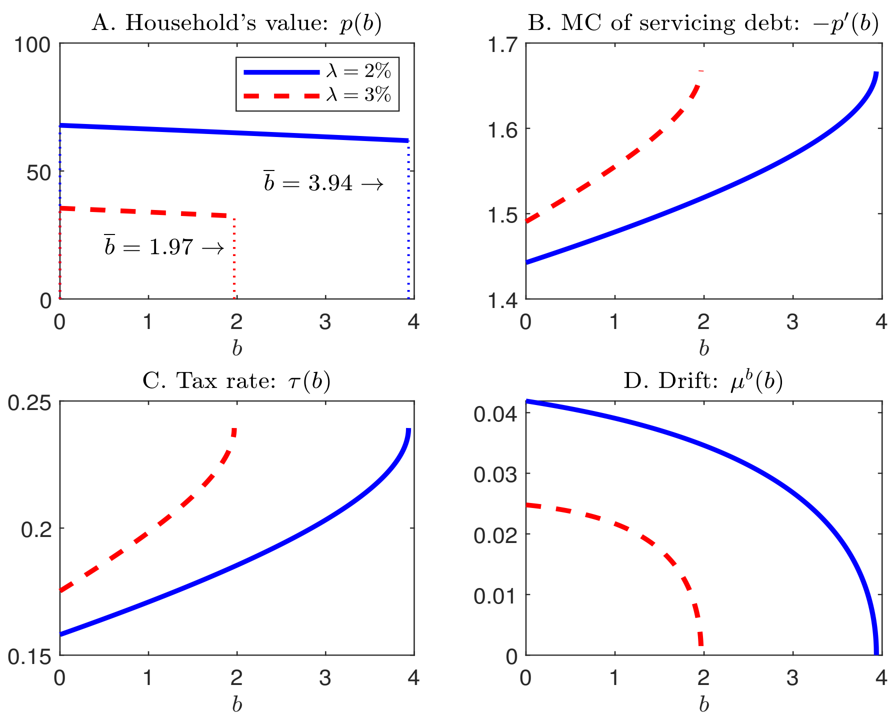
For both cases, as b increases, the tax rate $\tau(b)$ and the marginal cost of debt increase until debt has reached debt capacity $\bar{b} = 1.97$. While increasing ζ does not change the government's debt capacity, it does substantially increase the drift of the debt-GDP ratio $\mu^b(b)$, which in turn changes the time it takes for a government to reach its debt capacity, as we describe in Section 6.3.

Figure 6: **Effects of Interest Rate** r**.** All parameter values other than r are reported in Table 1.



Effects of risk-free rate r**.** Figure 6 compares outcomes in our baseline ($r = 1\%$) case with those in an $r = 0.5\%$ case. When r decreases across economies from 1% to 0.5%, a government's debt capacity $\bar{b}$ increases substantially from 1.97 to 2.62. Importantly, both the marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ decrease substantially for the lower r economy. Because interest payments are smaller, debt burden is smaller and tax distortions are also smaller. As a result, a government is more willing to borrow causing the drift of the debt-GDP ratio $\mu^b(b)$ to increase as r falls for all levels of b (panel D).

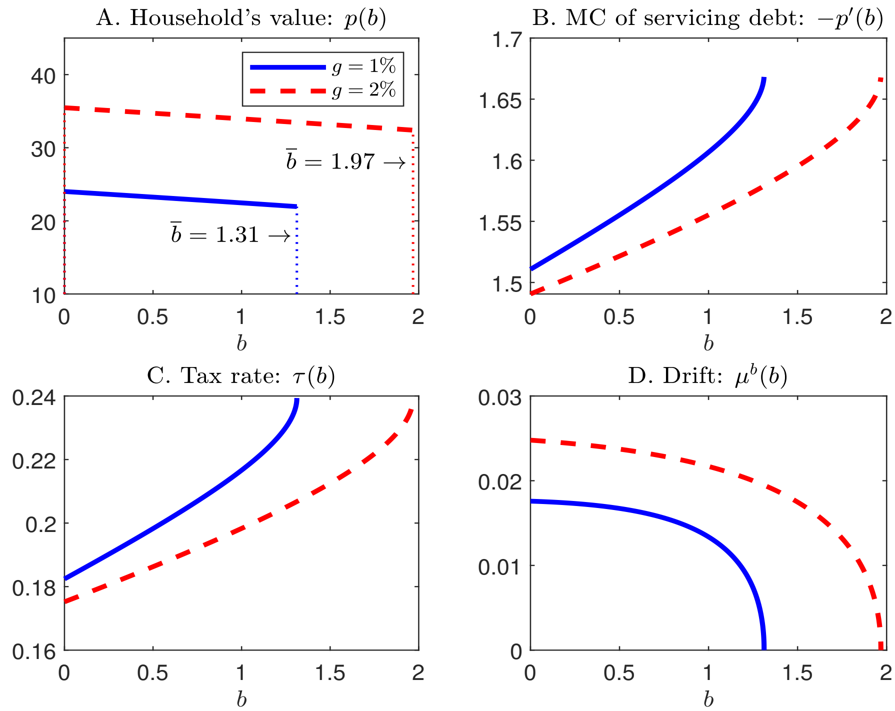
Figure 7: **Effects of Risk Premium λ.** All parameter values other than λ are reported in Table 1.



Effects of risk premium λ. Figure 7 compares outcome under our baseline ($\lambda = 3\%$) case with those of a $\lambda = 2\%$ case. When across economies λ decreases from 3% to 2%, a government's debt capacity $\bar{b}$ doubles from 1.97 to 3.94. Importantly, both the marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ decrease markedly as the risk premium λ falls. Because systematic risk management costs are smaller, the debt burden and tax distortions are smaller. As a result, a government is more willing to borrow causing the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to increase as risk premium falls for all levels of b (panel D).

Effects of Output Growth Rate g. Figure 8 compares outcomes under our baseline ($g = 2\%$) case with those from a $g = 1\%$ economy. When the growth rate across economies decreases from 2% to 1%, a government's debt capacity $\bar{b}$ decreases by about one third

Figure 8: **Effect of Average Output Growth Rate g.** All parameter values other than g are reported in Table 1.

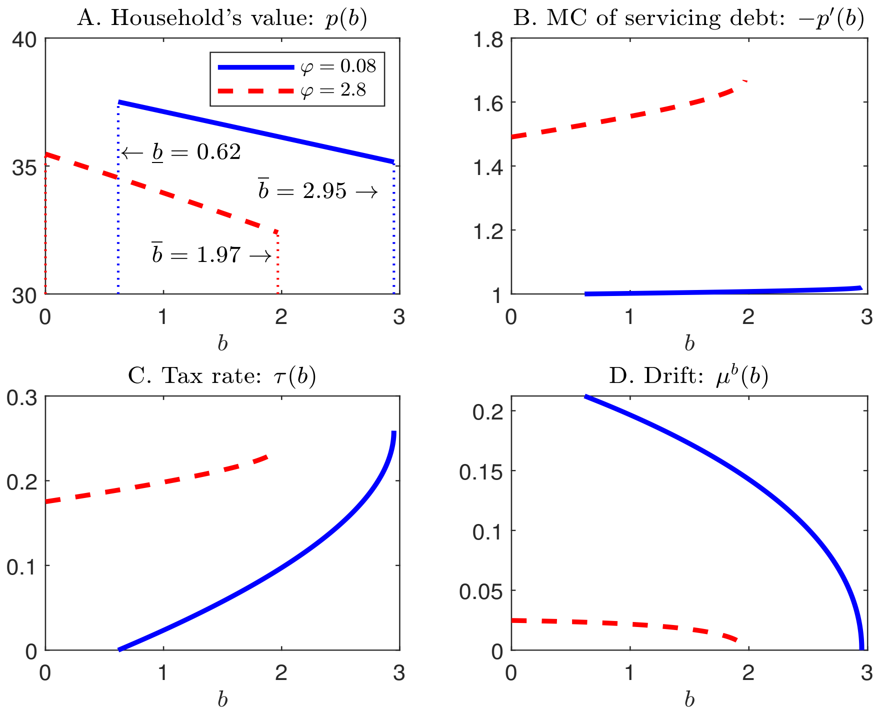


from 1.97 to 1.31. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase substantially as the growth rate falls from 2% to 1%. With slower growth, a government is less willing to borrow against the future, causing drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to fall for all levels of b (panel D). That government response has important implications about the time it takes for a government to reach its debt limit.

Effects of tax distortion cost φ. The parameter φ governs tax distortions in the deadweight loss function $c(\cdot)$. Figure 9 compares outcomes under our baseline ($\varphi = 2.8$) case with those from a $\varphi = 0.08$ case. When φ decreases from 2.8 to 0.08, a government's debt capacity $\bar{b}$ increases a little from 1.97 to 2.95 and the household's value function $p(b)$ increases. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both decrease. When taxes are less distortionary, a government is more willing to borrow against the future, causing lump-sum debt issuance threshold $\underline{b}$ to increase from 0 to 0.62 (panel A), and drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b)$ to increase at all levels of b (panel D).

Effects of default costs: (increasing tax distortion costs $\kappa \geqslant 1$). The parameter

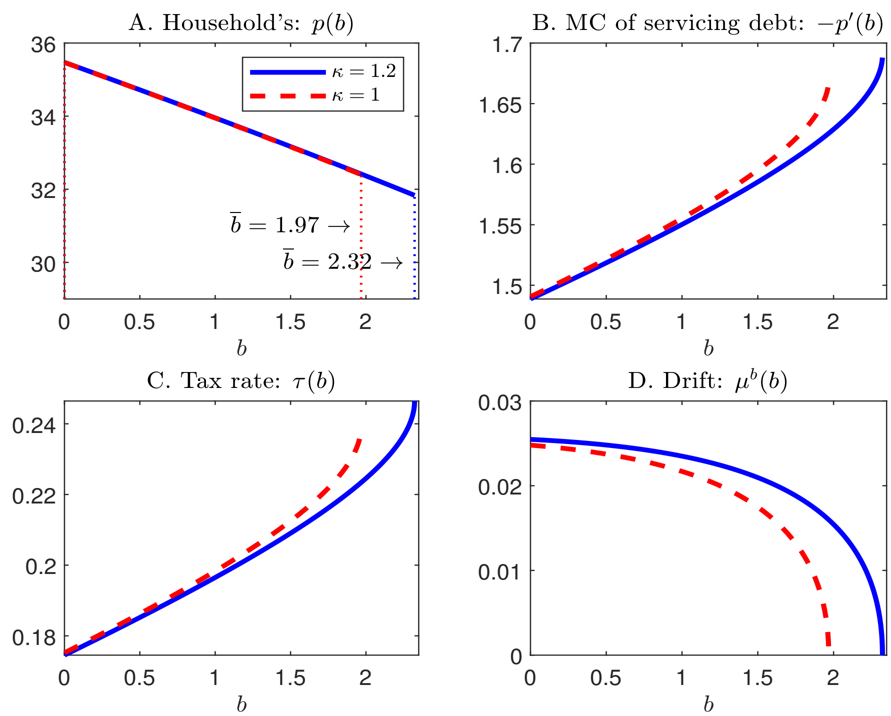
Figure 9: **Effect of Tax Distortion Cost φ.** All parameter values other than φ are reported in Table 1.



κ measures how much more distortionary taxes are in the balanced-budget regime than in the service-debt regime. Figure 10 compares outcomes under our baseline ($\kappa = 1$) case with those under a $\kappa = 1.2$ case. When across economies κ increases from 1 to 1.2, a government's debt capacity $\overline{b}$ *increases* from 1.97 to 2.32 and the household's value function $p(b)$ increases slightly. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both decrease. That is because when default is more costly, a government is more willing to repay debt, allowing it to borrow more. As κ increases across economies, the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ is higher for all levels of b (panel D).

Effects of default costs: output loss $(1-\alpha)$. The parameter α measures the recovery of output in the default regime. Figure 11 compares outcomes under our baseline ($\alpha = 0.94$) case with those under an $\alpha = 0.9$ case. When across economies output loss $(1-\alpha)$ increases from 6% to 10%, a government's debt capacity $\overline{b}$ *increases* markedly from 1.97 to 3.22, but the household's value function $p(b)$ increases only slightly. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both decrease. This is because when default is more costly, the

Figure 10: **Effects of Default Costs: (Increasing Tax Distortion Costs $\kappa \geqslant 1$).** All parameter values other than κ are reported in Table 1.

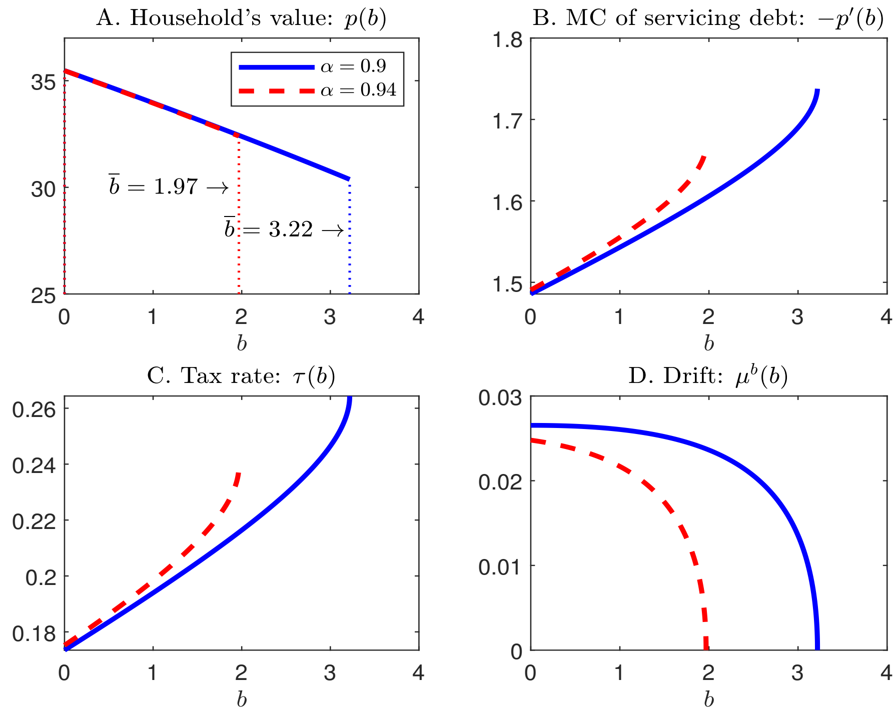


government is more willing to repay debt and hence is able to borrow more. Finally, the drift of the debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ is higher as we increase output loss $(1 - \alpha)$ for all levels of b (panel D).

Our comparative static results with respect to $(1-\alpha)$ and κ are similar because increasing $(1 - \alpha)$ directionally has the same effect as increasing κ. Both make default more costly, which in turn improves incentives to repay and therefore debt capacity.

Effects of government spending-GDP ratio γ. The parameter γ measures government spending as a fraction of output. Figure 12 compares outcomes under our baseline ($\gamma = 0.2$) case with those under a $\gamma = 0.3$ case. When across economies government spending γ increases from 0.2 to 0.3, a government's debt capacity $\bar{b}$ *decreases* slightly from 1.97 to 1.80, but the household's value function $p(b)$ decreases markedly. The marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase substantially. That is because when the government spending fraction is higher, a household's value in the balanced-budget regime becomes lower. Hence, a government is more willing to tax more in order to repay its debt.

Figure 11: **Effect of Default Costs: Output Recovery α.** All parameter values other than α are reported in Table 1.



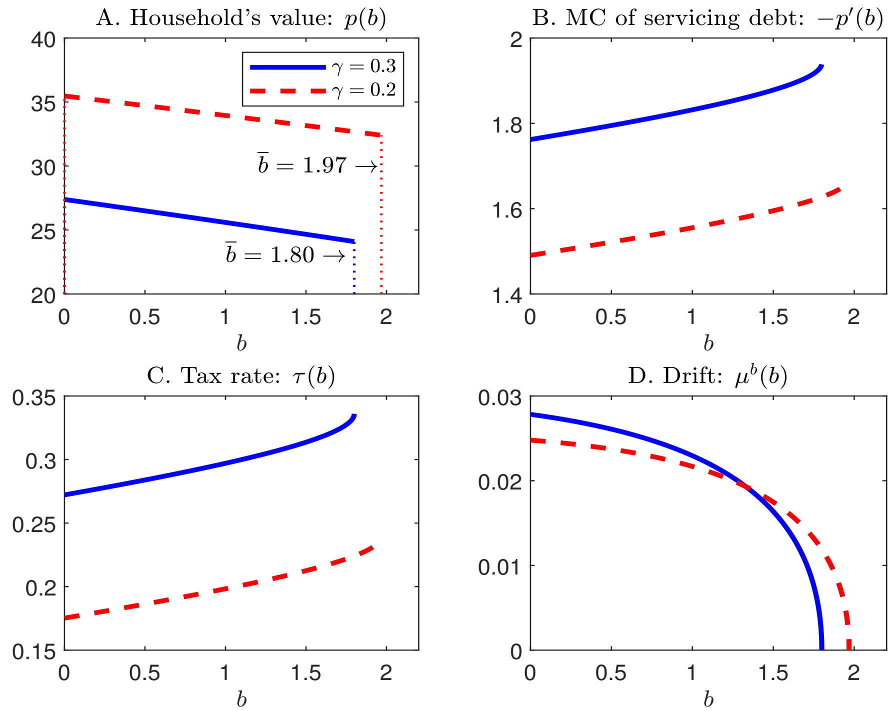
That enables it to borrow more.

Effects of expected balanced-budget-regime duration $1/\chi$**.** In our baseline Section 2 model, the government permanently stays in the balanced-budget regime once it enters. In reality, a sovereign after defaulting on its debt stochastically regains its access to capital markets. To capture a finite stochastic duration of staying in the balanced-budget regime, we assume that a government exits it at a constant (annual) rate, denoted by χ, following the sovereign-debt literature.[44] We set $\chi = 1/5$ per annum as a sovereign after default on average stays in the balanced-budget regime for four or five years (e.g., see the estimate in Aguiar and Gopinath, 2006). In Figure 13, we compare this $\chi = 0.2$ case with our baseline $\chi = 0$ case in which the balanced-budget regime is an absorbing state.

As we decrease the expected duration of being in the balanced-budget regime $1/\chi$ from ∞ to five years, the equilibrium debt capacity $\overline{b}$ decreases from 1.97 to 1.43 while the marginal cost of debt $-p'(b)$ and the tax rate $\tau(b)$ both increase. With a lower debt capacity (for the

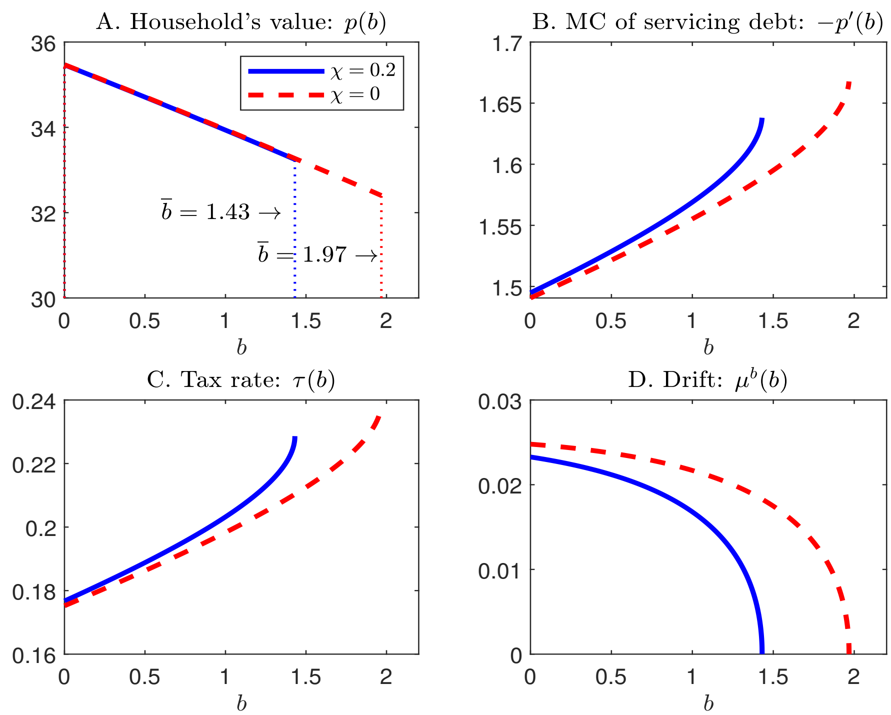
[44] In Appendix B, we provide technical details.

Figure 12: **Effect of Government Spending γ.** All parameter values other than γ are reported in Table 1.



$\chi = 0.2$ case), the government has less room to smooth taxes and hence has to tax more in order to honor its debt. Because higher taxes cause more distortions, the government's marginal cost of debt is higher. As a result of higher taxes, the government pays back its debt at a faster rate (for all admissible levels of b) causing the drift of its debt-GDP ratio $\dot{b}_t = \mu^b(b_t)$ to be lower for the $\chi = 0.2$ case than for our baseline $\chi = 0$ case (panel D).

Figure 13: **Effect of intensity to exit balanced-budget regime χ.** All parameter values other than χ are reported in Table 1.



Internet Appendix to
"A p Theory of Government Debt and Taxes"

Wei, Jiang, Thomas J. Sargent, Neng Wang, and Jinqiang Yang

I.A Dual

The dual formulation of our primal problem in Section 2 **ABCD** model describes a government that is devoted to maximizing the present value of its fiscal surplus stream. The dual can be interpreted as an abstract "tax-farmer" problem. We show that taxes and transfer payments for this dual problem are identical to those that emerge from the solution in Section 5 for the primal problem. In the dual problem, the government maximizes the present value of its future primary surpluses subject to a sequence of participation constraints that induce the representative household to consent to the government's fiscal plan.

I.A.1 Government's Value and Household's Promised Value

The dual government chooses a tax revenue process $\{\mathcal{T}_t; t \geqslant 0\}$ that provides a smooth flow payment $(Y_t - \mathcal{T}_t - C_t)$ and a cumulative payment process $\{J_t; t \geqslant 0\}$ to the impatient representative household. Optimal policies, $\{\mathcal{T}_t; t \geqslant 0\}$ and $\{J_t; t \geqslant 0\}$, depend on histories of idiosyncratic and systematic shocks $\{\mathcal{Z}_t^h, \mathcal{Z}_t^m; t \geqslant 0\}$. The maximum feasible tax rate that the planner can impose on the output process is $\overline{\tau} Y_t$ for all $t \geqslant 0$, i.e., $\mathcal{T}_t \leqslant \overline{\tau} Y_t$, the same as the constraint (9) that appeared in our primal dynamic debt management problem.

The planner maximizes the risk-adjusted present value of $(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s$, the difference between the government's primary surplus $((\mathcal{T}_s - \Gamma_s)\, ds)$ and its distribution to the household (dJ_s), at time 0. Let F_t denote the planner's optimal value function at time t:

$$F_t = \max \quad \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} \frac{\Lambda_s}{\Lambda_t} \left[(\mathcal{T}_s - \Gamma_s)\, ds - dJ_s \right] \right] . \tag{I.1}$$

We'll soon indicate the arguments with respect to which F_t is a maximum. We adopt an assumption like Green (1987), Phelan and Townsend (1991), and Atkeson (1991) that the planner is risk-neutral or has access to complete insurance markets. The same unique SDF Λ described by process (18) prevails as did in our section Section 5 primal problem. The same "small open economy" assumption rationalizes the exogeneity of that process. We assume that there is zero continuation value for the planner after $T^{\mathcal{D}}$. This assumption corresponds to our earlier assumption of no debt recovery upon default in the debt management problem.

Household's promised value $\{W_t; t \geqslant 0\}$. The dual problem uses the household's promised value, denoted by $\{W_t; t \geqslant 0\}$, as the key state variable.[1] The household's promised value at time t, W_t, equals the present value of all future payments:

$$W_t = \mathbb{E}_t \int_t^\infty e^{-\zeta(s-t)} \frac{\Lambda_s}{\Lambda_t} \left(dJ_s + (Y_s - (\mathcal{T}_s + C_s)) \left(1 - \mathbf{1}_s^{\mathcal{D}}\right) ds + [\widehat{Y}_s - (\widehat{\mathcal{T}}_s + \widehat{C}_s)] \mathbf{1}_s^{\mathcal{D}} ds \right). \quad \text{(I.2)}$$

Using the Martingale Representation Theorem, without loss of generality, we can represent the dynamics of $\{W_t; t \geqslant 0\}$ as:

$$dW_t = [(\zeta + r)W_t - (Y_t - \mathcal{T}_t - C_t) - \eta \Phi_t^m]\, dt - dJ_t - \Phi_t^h d\mathcal{Z}_t^h - \Phi_t^m d\mathcal{Z}_t^m\,. \quad \text{(I.3)}$$

The planner chooses $\{\Phi_t^h; t \geqslant 0\}$ and $\{\Phi_t^m; t \geqslant 0\}$, exposures of the household's promised value $\{W_t; t \geqslant 0\}$ to idiosyncratic and systematic risks, respectively.[2]

The dual government must respect a sequence of constraints that require the household to choose to continue to participate. Let $\underline{W}_t = \underline{W}(Y_t)$ denote the minimal threshold for the household's promised value W_t at which the household is willing to participate. Participation constraints are:

$$W_t \geqslant \underline{W}(Y_t), \quad t \geqslant 0\,. \quad \text{(I.4)}$$

We will determine $\underline{W}(Y_t)$ soon.

Next, we turn to the dual planner's choice of a lump-sum payout to the household and an associated upper boundary for W. There is a cost of deferring payments because the household is impatient ($\zeta \geqslant 0$) relative to the dual government planner. Deferring payments to the household increases W_t, which relaxes the participation constraint. This suggests an endogenous threshold level, $\overline{W}_t = \overline{W}(Y_t)$, above which it is optimal for the planner to make a payment to the household and to defer payments otherwise. Therefore, we set

$$dJ_t = \max\{W_t - \overline{W}(Y_t), 0\}\,. \quad \text{(I.5)}$$

[1]See DeMarzo and Sannikov (2006) and Sannikov (2008) for pioneering work on continuous-time recursive contracting formulations. See Ai and Li (2015) and Bolton, Wang, and Yang (2019) for continuous-time recursive formulations of contracting problems with limited commitment in Corporate Finance.

[2]As in our Section 5 primal debt management problem, the government and household both diversify away idiosyncratic risks and optimally choose aggregate risk exposures. So we use the risk adjustments called for by the SDF Λ given in (18), to evaluate risk premia for both of them. Note that the household is impatient, having a discount rate that exceeds the risk-free rate r by $\zeta \geqslant 0$.

Let $F(W_t, Y_t)$ denote the planner's value function that solves the optimization problem (I.1). In the payout region where $W_t > \overline{W}(Y_t)$,

$$F(W_t, Y_t) = F(\overline{W}(Y_t), Y_t) - \left(W_t - \overline{W}(Y_t)\right) \,, \tag{I.6}$$

and the threshold level $\overline{W}$ solves

$$\max_{\overline{W}} \quad F(\overline{W}, Y) + \overline{W} \,. \tag{I.7}$$

In the interior region where $W \in [\underline{W}, \overline{W}]$, the planner optimally sets $dJ_t = 0$ and the value function $F(W, Y)$ satisfies the HJB equation:

$$\begin{aligned} rF(W,Y) = \max_{\mathcal{T} \leqslant \overline{\tau} Y, \Phi^h, \Phi^m} \quad & (\mathcal{T} - \Gamma) + ((\zeta + r)W - (Y - \mathcal{T} - C(\mathcal{T}, Y)))\, F_W \\ & + (g - \rho\eta\sigma_Y) Y F_Y + \frac{\sigma_Y^2 Y^2 F_{YY}}{2} \\ & + \frac{((\Phi^h)^2 + (\Phi^m)^2) F_{WW}}{2} - \left(\psi_h \Phi^h + \psi_m \Phi^m\right) Y F_{WY}. \end{aligned} \tag{I.8}$$

We provide details now.

HJB equation for the planner's value function $F(W, Y)$. Using Ito's formula, we obtain the following SDF-adjusted dynamics for the planner's value function $F(W_t, Y_t)$:

$$d(\Lambda_t F(W_t, Y_t)) = \Lambda_t dF(W_t, Y_t) + F(W_t, Y_t) d\Lambda_t + < d\Lambda_t, dF(W_t, Y_t) >, \tag{I.9}$$

where the SDF Λ_t is given in (18) and

$$\begin{aligned} dF(W_t, Y_t) &= F_W dW_t + \frac{F_{WW}}{2} < dW_t, dW_t > + F_Y dY_t + \frac{F_{YY}}{2} < dY_t, dY_t > + F_{WY} < dW_t, dY_t > \\ &= \left[(\zeta W_t - (Y_t - \mathcal{T} - C_t) - \Phi^m \eta)\, F_W + gY F_Y + \frac{\sigma_Y Y^2 F_{YY}}{2} \right] dt \\ &\quad + \left[\frac{\left((\Phi^h)^2 + (\Phi^m)^2\right) F_{WW}}{2} - (\Phi^h \psi_h + \Phi^m \psi_m) Y F_{WY} \right] dt \\ &\quad - F_W (\Phi^h d\mathcal{Z}_t^h + \Phi^m d\mathcal{Z}_t^m) + Y F_Y (\psi_h d\mathcal{Z}_t^h + \psi_m d\mathcal{Z}_t^m) \,. \end{aligned} \tag{I.10}$$

Note that the process defined by

$$\int_0^t \Lambda_s \left(\mathcal{T}_s - \Gamma_t\right) ds + \Lambda_s dJ_s + \Lambda_t F(W_t, Y_t)$$

is a martingale under the physical measure $\mathbb{P}$. Therefore, its drift under $\mathbb{P}$ is zero:

$$\mathbb{E}_t\left[d\left(\Lambda_t F(W_t, Y_t)\right)\right] + \Lambda_t\left(\mathcal{T}_t - \Gamma_t\right) = 0. \tag{I.11}$$

Note that we have used the result that $dJ_t = 0$ in the interior region. Simplifying (I.11) gives the HJB equation (I.8) for the household's value function $F(W_t, Y_t)$.

We do not repeat FOC for the tax rate and other derivations contained in the main body. Below we provide the details for risk management policies.

Optimal hedging policies. The optimal idiosyncratic and systematic risk hedging demand functions, $\phi^h(w_t) = \Phi_t^h/Y_t$ and $\phi^m(w_t) = \Phi_t^m/Y_t$, are respectively given by

$$\phi^h(w) = \frac{\psi_h Y F_{WY}(W,Y)}{F_{WW}(W,Y)} = -\psi_h w \text{ and} \tag{I.12}$$

$$\phi^m(w) = \frac{\psi_m Y F_{WY}(W,Y)}{F_{WW}(W,Y)} = -\psi_m w\,. \tag{I.13}$$

Household promised value w_t dynamics. Applying Ito's lemma to $w_t = W_t/Y_t$, where W_t is given in (I.3) and Y_t is given in (3), we obtain:

$$\begin{aligned} dw_t &= [(\zeta + r + \rho\eta\sigma_Y - g)w_t - (1 - \theta_t - c(\theta_t))]dt + dj_t \\ &\quad + \left[\sigma_Y^2 w_t dt + \left(\sqrt{1-\rho^2}\sigma_Y \phi^h(w_t) + \rho\sigma_Y\phi^m(w_t)\right)dt\right] \\ &\quad -(\phi^h(w_t) + \sqrt{1-\rho^2}\sigma_Y w_t)d\mathcal{Z}_t^h - (\phi^m(w_t) + \rho\sigma_Y w_t)d\mathcal{Z}_t^m\,, \qquad \text{(I.14)} \\ &= \mu^w(w_t)dt + dj_t + \sigma^{w,h}(w_t)d\mathcal{Z}_t^h + \sigma^{w,m}(w_t)d\mathcal{Z}_t^m\,, \qquad \text{(I.15)} \end{aligned}$$

where $dj_t = 0$ in the interior region and

$$\begin{aligned} \mu^w(w_t) &= (\zeta + r + \lambda - g)w_t - (1 - \theta_t - c(\theta_t))\,, \qquad \text{(I.16)} \\ \sigma^{w,h}(w) &= (\phi^h + \psi_h w) = 0\,, \qquad \text{(I.17)} \\ \sigma^{w,m}(w) &= (\phi^m + \psi_m w) = 0. \qquad \text{(I.18)} \end{aligned}$$

Therefore, the w_t process evolves deterministically as:

$$\dot{w}_t = (\zeta + r + \lambda - g)w_t - (1 - \theta(b_t) - c(\theta(b_t)))\,. \tag{I.19}$$

Household promised value in balanced-budget regime: $\widehat{w}$. In the balanced-budget regime, the scaled promised value $\widehat{w}$ satisfies the following equation:

$$(\zeta + r)\widehat{w} = 1 - \gamma/\alpha - \kappa c(\gamma/\alpha) + (g - \rho\eta\sigma_Y)\widehat{w}\,, \tag{I.20}$$

which yields

$$\widehat{w} = \frac{1 - \gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + r + \lambda - g}\,. \tag{I.21}$$

Planner's Optimal Value Function

Using homogeneity again, we can simplify the dual to a one-dimensional problem. Let $w_t = W_t/Y_t$ denote the scaled household's value and let

$$F(W_t, Y_t) = f(w_t) \cdot Y_t\,. \tag{I.22}$$

Let $\overline{w}_t = \overline{W}_t/Y_t$ denote the scaled upper boundary of w. We can show that $\overline{w}_t$ is constant so that we can drop the time subscript if we want. The scaled optimal lump-sum transfer to the household for w_t, $dj_t = dJ_t/Y_t$, at any t is

$$dj_t = \max\{w_t - \overline{w}_t, 0\}. \tag{I.23}$$

Interior region: $w_t \in [\underline{w}, \overline{w}]$**.** Here there is no lump-sum transfer: $dj_t = 0$. Let $\theta_t = \theta(w_t) = \mathcal{T}_t/Y_t$ denote the optimal tax rate. Substituting (I.22) into (I.8) and simplifying yields the following implicit equation for $\theta(w)$:

$$1 + c'(\theta(w)) = -1/f'(w). \tag{I.24}$$

Using the optimal tax policy (I.24) and the optimal hedging strategies, (I.12) and (I.13), we obtain the following deterministic dynamics for the scaled promised value w_t:

$$\dot{w}_t \equiv \mu^w_t = \mu^w(w_t) = (\zeta + r + \lambda - g)w_t - (1 - \theta_t - c(\theta_t))\,. \tag{I.25}$$

Substituting $F(W_t, Y_t) = f(w_t) \cdot Y_t$ from (I.22) and the optimal policy functions (I.24), (I.12), and (I.13) for $\theta(w)$, $\phi^h(w)$, and $\phi^m(w)$, respectively, into the HJB equation (I.8), we obtain the following first-order nonlinear differential equation for the planner's scaled value $f(w)$:

$$(r + \lambda - g)f(w) = \tau(w) - \gamma + \left[(\zeta + r + \lambda - g)w - (1 - \theta(w) - c(\theta(w)))\right] f'(w)\,. \tag{I.26}$$

Lump-sum payout region: $w > \overline{w}$**.** Here the planner's value function is $f(w) = f(\overline{w}) +$

$\overline{w} - w$. The upper boundary $\overline{w}$ is constant and solves

$$\max_{\overline{w}} \quad f(\overline{w}) + \overline{w}\,. \tag{I.27}$$

Participation constraint and balanced primary budgets. At any time t, the household is free to enter autarky, in which case output immediately drops to $\widehat{Y}_t = \alpha Y_t$ and the household pays for public spending period-by-period so that $\widehat{\mathcal{T}}_t = \Gamma_t$. The household's value in this regime, $\widehat{W}(\widehat{Y}_t)$, is

$$\widehat{W}(\widehat{Y}_t) = \mathbb{E}\int_t^{\infty} e^{-\zeta(s-t)}\frac{\Lambda_s}{\Lambda_t}\left(\widehat{Y}_s - \Gamma_s - \widehat{C}(\Gamma_s, \widehat{Y}_s)\right) dt\,. \tag{I.28}$$

The participation constraint requires that the lower boundary of W_t in the interior region, $\underline{W}(Y_t)$, is greater than or equal to the value function in the balanced-budget regime $\widehat{W}(\widehat{Y}_t)$:

$$W_t \geqslant \underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)\,. \tag{I.29}$$

The inequality $\underline{W}(Y_t) \geqslant \widehat{W}(\widehat{Y}_t)$ holds with equality when the tax constraint (9) is not binding. Otherwise, the tax constraint (9) pins down the lower boundary $\underline{W}(Y_t)$.

Let $\widehat{w}_t = \widehat{W}(\widehat{Y}_t)/\widehat{Y}_t$. Using homogeneity and solving (I.28), we obtain:

$$\widehat{w} = \frac{1 - \gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + r + \lambda - g}\,. \tag{I.30}$$

Then the scaled promised outside value $\underline{w}$ is

$$\underline{w} = \alpha\widehat{w}\,, \text{ when the tax constraint (9) does not bind.} \tag{I.31}$$

Otherwise, (9) binds at the boundary and $\underline{w}$ is the root of the following equation:

$$\theta(\underline{w}) = \overline{\tau}\,. \tag{I.32}$$

To ensure that $w \geqslant \underline{w}$, using the same reasoning as deployed for our Section 5 primal formulation, we obtain the following zero-drift condition for w at $\underline{w}$:

$$\mu^w(\underline{w}) = (\zeta + r + \lambda - g)\underline{w} - (1 - \theta(\underline{w}) - c(\theta(\underline{w}))) = 0. \tag{I.33}$$

The following theorem describes the optimal contract.

Theorem I.A.1. *Under the $r + \lambda > g$ condition given in (28), $\kappa \geqslant 1$, $\alpha \leqslant 1$, and the condition $1 - \gamma/\alpha - \kappa c(\gamma/\alpha) \geqslant 0$ given in (52), the scaled value function in the no-default regime, $f(w)$, satisfies the nonlinear first-order differential equation:*

$$(r + \lambda - g)f(w) = \tau(w) - \gamma + [(\zeta + r + \lambda - g)w - (1 - \theta(w) - c(\theta(w)))]\, f'(w)\,, \tag{I.34}$$

subject to the zero-drift condition (I.33) and one of the following two conditions for the scaled promised outside value $\underline{w}$:

$$\underline{w} \;=\; \alpha\widehat{w}\,, \quad \textit{when the tax constraint (9) does not bind}; \tag{I.35}$$

$$\theta(\underline{w}) \;=\; \overline{\tau}\,, \quad \textit{when the tax constraint (9) binds}. \tag{I.36}$$

The scaled value $\widehat{w}$ in the balanced-budget regime is

$$\widehat{w} \;=\; \frac{1 - \gamma/\alpha - \kappa c(\gamma/\alpha)}{\zeta + (r + \lambda - g)}\,. \tag{I.37}$$

The lump-sum payout boundary $\overline{w}$ is given by (I.27), and the optimal lump-sum payout policy, dj_t, is given by (I.23). The optimal tax rate policy $\theta(w)$ is given by (I.24) and the scaled promised value $\{w_t\}$ evolves deterministically at the rate of $\dot{w}_t$ described by (I.25).

I.A.2 Taxes and Debts in Primal and Dual

Primal and dual problems yield identical tax outcomes with probability one. The state variable in the primal government debt management problem (scaled debt, b) equals the value function (scaled dual planner's value, $f(w)$) in the dual planner's problem. By symmetry, the state variable in the dual planner's problem (promised value for the household, w) equals the value function (investors' value, $p(b)$) in the primal government debt management problem. Thus,

$$b = f(w) \quad \text{and} \quad w = p(b). \tag{I.38}$$

Together these equations imply $f \circ p(b) = b$. The composition of $p(\,\cdot\,)$ from the primal debt management problem with $f(\,\cdot\,)$ from the dual planner's problem equals an identity function. Table 2 summarizes one-to-one mappings for state variables, value functions, policy rules in the primal and dual problems.

Equivalence of Primal and Dual

The government's debt management problem (20) is equivalent to the planner's value-

Table 2

Comparison of Primal and Dual Optimization Problems

	Primal Debt Management	Dual Planner's Allocation
A. State variables	b	w
Drift	$\dot{b}_t$ given in (56)	$\dot{w}_t$ given in (I.25)
Admissible region	$b \in [\underline{b}, \overline{b}]$	$w \in [\underline{w}, \overline{w}]$
B. Value function	$p(b)$	$f(w)$
Interior region	ODE given in (60)	ODE given in (I.26)
C. Policy rules		
lump-sum transfer	du given in (59)	dj given in (I.23)
Payout boundaries	$\underline{b}$ given in (58)	$\overline{w}$ given in (I.27)
Tax rates	$\tau(b)$ given in (55)	$\theta(w)$ given in (I.24)
D. Limited commitment		
Boundary condition	$\mu^b(\bar{b}) = 0$	$\mu^w(\underline{w}) = 0$
Default value	$\widehat{p}$ given in (63)	$\widehat{w}$ given in (I.30)
Non-binding-tax-constraint case	$p(\bar{b}) = \alpha \widehat{p}$	$\underline{w} = \alpha \widehat{w}$
Binding-tax-constraint case	$\tau(\bar{b}) = \overline{\tau}$	$\theta(\underline{w}) = \overline{\tau}$

maximizing problem (I.1). The key implications are: 1.) the credible debt capacity, $\overline{B}(Y)$, in the primal problem equals the planner's value when the limited-commitment constraint binds, $F(\underline{W}, Y)$ in the dual problem: $\overline{B}(Y) = F(\underline{W}, Y)$; 2.) the lump-sum debt-issuance and payout boundary, $\underline{B}(Y)$, equals the planner's value when the planner makes a lumpy payouts, $F(\underline{W}, Y)$ in the dual problem: $\underline{B}(Y) = F(\overline{W}, Y)$; 3.) the value function $P(B, Y)$ in the primal problem characterized by the HJB equation (39) and associated FOCs maps to the value function $F(W, Y)$ in the dual problem characterized by the HJB equation (I.8) and associated FOCs as follows: $P(B_t, Y_t) = W_t$ and $B_t = F(W_t, Y_t)$.

Using the homogeneity property, we obtain the following mapping for scaled variables and value functions:

$$b = f(w) \quad \text{and} \quad w = p(b). \tag{I.39}$$

Additionally, we have the following results at the boundaries:

$$\bar{b} = f(\underline{w}), \tag{I.40}$$

and

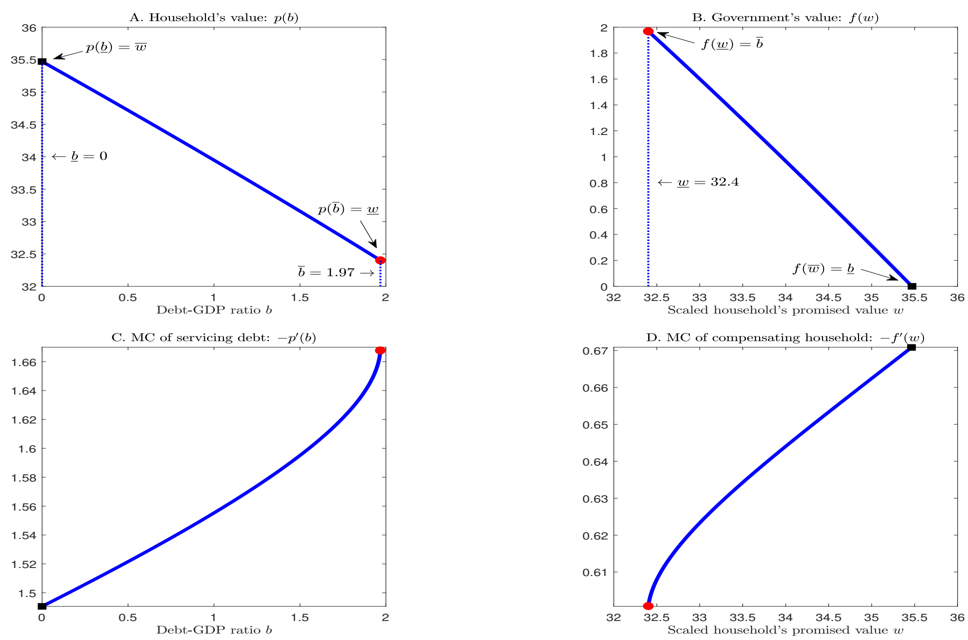
$$\underline{b} = f(\overline{w}). \tag{I.41}$$

Next, we demonstrate the equivalence between the two problems by showing that by substituting $b = f(w)$ into the ODE for $p(b)$, we obtain the ODE for $f(w)$, and vice versa.

Substituting (I.39) and (I.40) into ODE (60) for $f(b)$, we obtain the ODE (I.26) for $p(w)$. Substituting (I.39) and (I.40) into the constraint (57) for $\overline{b}$ and ODE (63) for the default value $\widehat{f}$, we obtain the constraint (I.33) for w, and ODE (I.20) for the default value $\underline{w}$. Substituting (I.39) and (I.41) into the constraint (58) for $\underline{b}$, we obtain constraint (I.27) for $\overline{w}$. Substituting (I.39) into the optimal tax policy (55) in the government debt problem, we obtain the optimal tax policy (I.24) in the dual planner's problem.

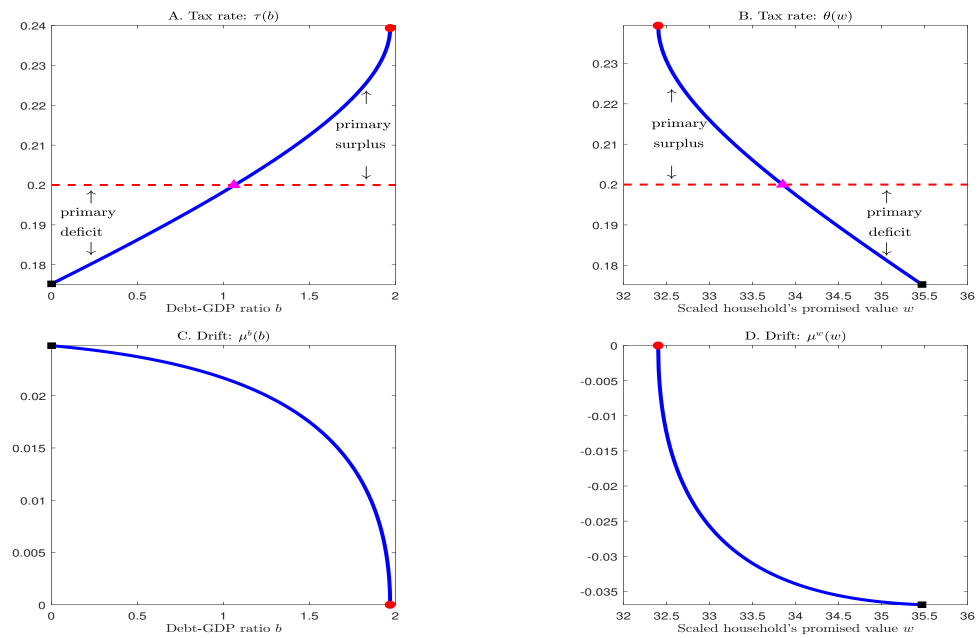
I.A.3 Primal and Dual in Pictures

Figure I-1: **Household's Value $p(b)$, Planner's Value $f(w)$, Marginal Cost of (Servicing) Debt $-p'(b)$, and Marginal Cost of Compensating Household $-f'(w)$.** Debt capacity is $\overline{b} = 1.97$ and there is no lump-sum debt issuance and payout: $\underline{b} = 0$. Parameter value are reported in Table 1.



We now illustrate the equivalence between the (primal) government debt management problem and the (dual) government profit-maximization problem. Panels A and C of Figure I-1 plot the household's value $p(b)$ and the marginal cost (MC) of servicing debt $-p'(b) = P_B(B, Y)$, respectively. The household's value $p(b)$ is decreasing and concave in b because as b increases the household becomes more constrained. As we increase b from its lower bound $b = 0$ to the government's debt capacity $b = \overline{b} = 1.97$, $p(b)$ decreases from $p(0) = 35.5$ to

Figure I-2: **Optimal Tax Rate $\tau(b)$, Optimal Tax Rate $\theta(w)$, Drift of Debt-GDP Ratio $\mu^b(b)$, and Drift of Scaled Promised Value $\mu^w(w)$.** Lower bound of promised value $\overline{w} = 32.4$ and lumpy payment boundary: $\underline{w} = 35.5$. Parameter value are reported in Table 1.



$p(\overline{b}) = 32.4$ and the MC of servicing debt $-p'(b)$ increases from $-p'(0) = 1.49$ to $-p'(\overline{b}) = -p'(1.97) = 1.67$ (panel C). That the MC of servicing debt exceeds one reflects costs of tax distortions and limited commitment. At the current US debt-GDP ratio of 1.08, the MC of servicing one dollar of debt is about $-p'(1.08) = 1.56$ dollars.

Panels B and D of Figure I-1 plot the government's value $f(w)$ and the marginal cost (MC) of compensating households $-f'(w) = -F_W(W, Y)$, respectively, for the section I.A dual problem. The planner's value $f(w)$ is decreasing and concave in the (scaled) household's promised value w. As the participation constraint limits the government more, the MC of compensating households $-f'(w)$ increases.

We can illustrate equivalence of primal and dual problems by rotating panel A (away from its plane) and swapping x and y axes. Doing so generates panel B. As a result, the red dot in panel A corresponds to the red dot in panel B: $p(\overline{b}) = \underline{w}$ and $f(\underline{w}) = \overline{b}$. Similarly, the black square in panel A corresponds to the black square in panel B. Indeed, for all $b \in [0, \overline{b}]$, we have $f \circ p(b) = b$ so that the composition of $p(\cdot)$ from the primal debt management problem with $f(\cdot)$ from the dual planner's problem is an identity function.

Equivalence between primal and dual problems implies $f \circ p(b) = b$, so we also have $p'(w) \times f'(b) = 1$. Since tax distortions make the MC of servicing debt exceeds one ($-f'(b) > 1$), the marginal cost (MC) of compensating households by cutting taxes must be less than one $-p'(w) < 1$. As we increase w from $\underline{w} = 32.4$ to $\overline{w} = 35.5$, the MC $-f'(w)$ increases from $-f'(w) = 0.60$ at $w = \underline{w} = 32.4$ to $-f'(w) = 0.67$ at $w = \overline{w} = 35.5$. The MC of cutting taxes is less than one for the dual government because cutting taxes also reduces distortions and relaxes the household's participation constraint. The higher the value of household's promised value w, the less financially constrained is the household and the smaller the benefit from reducing distortions by cutting taxes.

In panels A and B of Figure I-2, we plot optimal tax rate function $\tau(b)$ and $\theta(w)$ in the primal and dual formulations. The optimal tax rate $\tau(b)$ increases with b and reaches its maximum value $\tau(\overline{b}) = 0.24$ at the debt limit $\overline{b} = 1.97$ (panel A). For sufficiently low b, the government runs a primary deficit by keeping taxes low. When debt is sufficiently high ($b > 1.06$), the government runs a primary surplus by increasing the tax rate (at an increasing rate) in order to bring down the drift of b (panel C). In the limit, the economy settles at $\overline{b} = 1.97$. At the current debt-output ratio (1.08), the optimal tax rate on output is about $\tau(1.08) = 20\%$.

Panel B for the dual problem shows that $\theta(w)$ decreases with w. This happens because the government's power to tax the household decreases as the household's value w increases. Red dots in panels A and B describe the same outcomes, as do black squares.

Panels C and D plot the drift of b and the drift of w, respectively. Note that $\dot{b}_t$, the rate at which the debt-GDP ratio b increases, decreases with the level of b_t. As b increases, both the marginal cost of servicing debt $-p'(b)$ and the tax rate $\tau(b)$ increase. As a result, the debt-GDP ratio increases at a slower rate (i.e., $\dot{b}_t$ decreases) until it eventually reaches zero at debt capacity: $\mu^b(\overline{b}) = 0$ (panel C). This occurs because the government cannot exceed its debt limit. Correspondingly, the drift of scaled promised value in the absolute value $|\mu^w(w_t)|$ decreases as w_t decreases. As w decreases, the promised value w decreases at a slower rate until it reaches zero at the lowest promised value $\underline{w}$: $\mu^w(\underline{w}) = 0$ (panel D).