The Need for Fees at a DEX: How Increases in Fees Can Increase DEX Trading Volume

Abstract

We model endogenous trading and liquidity provision at a decentralized exchange (DEX) and demonstrate that increasing DEX trading fees can increase DEX trading volume. DEXs employ a mechanical pricing rule whereby price impacts decrease with inventory which DEXs acquire only by offering fee revenues to investors. Consequently, higher DEX fees can incentivize higher inventory, thereby reducing price impacts. Moreover, the price impact reduction can offset the fee increase so that the trading costs for marginal DEX traders decline despite paying higher trading fees. In turn, lower trading costs drive trading activity to the DEX, generating an increase in trading volume.

Keywords: Decentralized Exchange, DEX, Automated Market Makers, AMM, Fees, Liquidity

1 Introduction

A decentralized exchange (DEX) is an innovation in decentralized finance that allows agents to exchange various cryptoassets without relying on a trusted intermediary. These types of exchanges are facilitated by smart contracts which are deployed on a blockchain (e.g., Ethereum). The smart contracts allow investors to supply liquidity to the exchange in the form of cryptoasset inventory from which agents can trade for a fee paid to the liquidity providers. The DEX is decentralized in the sense that any agent can take part in liquidity provision and trading, while the smart contract itself is immutable, guaranteeing that the rules of the exchange cannot be changed after its creation. Consequently, the usage of smart contracts provides some protection for users relative to a centralized exchange (CEX) in that the DEX possesses no discretion beyond the smart contract code and therefore cannot misappropriate user funds.

In this paper, we provide an economic model of a DEX with the aim of understanding the role that DEX trading fees play in the adoption of the DEX as a trading platform. Our main finding is that an increase in DEX trading fees can increase the equilibrium DEX trading volume and therefore the use of the DEX. This result is particularly noteworthy for the practical design of decentralized exchanges in aiming to maximize their use and adoption.

We derive our main finding, that increases in DEX trading fees can increase DEX trading volume, by showing that an increase in DEX trading fees can reduce overall DEX trading costs, thereby driving trading activity from competing exchanges to the DEX. In order to understand why this result holds, first note that DEXs employ a mechanical pricing rule which implies a mechanical price impact for any trade at the DEX. In turn, the cost of trading with the DEX arises not only from the trading fee, but also from the price impact of the trade. We show that due to the form of the mechanical pricing rule, the price impact of DEX trading is strictly decreasing in the DEX inventory level which is the amount of funds provided by outside investors that are used as liquidity to facilitate trades. Therefore, whenever increases in the DEX fee level lead to an increase in the DEX inventory then the traders will always be guaranteed a lower price impact from trading at the DEX. Importantly, DEXs acquire inventory by offering a pro-rata share of DEX trading fees to the investors who provide such inventory. As a consequence, an increase in the DEX fee level can result in an increase in the overall DEX fee revenue, increasing the investor return from financing DEX inventory. Whenever this is the case, an increase in DEX fees will lead to a higher level of equilibrium DEX inventory, lower price impacts, and, when the reduction in price impact is sufficiently large, higher equilibrium DEX trading volume.

Formally, we model a one-period setting consisting of two types of agents, investors and traders, and two types of trading exchanges, a DEX and a CEX.[1] A unit measure of investors arrive at the beginning of the period. Each investor possess a unit of capital which she decides to invest either in the DEX (i.e., to provide inventory) or in an alternative investment opportunity which generates an exogenous and known expected return. Subsequently, liquidity traders with heterogeneous trading demand arrive according to a Poisson process and trade at either the DEX or the CEX, selecting whichever exchange offers the lower trading cost. Finally, at the end of the period, each investor who invested in the DEX receives a pro-rata share of all DEX trading fees, whereas each investor who invested in her alternative investment opportunity receives their known exogenous return.

Our model entails two sources of trading costs: trading fees and price impacts. Each exchange charges an exogenous proportional fee on the size of the trade. The CEX always offers execution at fair value, without price impact. In contrast, the DEX employs a standard mechanical pricing rule (see John et al. 2023) such that the execution price at the DEX approaches fair value for arbitrarily small trade sizes but diverges from fair value as the trade size diverges from zero (i.e., non-zero price impact).

Our main result, Proposition 4.1, establishes that an increase in the DEX fee level generates an increase in the DEX trading volume so long as the initial DEX fee level is sufficiently small. This result arises because the referenced increase in the DEX fee level reduces the

[1] We model the CEX as an outside option to the DEX and therefore this could equivalently represent a dealer.

price impact of DEX trading to such an extent that the overall DEX trading cost falls for the marginal trader who would be indifferent between the DEX and the CEX in the absence of a DEX fee level increase. In turn, since traders select between the DEX and the CEX on the basis of whichever provides the lower trading cost, the referenced increase in the DEX fee level drives traders to the DEX from the CEX, implying an increase in the equilibrium DEX trading volume. We establish the described channel that generates our main result via Propositions 4.2 and 4.3. More specifically, Proposition 4.2 establishes that an increase in the DEX fee level reduces trading costs for marginal traders so long as the initial DEX fee level is sufficiently small, whereas Proposition 4.3 establishes that an increase in the DEX fee level reduces the price impact of DEX trading so long as the initial DEX fee level is sufficiently small.

The result of Proposition 4.3, that an increase in the DEX fee level reduces the DEX price impact when the DEX fee level is sufficiently small, arises because the DEX's mechanical pricing rule embeds a negative relationship between the DEX price impact and the DEX inventory level, and the DEX inventory level is increasing in the DEX fee level when the DEX fee level is sufficiently small. More explicitly, the DEX acquires inventory by offering a pro-rata share of DEX trading fee revenues to investors in exchange for financing the capital which is used as DEX inventory. Therefore, an increase in the DEX fee level can increase overall DEX fee revenues and thus the return on investment from financing the DEX inventory. Then, given that investors select the investment that generates the highest expected return, the referenced increase in the DEX fee level endogenously increases DEX investment which increases DEX inventory and, as mentioned above, reduces the price impact of DEX trading. We formally establish the referenced relationships in Propositions 4.4 - 4.6. In particular, we demonstrate that increases in the DEX investment level monotonically reduce the price impact of DEX trading in Proposition 4.4, whereas we establish that, for a sufficiently small initial DEX fee level, an increase in the DEX fee level increases the DEX investment return and also the DEX investment level in Propositions 4.5 and 4.6.

Finally, we consider two extensions of our main model and show that our main insight, that DEX trading volume is increasing in DEX fees when the DEX fee is sufficiently small, is robust to these extensions. In the first extension, we allow traders to optimally split their trades between the DEX and the CEX. We derive the optimal trade splitting strategy in that context and demonstrate that our main result continues to hold when traders trade under the optimal splitting strategy. In our second extension, we allow the CEX to optimally respond to changes in the DEX fee level. Formally, we assume that after the DEX fee level is set, the CEX selects the CEX fee level (as a function of the DEX fee level) in order to maximize CEX trade revenue. We then show that even when the CEX fee level is set strategically, our main result continues to hold.

As a qualification, we emphasize that our results apply specifically to DEXs and do not necessarily generalize to CEXs. In more detail, there are mixed results in the literature regarding the effect of CEX fee levels upon CEX trading volume (see, e.g., Colliard and Foucault 2012, Foucault et al. 2012 and Malinova and Park 2015), which our paper cannot address given the exogenous nature of price impacts at the DEX. Explicitly, while we show unambiguously that DEX trading volumes first increase and then decrease in DEX fees, the effect of increasing CEX fees upon CEX trading volumes remains unclear, both in terms of the academic literature and the perception of practitioners and regulators (see Securities and Exchange Commission 2018).[2]

Our paper adds to the literature on the economics of blockchains.[3] More specifically, we add to a recent strand of that literature which examines Decentralized Finance (DeFi) applications including DEXs. Prominent papers from the DeFi literature include Cong et al. (2021), Cong et al. (2022), Kozhan and Viswanath-Natraj (2021) and Mayer (2022),

[2]The ambiguity regarding the effect of CEX fees upon CEX trading costs and CEX trading fees was seen as sufficiently pressing by the SEC that the SEC launched a pilot study. The SEC noted, "the mixed results from the academic literature and the disagreement among commenters" led them to propose the study. Further, one of the main inquiries that the pilot aimed to address was to "better understand the cumulative effects of changes in transactions fees and rebates on spreads and trading costs."

[3]John et al. (2023) and John et al. (2022) provide surveys of the literature. Notable early works from that literature include Biais et al. (2019), Easley et al. (2019), Makarov and Schoar (2019), Huberman et al. (2021), Saleh (2021) and Biais et al. (2022).

whereas prominent papers that focus on DEXs specifically include Park (2021), Barbon and Ranaldo (2022), Lehar and Parlour (2022), Lehar et al. (2022) and Milionis et al. (2022). Our paper relates most closely to Capponi and Jia (2021), which also theoretically examines a DEX in the absence of asymmetric information. Our work differs particularly from Capponi and Jia (2021) in that we allow traders to not only trade a DEX but also at a CEX. Moreover, we model traders as heterogeneous in trading demand, and we thereby derive a separating equilibrium in which each trader selects her optimal trading venue, DEX or CEX. Importantly, our focus is on examining the equilibrium relationship between the DEX fee level and the DEX trading volume, and our model generates endogenous trading volume for both the DEX and the CEX, each as a function of the trading fees charged by the DEX and CEX.

2 Model

In our setting, time is indexed by $t \in [0, 1]$. There are two types of agents: investors and traders. A unit measure of investors arrive at $t = 0$. Each investor possesses a unit of capital. Upon arrival, each decides whether to provide her capital to a decentralized exchange (DEX) or to invest in an alternative investment instead. Thereafter, liquidity traders arrive randomly over $(0, 1)$. Each arrival decides whether to trade at the DEX or to trade at a centralized exchange (CEX). Within this interval, the framework is similar to that of the sequential trade models (beginning with Glosten and Milgrom 1985 and Easley and O'Hara 1987). As in Easley et al. (1996) and others, arrivals follow a Poisson Process. At $t = 1$, all investor pay-offs are realized.

2.1 Exchanges

We model two exchanges, one CEX and one DEX, each of which offers trading of a single cryptoasset against a USD-equivalent token. We will regularly reference Exchange $i \in \{C, D\}$

where Exchange $i = C$ refers to the CEX while Exchange $i = D$ refers to the DEX. Hereafter, for exposition, we refer to the single cryptoasset as ETH and to the USD-equivalent token as USD.[4] We assume that at time $t = 0$ (before investors provide liquidity) the dollar value of ETH is known and equal to $V > 0$ and that this value remains constant over the interval $[0, 1)$. At time $t = 1$ we assume that the dollar value of ETH jumps to $V' = V \cdot R$ where $R \geqslant 0$ is the gross return on ETH drawn at $t = 1$ from a known distribution F with support $\mathbb{R}_+$.

In general, trading ETH entails two costs: a cost arising from the price of the ETH and a cost arising from fees charged on the trading of ETH. More explicitly, we denote by $P_i(\delta) \geqslant 0$ (defined precisely below) the per unit ETH price (in USD) for a trade of δ ETH at exchange i so that a trade of $\delta \in \mathbb{R}$ ETH entails a direct cost of $\delta \times P_i(\delta)$ USD at exchange $i \in \{C, D\}$. When $\delta > 0$ the trade corresponds to a buy of $|\delta|$ ETH; $\delta < 0$ corresponds to a sale of $|\delta|$ ETH. Note that the sign of δ carries through in the direct cost (as an expense for a buyer or reduced proceeds for a seller). In addition to the direct cost, trading with an exchange also entails an indirect cost arising from the trading fee charged by the exchange which is proportional to the size of the trade. More formally, recall that $V > 0$ denotes the fair value of ETH (in USD) and let $f_i \geqslant 0$ denote the proportional trading fee charged at exchange $i \in \{C, D\}$. Then, a trade of δ ETH entails a proportional fee of $f_i \times |\delta|$ (denominated in ETH) at exchange i so that the overall fee for a trade of δ ETH equals $f_i \times |\delta| \times V$ in USD. Formally, the overall cost of trading δ ETH at exchange i, which we denote by $\Psi_i(\delta)$, is given as follows:

$$\underbrace{\Psi_i(\delta)}_{\text{Total Trading Cost}} = \underbrace{P_i(\delta) \times \delta}_{\text{Execution Price Cost}} + \underbrace{f_i \times |\delta| \times V}_{\text{Trading Fees Cost}} \tag{1}$$

The key difference between a CEX and a DEX arises in the specification of the execution

[4] Formally, the reader should consider the trading as being ETH against a stablecoin pegged to USD. Such pairs (e.g., ETH-USDC, ETH-USDT) are offered by both centralized and decentralized exchanges.

price (i.e., $P_i(\delta)$). The CEX always prices ETH at its known fair value V:

$$P_C(\delta) = V \tag{2}$$

In contrast, ETH pricing at the DEX is mechanical and determined according to an exogenous function known as an Automated Market Maker (AMM) function. The referenced mechanical function determines a price as a function of not only the trade size, δ, but also the DEX inventory levels for ETH and USD. We assume the DEX employs the most common AMM function used in practice, the Constant Product Automated Market Maker (CPAMM) function, which implies the following pricing function (see John et al. 2023):

$$P_D(\delta) := \Xi(I_{USD}, I_{ETH}, \delta) \equiv \begin{cases} \frac{I_{USD}}{I_{ETH}-\delta} & \text{if } \delta < I_{ETH} \\ \infty & \text{if } \delta \geqslant I_{ETH} \end{cases} \tag{3}$$

where I_{USD} and I_{ETH} denote the USD and ETH inventory levels at the DEX respectively and $\Xi(I_{USD}, I_{ETH}, \delta)$ is the functional form of the CPAMM pricing function. We follow the specification of UniSwap V2 and require that investors who provide liquidity do so by adding both ETH and USD to the inventory in a fixed proportion (see John et al. 2023 for details). Moreover, we ensure the absence of arbitrage across the DEX and CEX by requiring that this fixed proportion is such that the marginal ETH price at the DEX is initially aligned with the ETH price at the CEX:

$$\lim_{\delta \to 0^+} P_D(\delta) = V = P_C(\delta) \tag{4}$$

2.2 Traders

We assume that there exist two types of traders: liquidity traders and opportunistic traders. Liquidity traders randomly arrive over the interval. Each liquidity trader has an exogenous trading demand to trade immediately and will trade either at the CEX or the DEX (whichever minimizes her trading cost). If a liquidity trader goes to the DEX, their execution pushes the DEX price away from fair value. Opportunistic traders immediately act to restore the DEX price. Intuitively, they correspond to high-frequency value arbitrageurs in that they continuously monitor the DEX market and trade whenever mispricing generates opportunities. They are similar to the arbitrageurs in Foucault et al. (2017) and Aquilina et al. (2021), but as there is no private information in this model, their activities are not (in the sense of these papers) "toxic". Since the fair value remains constant within the interval, perturbations in the DEX price are a consequence of the arriving liquidity traders' demands. These perturbations are transient and there are no permanent price changes until $t = 1$ when, as discussed earlier, there is a change in the ETH fair value.

In more detail, the DEX pricing function (see Equation 3) mechanically implies that the ETH price at the DEX moves in the direction of a trade (i.e., a buy increases ETH prices, whereas a sell decreases ETH prices) so that, even though DEX and CEX marginal prices are initially aligned, a liquidity trade in one direction produces an opportunity to trade in the opposite direction at a price which is favorable relative to fair value. We assume that such opportunities are seized upon immediately so that any movement in the marginal ETH price at the DEX away from fair value is subsequently traded away by traders who wait opportunistically for such price movements before executing their trade to benefit from lower trading costs. We refer to such traders that seize favorable trading opportunities as opportunistic traders, and note that such traders can be interpreted as a type of liquidity trader that is sufficiently time insensitive so that they can wait for the price to move in a favorable direction before executing a trade without incurring a large opportunity cost.

Formally, we assume liquidity traders arrive randomly over time $t \in (0, 1)$ according to a Poisson Process with unit intensity. We index the N liquidity traders who arrive over the interval by $j \in \{1, ..., N\}$. Liquidity Trader j possesses trading demand $\delta_j \sim U[-1, 1]$ where $\delta_j > 0$ represents the need to buy $|\delta_j|$ ETH, while $\delta_j < 0$ represents the need to sell $|\delta_j|$ ETH.

Each Liquidity Trader decides whether to trade at the DEX or at the CEX by maximizing their payoff, $\Pi_i(\delta_j)$, from trading δ_j at Exchange $i \in \{C, D\}$ given by

$$\Pi_i(\delta_j) = \gamma \cdot |\delta_j| + V \cdot \delta_j - \Psi_i(\delta_j)$$

where $\gamma > 0$ represents the per unit benefit of trading so that $\gamma \cdot |\delta_j|$ represents the total benefit received from the trade (denominated in USD), and $V \cdot \delta_j - \Psi_i(\delta_j)$ is the net cost of trading δ_j at Exchange i. Note that $V \cdot \delta_j - \Psi_i(\delta_j)$ is the net cost of trading because $V \cdot \delta_j$ corresponds to the fair value of the ETH being bought or sold, whereas $\Psi_i(\delta_j)$ corresponds to the overall cost of the associated trade, defined in Equation (1). In addition, we endow each trader with an outside option that yields her a payoff of zero and assume that each agent trades at her preferred Exchange $i \in \{C, D\}$ over utilizing the outside option whenever she is indifferent (i.e., whenever $\max_{i \in \{C,D\}} \Pi_i(\delta_j) = 0$). This outside option will not be relevant whenever $f_C < \gamma$ as in this case no trader will utilize the outside option. Importantly, while we take f_C as exogenously given in Sections 3 and 4, we consider the extension whereby f_C is optimally chosen as a function of f_D in Section 5.2 and show that our results are robust to allowing the CEX to strategically set fees. Furthermore, in this extension (where the outside option becomes relevant) we show that it will never be optimal to set a CEX fee $f_C > \gamma$, even when the CEX is a monopolist (e.g., when $f_D = +\infty$), and therefore we assume, without loss of generality, that $f_C \leqslant \gamma$ throughout.[5]

We note that choosing Exchange i to maximize the payoff $\Pi_i(\delta_j)$ is equivalent to selecting the exchange that minimizes the total cost of trading for Liquidity Trader j as follows:

$$i(j) = \underset{i \in \{C,D\}}{\operatorname{argmin}} \ \Psi_i(\delta_j) \tag{5}$$

In turn, we define $\mathcal{D}$ as the random set of liquidity traders who optimally trade at the

[5]In addition, we assume $\gamma \leqslant 25\%$ and $V \geqslant \frac{1}{2}$. These assumptions simplify our equilibrium solution but are not necessary.

DEX, stated explicitly as follows:

$$\mathcal{D} = \{j : i(j) = D\} \tag{6}$$

We note here that while we focus on the case whereby liquidity traders trade their entire volume at either the DEX or CEX, we allow for optimal order splitting in Section 5.1 and show that our results are robust to this extension.

As discussed, we assume that opportunistic traders arrive immediately after liquidity traders to seize favorable trading opportunities until their trading realigns the marginal DEX price with the CEX price which is equal to the ETH fair value. Due to the mechanical pricing rule at a DEX, this re-alignment occurs only after the DEX has experienced trading of an equal magnitude but opposite direction as the liquidity trade that generated the opportunity. Whether such trading occurs across multiple opportunistic trades or a single opportunistic trade is without loss of generality, so we assume that the re-alignment occurs through a single trade for exposition; more formally, we assume that every liquidity trade of size δ_j made at the DEX is immediately reversed by an opportunistic trade of size $-\delta_j$.

2.3 Investors

We assume that there exists a unit measure of investors indexed by $k \in [0, 1]$. Each Investor k possesses a unit of capital (denominated in USD) and at time $t = 0$ chooses whether to invest that capital in the DEX or in an alternative investment. Investing in the DEX generates a return equal to the pro-rata share of fees generated by the DEX plus any profit/loss on the DEX inventory. More explicitly, the total net return from investing at the DEX is given as $r_{P\&L} + r_D$ where $r_{P\&L}$ denotes the net expected return due to the profit/loss on the initial inventory investment, whereas r_D denotes the endogenous net expected return accrued from the pro-rata share of fee revenue received by each investor. More formally, $r_{P\&L}$ is given explicitly as follows:[6]

$$r_{P\&L} = \mathbb{E}[\sqrt{R}] - 1 \tag{7}$$

whereas r_D is given explicitly as follows:

$$r_D = \frac{\text{Total Expected Fees}}{\text{Total Invested Capital}} = \frac{2 \times \mathbb{E}[\sum_{j:j\in\mathcal{D}} f_D \times |\delta_j| \times V]}{I} \tag{8}$$

where I corresponds to the total DEX investment (in USD) and $\mathbb{E}[\sum_{j:j\in\mathcal{D}} f_D \times |\delta_j| \times V]$ corresponds to the expected fee revenue from liquidity traders. The multiplicative factor of 2 in the numerator of Equation (8) reflects the fact that any fee paid by a liquidity trader at the DEX is duplicated by fees from opportunistic trading; all of our results hold even without this factor of 2.

We assume that each investor is risk neutral and therefore invests in the investment opportunity that provides her with the highest expected return. In particular, denoting by ρ_k the return from Investor k's alternative investment opportunity, then Investor k invests in the DEX if and only if $r_{P\&L} + r_D \geqslant \rho_k$. Letting $\tilde{\rho}_k := \rho_k - r_{P\&L}$ further implies that Investor k invests at the DEX if and only if $r_D \geqslant \tilde{\rho}_k$ which states that Investor k invests in the DEX if and only if the net expected return from fees, r_D, exceeds the expected return from utilizing Investor k's alternative investment opportunity minus the expected profit/loss on Investor k's initial DEX inventory investment. Of particular note, the net expected return on DEX

[6] To understand Equation (7), note that DEX inventory value is given by $\Pi = I_{USD} + V \cdot I_{ETH}$, whereas prices being aligned across the DEX and CEX as per Equation (4) requires $V = \frac{I_{USD}}{I_{ETH}} \Leftrightarrow I_{USD} = V \cdot I_{ETH}$. The aforementioned two equations then imply $\Pi = 2 \cdot V \cdot I_{ETH}$ so that the net expected return of investing in DEX inventory is $r_{P\&L} = \mathbb{E}[\frac{\Pi'}{\Pi}] - 1 = \mathbb{E}[\frac{V'}{V} \cdot \frac{I'_{ETH}}{I_{ETH}}] - 1$. Further, the constant product nature of the AMM function requires $I_{USD} \cdot I_{ETH} = I'_{USD} \cdot I'_{ETH}$ (see John et al. 2023) so that, applying $I_{USD} = V \cdot I_{ETH}$ from Equation (4) yields $V \cdot I_{ETH}^2 = V' \cdot (I'_{ETH})^2$ and thus $\frac{I'_{ETH}}{I_{ETH}} = \sqrt{\frac{V}{V'}}$. Applying this expression to $r_{P\&L}$ then yields Equation (7):

$$r_{P\&L} = \mathbb{E}[\frac{V'}{V}\frac{I'_{ETH}}{I_{ETH}}] - 1 = \mathbb{E}[\sqrt{\frac{V'}{V}}] - 1 = \mathbb{E}[\sqrt{R}] - 1$$

Note that the realized net return from investing in DEX inventory, $\sqrt{R} - 1$, is always lower than the realized net return of an ETH-USD value-weighted portfolio – i.e., $\sqrt{R} - 1 \leqslant \frac{1}{2}(R-1)$ for all $R \geqslant 0$. This lower realized return is commonly known as impermanent loss and arises because ETH value fluctuations render DEX prices as stale so that opportunistic traders extract value from DEX investors by trading against the DEX until the DEX price mechanically adjusts to the new fair value (see Capponi and Jia 2021).

inventory, $r_{P\&L}$, is exogenous and does not vary across investors. In turn, we can subsume $r_{P\&L}$ into the alternative investment opportunity without loss of generality. More explicitly, we specify $\tilde{\rho}_k = \rho_k - r_{P\&L} \sim G[0,1]$ where G is given explicitly by $G(x) = x^{\frac{1}{\theta}}$ with $\theta > 1$. Then, the equilibrium DEX investment is given as follows:

$$I = G(r_D) = (r_D)^{\frac{1}{\theta}} \tag{9}$$

As a note regarding exposition, we hereafter refer to r_D as the *DEX investment return* despite there being an additional component, $r_{P\&L}$, within the total DEX investment return. We employ such language because, as discussed, $r_{P\&L}$ is exogenous and thus an increase (decrease) in r_D always corresponds to an identical increase (decrease) in the DEX investment return, irrespective of $r_{P\&L}$. It is thus convenient to omit reference to the exogenous component of the DEX investment return, $r_{P\&L}$, and to refer to the endogenous component, r_D, as the DEX investment return directly.

3 Model Solution

Formally, an equilibrium is a DEX investment return $r_D^\star$, a DEX investment level $I^\star$, a DEX USD inventory level $I_{USD}^\star$, a DEX ETH inventory level $I_{ETH}^\star$, a DEX pricing function $P_D^\star(\delta) := \Xi(I_{USD}^\star, I_{ETH}^\star, \delta)$, and a set of traders that trade at the DEX $D^\star$ such that each trader optimally selects the exchange at which she trades and each investor invests in the DEX iff it is optimal. More explicitly, an equilibrium is defined by the requirement that Equations (1) - (9) must all hold simultaneously with the equilibrium solutions replacing the associated endogenous object in each equation. Finally we note that our main focus is on examining the implications of the DEX fee level, f_D, upon equilibrium objects. As such, we explicitly state the dependence of all equilibrium objects on the DEX fee level. Moreover, we omit discussion regarding the trivial case of $f_D \geqslant f_C$ and restrict ourselves to $f_D \in [0, f_C)$; note that $f_D \geqslant f_C$ is a trivial case because it implies that DEX trading costs exceed CEX trading costs for all traders so that no trading occurs at the DEX in equilibrium.

Turning to our equilibrium solution, we solve for a symmetric equilibrium in that we require that all liquidity traders of the same size must trade at the same exchange. Explicitly, we require that the set of liquidity traders trading at the DEX, $\mathcal{D}^\star$, is of the following form:

$$\mathcal{D}^\star = \{j : \delta_j \in \Delta^\star\} \tag{10}$$

where $\Delta^\star \subseteq [-1, 1]$ denotes the range of trade sizes such that a liquidity trader trades at the DEX with $[-1, 1]$ being the support for the distribution that generates trade sizes.

As an intermediate step to solving for an equilibrium, we begin with the following result that derives the optimal behaviour of the liquidity traders while taking as given the investment level of the DEX, $I^\star(f_D)$:

Proposition 3.1. *Optimal Trading Strategy*

Denote by $I^\star(f_D)$ the equilibrium DEX investment level. The optimal strategy for Liquidity Trader j is:

$$i^\star(j) = \begin{cases} D & \text{if } \delta_j \in \Delta^\star(f_D) \\ C & \text{Otherwise} \end{cases} \tag{11}$$

where $\Delta^\star(f_D) := [\delta^\star_-(f_D), \delta^\star_+(f_D)]$ with the bounds $\delta^\star_-(f_D) < 0$ and $\delta^\star_+(f_D) > 0$ given explicitly as:

$$\delta^\star_\pm(f_D) = -\frac{(f_C - f_D)}{1 \pm (f_C - f_D)} \cdot \frac{I^\star(f_D)}{2V} \tag{12}$$

The endogenous quantities given by Equation (12), $\delta^\star_\pm(f_D)$ and $\delta^\star_-(f_D)$, determine the equilibrium buy and sell cut-off sizes respectively in that traders with an ETH buy order (i.e., $\delta > 0$) trade at the DEX if and only if $\delta \leqslant \delta^\star_+(f_D)$, whereas traders with an ETH sell order (i.e., $\delta < 0$) trade at the DEX if and only if $\delta \geqslant \delta^\star_-(f_D)$. This derived structure, characterized by cut-offs, implies that traders with larger trade sizes (in absolute magnitude) prefer trading at the CEX relative to the DEX, which is a necessary feature of any symmetric equilibrium. In particular, the DEX ETH price is increasing in trade size (i.e., $\frac{dP_D(\delta)}{d\delta} > 0$), whereas the CEX ETH price is always equal to fair value (i.e., $P_C(\delta) = V$ and thus $\frac{dP_C(\delta)}{d\delta} = 0$) so that the DEX entails an increasing average cost of trading (i.e., $\frac{\Psi_D(\delta)}{|\delta|}$ increases in $|\delta|$) while the CEX entails a constant average cost of trading (i.e., $\frac{\Psi_C(\delta)}{|\delta|}$ is constant in $|\delta|$). Hence, the DEX necessarily will only be optimal for smaller trade sizes (i.e., $\Delta^\star(f_D)$ must be of the form $[\delta_-^\star(f_D), \delta_+^\star(f_D)]$). Our next result builds upon Proposition 3.1, deriving a unique non-trivial equilibrium:[7]

Proposition 3.2. *Unique Equilibrium*

There exists a unique non-trivial symmetric equilibrium which is given as follows:

- *Equilibrium Investment Return at the DEX*

  The equilibrium expected return from investing in the DEX is:

$$r_D^\star(f_D) = \left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{\theta}{\theta - 1}} \tag{13}$$

- *Equilibrium Investment Level at the DEX*

  The equilibrium DEX investment level is:

$$I^\star(f_D) = \left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{14}$$

- *Equilibrium Inventory Levels at DEX*

  Equilibrium inventory of ETH and USD are functions of DEX investment as follows:

[7] A non-trivial equilibrium is defined as an equilibrium that features non-zero DEX trading volume. Note that there always exists a trivial equilibrium with zero DEX trading volume. In particular, if the DEX were to possess no investment, then trading costs would be infinite, no trading would occur at the DEX, and therefore investment returns would be zero, supporting the optimality of zero investment and ensuring that such a trivial equilibrium always exists. We omit discussion regarding this trivial equilibrium because its properties are straight-forward and well-known in the more general context of settings with positive network effects.

$$I^{\star}_{USD}(I) = \frac{I}{2}, \qquad I^{\star}_{ETH}(I) = \frac{I}{2V} \tag{15}$$

so that applying the equilibrium investment level from Equation (14) to Equation (15) yields the explicit equilibrium inventory solutions:

$$I^{\star}_{ETH}(f_D) = \frac{1}{2V}\left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{16}$$

and

$$I^{\star}_{USD}(f_D) = \frac{1}{2}\left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{17}$$

- <u>*Equilibrium Pricing Function at the DEX*</u>

  The equilibrium DEX pricing function depends upon DEX investment as follows:

$$P_D(I, \delta) = \Xi(I^{\star}_{USD}(I), I^{\star}_{ETH}(I), \delta) \tag{18}$$

where $\Xi(\cdot, \cdot, \cdot)$, $I^{\star}_{USD}(I)$ and $I^{\star}_{ETH}(I)$ are given by Equations (3) and (15).

In turn, applying the equilibrium investment level from Equation (14) to Equation (18) yields the equilibrium DEX pricing function:

$$P^{\star}_D(f_D, \delta) := P_D(I^{\star}(f_D), \delta) = \Xi(\frac{1}{2}I^{\star}(f_D), \frac{1}{2V}I^{\star}(f_D), \delta) \tag{19}$$

Since our primary object of interest is the equilibrium trading volume, we provide the following corollary which derives the equilibrium DEX trading volume, $T^{\star}(f_D)$:

Corollary 3.3. <u>*Equilibrium DEX Trading Volume*</u>

The equilibrium expected DEX trading volume (denominated in USD), $T^{\star}(f_D)$, is given as follows:

$$T^\star(f_D) = \frac{V}{2} \cdot (\delta_+^\star(f_D)^2 + \delta_-^\star(f_D)^2) \tag{20}$$

with $\delta_\pm^\star(f_D)$ being given in Equation (12).

4 Results

We begin with our main result, Proposition 4.1, which establishes that increases in fees charged to traders at the DEX can increase the equilibrium DEX trading volume:

Proposition 4.1. *DEX Trading Volume Can Increase in DEX fees*
The equilibrium expected trading volume, $T^\star(f_D)$, first increases and then decreases in the DEX fee level, f_D. More formally, there exists $\tilde{f} \in (0, f_C)$ such that $\frac{dT^\star}{df_D} > 0$ for $f_D \in (0, \tilde{f})$, whereas $\frac{dT^\star}{df_D} < 0$ for $f_D \in (\tilde{f}, f_C)$.

Explicitly, Proposition 4.1 establishes that there exists a non-zero fee level, $\tilde{f}$, such that increases in the DEX fee level up to $\tilde{f}$ will always lead to an increase in the DEX trading volume. Proposition 4.1 arises because an increase in the DEX fee level can decrease the overall cost of trading at the DEX. In turn, since a trader optimally trades at the exchange that charges her the lowest trading cost (see Equation 5), an increase in the DEX fee level can generate increases in trading volume as per Proposition 4.1 specifically because such DEX fee increases reduce DEX trading costs. We formalize this point with our next result:

Proposition 4.2. *DEX Trading Costs Can Decrease in DEX Fees*
Let $\Psi_D^\star(f_D, \delta)$ denote the equilibrium DEX trading cost for a trader with trade size δ, given explicitly as:

$$\Psi_D^\star(f_D, \delta) = P_D^\star(f_D, \delta) \times \delta + f_D \times |\delta| \times V \tag{21}$$

where $P_D^\star(f_D, \delta)$ is given by Equation (19).

Then, the following results hold:

1.) There exists $\hat{f}_+ \in (0, f_C)$ *such that cost of trading with the DEX for the marginal buy trader (i.e., the trader with size* $\delta^\star_+(f_D) > 0$*) is decreasing in* f_D *when* $f_D \in (0, \hat{f}_+)$ *and increasing in* f_D *when* $f_D \in (\hat{f}_+, f_C)$*. In particular,* $\frac{\partial \Psi^\star_D}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta^\star_+(f_D))} < 0$ *when* $f_D \in (0, \hat{f}_+)$ *and* $\frac{\partial \Psi^\star_D}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta^\star_+(f_D))} > 0$ *when* $f_D \in (\hat{f}_+, f_C)$*.*

2.) There exists $\hat{f}_- \in (0, f_C)$ *such that cost of trading with the DEX for the marginal sell trader (i.e., the trader with size* $\delta^\star_-(f_D) < 0$*) is decreasing in* f_D *when* $f_D \in (0, \hat{f}_-)$ *and increasing in* f_D *when* $f_D \in (\hat{f}_-, f_C)$*. In particular,* $\frac{\partial \Psi^\star_D}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta^\star_-(f_D))} < 0$ *when* $f_D \in (0, \hat{f}_-)$ *and* $\frac{\partial \Psi^\star_D}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta^\star_-(f_D))} > 0$ *when* $f_D \in (\hat{f}_-, f_C)$*.*

Proposition 4.2 establishes that an increase in the DEX fee level can reduce the overall DEX trading cost for a trader who would have been indifferent between trading at the DEX and trading at the CEX in the absence of such a DEX fee level increase. This result focuses upon traders who would have been indifferent between the DEX and the CEX in the absence of the DEX fee level change because changes in the trading costs of such marginal traders directly imply changes in our primary equilibrium quantity of interest, the DEX trading volume. More specifically, if the DEX trading cost falls for a trader who would have been indifferent between the DEX and the CEX in the absence of the DEX fee level increase, then that trader strictly prefers to trade at the DEX as opposed to the CEX after the DEX fee level increases. When this is the case, the trade size of the marginal trader (in absolute magnitude) must increase in response to an increase in DEX fees. In turn, an increase in the DEX fee level can generate an increase in DEX trading volume as per Proposition 4.1 precisely because the increase in the DEX fee level decreases DEX trading costs as per Proposition 4.2.

To understand Proposition 4.2, we reiterate that fees are not the only cost associated with trading at a DEX. In particular, Equation (1) highlights that the overall trading cost depends not only on the fee f_D but also on the price at which the cryptoasset is being traded $P^\star_D(f_D, \delta)$. Therefore, if an increase in the DEX fee level leads to a lower price impact (i.e., if $\frac{\partial P^\star_D}{\partial \delta}$ decreases in f_D), then an increase in the DEX fee level will reduce the overall DEX trading cost provided that the cost of paying a higher fee can be offset by the decrease in the trading cost due to trading at a price that is more favorable (i.e., a lower price impact). Our next result formally establishes such a channel whereby an increase in the DEX fee level reduces the DEX price impact so long as DEX fees are initially not too large:

Proposition 4.3. *DEX Price Impacts Can Decrease in DEX fees*
The equilibrium price impact at the DEX first decreases and then increases in the DEX fee level. More formally, letting

$$\lambda^\star(f_D, \delta) = \frac{\partial P_D^\star(f_D, \delta)}{\partial \delta} \tag{22}$$

denote the equilibrium price impact at the DEX. Then, there exists $\tilde{f} \in (0, f_C)$ *such that* $\frac{\partial \lambda^\star}{\partial f_D} < 0$ *for* $f_D \in (0, \tilde{f})$ *and* $\frac{\partial \lambda^\star}{\partial f_D} > 0$ *for* $f_D \in (\tilde{f}, f_C)$*. This* $\tilde{f}$ *applies uniformly for all feasible trade sizes (i.e., for* $\delta < I_{ETH}^\star$*).*

Proposition 4.3 defines equilibrium DEX price impact as the sensitivity of the DEX price to trade size (see Equation 22), and then establishes that the DEX price impact is decreasing in the DEX fee level whenever the initial DEX fee level is sufficiently small (i.e., $\frac{\partial \lambda^\star}{\partial f_D} < 0$ for $f \in (0, \tilde{f})$).

The relationship between DEX fee levels and price impact is important because it affects the overall DEX trading cost which in turn affects DEX trading volume. In particular, DEX trading prices mechanically move in the direction of a trade (see Equation 3) so that a larger price impact (i.e., a larger $\lambda^\star$) entails that a given buy order (i.e., $\delta > 0$) would involve a higher price and also entails that a given sell order (i.e., $\delta < 0$) would involve a lower price. As the trading cost is increasing in the price for a buy order but decreasing in the price for a sell order (see Equation 1), a larger price impact entails a higher cost for all traders. Thus, the result of Proposition 4.3, that price impacts decline for DEX fee levels up to a point (i.e., $\frac{\partial \lambda^\star}{\partial f_D} < 0$ for $f \in (0, \tilde{f})$), clarifies that increases in the DEX fee level can reduce the execution price component of the DEX trading cost. This is precisely the mechanism whereby increases in the DEX fee level can reduce overall DEX trading costs (Proposition

4.2) and also increase DEX trading volume (Proposition 4.1).

The relationship that Proposition 4.3 establishes between the DEX fee level and the DEX price impact arises due to two intermediate relationships. First, the mechanical pricing rule of a DEX (i.e., Equation 3) implies that an increase in total DEX investment always reduces DEX price impacts. Second, all DEX fee revenues are paid to investors which creates the incentive for investors to provide DEX investment (see Equation 8) so that increases in the DEX fee level can lead to increases in overall DEX investment. Then, putting the two aforementioned relationships together, an increase in the DEX fee level can generate increases in total DEX investment which, in turn, reduces DEX price impacts (Proposition 4.3) and therefore promotes higher DEX trading volume (Proposition 4.1). We proceed by formalizing the referenced intermediate relationships with Proposition 4.4 demonstrating the first relationship that increases in DEX investment decrease DEX price impacts, and Propositions 4.5 - 4.6 establishing the second relationship that increases in the DEX fee level can increase DEX investment.

Proposition 4.4. *DEX Price Impacts Always Decrease in DEX Investment*

The price impact is monotonically decreasing in the DEX investment level for all feasible trade sizes. More explicitly, $\frac{\partial \lambda(I,\delta)}{\partial I} < 0$ for all investment levels, I, and for all feasible trade sizes, $\delta < I^{\star}_{ETH}(I) = \frac{I}{2V}$, where $\lambda(I,\delta)$ denotes the price impact given an arbitrary investment level, $I > 0$:

$$\lambda(I,\delta) = \frac{\partial P_D(I,\delta)}{\partial \delta}$$

with $P_D(I,\delta)$ being defined in Equation (18).

Proposition 4.4 establishes that an increase in the DEX investment level unambiguously reduces the DEX price impact (i.e., $\frac{\partial \lambda}{\partial I} < 0$). This result arises due to the mechanical pricing function of the DEX (see Equation 3). To clarify this point, note that $P_D(I,\delta)$, which is defined in Equation (18), can be derived explicitly from Equations (3) and (15) as follows:

$$P_D(I,\delta) = \Xi(I^{\star}_{USD}(I), I^{\star}_{ETH}(I), \delta) = \frac{I \cdot V}{I - 2 \cdot V \cdot \delta} \tag{23}$$

In turn, the price impact, $\lambda(I,\delta)$, as a function of DEX investment, I, and trade size, δ, is given as follows:

$$\lambda(I,\delta) = \frac{\partial P_D(I,\delta)}{\partial \delta} = \frac{2 \cdot I \cdot V^2}{(I - 2 \cdot V \cdot \delta)^2} \tag{24}$$

so that direct verification reveals that the DEX price impact monotonically decreases in investment (i.e., $\frac{\partial \lambda}{\partial I} < 0$) whenever the trade size is feasible (i.e., when $\delta < I^{\star}_{ETH}(I) = \frac{I}{2V}$) as per Proposition 4.4. We ignore $\delta > I^{\star}_{ETH}(I) = \frac{I}{2V}$ and label such trade sizes as infeasible because in such a case there is insufficient inventory for the trade size and the price is consequently infinite (see Equation 3), which ensures that the DEX would not allow a trade of such size.

Although the particulars are specific to the AMM in our model, the association between capital (inventory) and liquidity arises in many models of traditional market-making, notably Brunnermeier and Pedersen (2009). The association is also of significant practical and regulatory importance, as low levels of market-making capital in the wake of the great financial crisis are sometimes viewed as impairing liquidity (Bao et al. 2018, for example).

Having established that increases in DEX investment decrease DEX price impacts (Proposition 4.4), we turn to demonstrating that increases in the DEX fee level can increase DEX investment:

Proposition 4.5. <u>*DEX Investment Can Increase in DEX Fee Levels*</u>
The equilibrium DEX investment, $I^{\star}(f_D)$, first increases and then decreases in the DEX fee level, f_D. More formally, there exists $\tilde{f} \in (0, f_C)$ such that $\frac{dI^{\star}}{df_D} > 0$ for $f_D \in (0, \tilde{f})$, whereas $\frac{dI^{\star}}{df_D} < 0$ for $f_D \in (\tilde{f}, f_C)$ with $I^{\star}(f_D)$ being given in Equation (14).

Proposition 4.5 establishes that increases in the DEX fee level can increase DEX investment up to some fee level $\tilde{f}$ (i.e., $\frac{dI^{\star}}{df_D} > 0$ for $f_D \in (0, \tilde{f})$). This result arises because DEXs acquire investment by offering investors a pro-rata share of all trading fees from the DEX in exchange for those investments (see Equation 8); in particular, for $f < \tilde{f}$, an increase in the DEX fee level increases DEX investment by increasing the DEX investment return through an increase in the overall trading fee revenue generated by the DEX. We formalize the point that such increases in the DEX fee level generate an increase in the DEX investment return with our final result:

Proposition 4.6. *DEX Investment Returns Can Increase in DEX Fee Levels*
The equilibrium DEX investment return, $r_D^\star(f_D)$, first increases and then decreases in the DEX fee level. More formally, there exists $\tilde{f} \in (0, f_C)$ such that $\frac{dr_D^\star}{df_D} > 0$ for $f_D \in (0, \tilde{f})$, whereas $\frac{dr_D^\star}{df_D} < 0$ for $f_D \in (\tilde{f}, f_C)$.

Collectively, our results establish an important feature of a DEX. More precisely, we demonstrate that an increase in fees at a DEX can increase trading volume at the DEX. Our main result, Proposition 4.1, establishes this finding, whereas our remaining results clarify the associated economic channel. In more detail, an increase in the DEX fee level can increase DEX investment returns (Proposition 4.6) and thereby DEX investment (Proposition 4.5), which generates a reduction in the DEX price impact (Propositions 4.3 and 4.4) and thereby a reduction in DEX trading costs (Proposition 4.2). In turn, the reduction in DEX trading costs drives trading activity from the CEX to the DEX, leading to an increase in equilibrium DEX trading volume as per Proposition 4.1.

5 Extensions

In this section we demonstrate the robustness of our results to natural extensions of our main model. Given that our main insight is that increasing DEX fees can lead to an increase in trading volume, we focus on replicating that main result in this section. In particular, in Section 5.1 we allow liquidity traders to split their trade between the DEX and CEX in order to lower their cost of trading. We derive the optimal order splitting strategy and show that our main insight holds when liquidity traders trade under this strategy. In Section 5.2 we allow for the CEX to optimally set its fee in response to the DEX fee with the objective of maximizing CEX trading revenue, and we demonstrate that our main insight continues to hold in that context.

5.1 Order Splitting

If traders have seamless access to trading at both the CEX and the DEX then it will be optimal for them to split their trade across the CEX and DEX in order to minimize the total cost of trading. In particular, for any $\delta \in [-1, 1]$, denote by $\mu^\star(\delta)$ the optimal fraction of the trade demand, δ, traded at the DEX, with the remaining fraction $1 - \mu^\star(\delta)$ being traded at the CEX. Then, $\mu^\star(\delta)$ is determined as follows:

$$\mu^\star(\delta) = \underset{\mu \in [0,1]}{\operatorname{argmin}} \quad \Psi_D(\mu \cdot \delta) + \Psi_C((1 - \mu) \cdot \delta) \tag{25}$$

The next result formalizes the optimal order splitting strategy.

Proposition 5.1. *Optimal Order Splitting Strategy and Trade Volume*

The optimal order splitting strategy $\mu^\star(\delta)$ is given by

$$\mu^\star(\delta) = \begin{cases} \frac{I^\star}{2V\delta}\left(1 - \sqrt{\frac{1}{1-(f_C - f_D)}}\right) & \delta \in [-1, \tilde{\delta}_-] \\ 1 & \delta \in [\tilde{\delta}_-, 0] \\ 1 & \delta \in [0, \tilde{\delta}_+) \\ \frac{I^\star}{2V\delta}\left(1 - \sqrt{\frac{1}{1+(f_C - f_D)}}\right) & \delta \in [\tilde{\delta}_+, 1] \end{cases}$$

where

$$\tilde{\delta}_+ = \frac{I^\star}{2V}\left(1 - \sqrt{\frac{1}{1 + (f_C - f_D)}}\right) \qquad \text{and} \qquad \tilde{\delta}_- = \frac{I^\star}{2V}\left(1 - \sqrt{\frac{1}{1 - (f_C - f_D)}}\right)$$

The equilibrium DEX trading volume under the optimal equilibrium trading strategy $\mu^\star(\delta)$ *is given by*

$$\tilde{T}^\star(f_D) = \frac{1}{2}\left(\tilde{\delta}_+ \cdot (1 - \frac{\tilde{\delta}_+}{2}) - \tilde{\delta}_- \cdot (1 + \frac{\tilde{\delta}_-}{2})\right)$$

Having derived the optimal trading strategy, we can now examine the equilibrium trading volume. In particular, our next result generalizes our main result by establishing that equilibrium DEX trading volume under the optimal trading strategy is increasing in DEX fees provided that DEX fees are sufficiently small.

Proposition 5.2. *DEX Volume Can Increase in DEX Fees With Optimal Order Splitting* *When liquidity traders utilize the optimal order splitting strategy,* $\mu^\star(\delta)$*, then there exists* $\tilde{f} > 0$ *such that the equilibrium DEX trade volume* $\tilde{T}^\star(f_D)$ *is strictly increasing in DEX fees whenever* $f_D \in [0, \tilde{f})$*.*

5.2 Optimal CEX Fees

Throughout the paper we have assumed that the CEX fee is exogenous. In practice though, a CEX may optimally set its fee in response to the DEX fee. In this section, we will show that our main insight is robust to this feature. In order to do so, we will allow that the CEX optimally sets its fee f_C as a function of f_D in order to maximize expected CEX fee revenue. More formally, the CEX optimally sets its fee as follows:

$$f_C^\star(f_D) := \underset{f_C \geqslant 0}{\operatorname{argmax}} \; V \cdot f_C \cdot T_C^\star(f_C, f_D) \tag{26}$$

where

$$T_C^\star(f_C, f_D) = \frac{1}{2}\left(\int_{\delta_+^\star(f_C,f_D)}^{1} x\,dx + \int_{\delta_-^\star(f_C,f_D)}^{-1} x\,dx\right) = \frac{1}{2} \cdot \left(1 - \frac{(\delta_+^\star(f_C,f_D))^2 + (\delta_-^\star(f_C,f_D))^2}{2}\right)$$

represents the expected trading volume at the CEX given that the CEX fee is $f_C \leqslant \gamma$ and the DEX fee is f_D. Note that $f_C > \gamma$ implies that $T_C^\star(f_C, f_D) = 0$ because, in such a case,

$\Pi_C(\delta) < 0$ for all $\delta \in [-1, 1]$ and therefore all traders prefer their outside option over trading at the CEX whenever $f_C > \gamma$. This implies that setting $f_C = \gamma$ generates a weakly larger payoff for the CEX than setting $f_C > \gamma$ for any DEX fee f_D and therefore $f_C^{\star}(f_D) \leqslant \gamma$ for all f_D which demonstrates that our restriction of $f_C \leqslant \gamma$ in our main model is without loss of generality.

Our final result formalizes that our main result is robust to the CEX setting its fee strategically in response to the DEX fee:

Proposition 5.3. *DEX Trade Volume Can Increase in DEX Fees under Optimal CEX Fees* *Suppose that the CEX sets fees, $f_C^{\star}(f_D)$, as a function of DEX fees, f_D, in order to solve (26). Then, there exists $\tilde{f} > 0$ such that equilibrium DEX trading volume $T^{\star}$ is increasing in DEX fees when $f_D \in [0, \tilde{f})$.*

6 Conclusion

We provide an economic model of a DEX. Our model is specifically aimed at clarifying the implications of varying DEX fee levels upon equilibrium quantities such as DEX trading volume and DEX trading costs. Of particular note, we demonstrate that increases in DEX fee levels can reduce DEX trading costs and thereby increase DEX trading volume. This result arises due to the fact that an increase in DEX fees can induce an increase in the return to DEX liquidity provision. Once this is the case, higher DEX fees lead to more DEX inventory which lowers the cost of trading with the DEX due to the role of inventory in the AMM pricing function. These insights generate important considerations for the design and management of DEX fees in order to support the maximal trade volume and thus return from investment at a DEX.

References

Aquilina, M., E. Budish, and P. O'Neill. 2021. Quantifying the High-Frequency Trading "Arms Race"*. The Quarterly Journal of Economics 137:493–564. URL https://doi.org/10.1093/qje/qjab032.

Bao, J., M. O'Hara, and X. Zhou. 2018. The Volcker Rule and corporate bond market making in times of stress. Journal of Financial Economics 130:95–113. URL https://www.sciencedirect.com/science/article/pii/S0304405X18301491.

Barbon, A., and A. Ranaldo. 2022. On The Quality Of Cryptocurrency Markets Centralized Versus Decentralized Exchanges. Working Paper .

Biais, B., C. Bisière, M. Bouvard, and C. Casamatta. 2019. The Blockchain Folk Theorem. Review of Financial Studies 32(5):1662–1715.

Biais, B., C. Bisière, M. Bouvard, C. Casamatta, and A. J. Menkveld. 2022. Equilibrium Bitcoin Pricing. Journal of Finance Forthcoming.

Brunnermeier, M., and L. H. Pedersen. 2009. Market Liquidity and Funding Liquidity. Review of Financial Studies 22:2201–2238.

Capponi, A., and R. Jia. 2021. The Adoption of Blockchain-based Decentralized Exchanges. Columbia University Working Paper .

Colliard, J.-E., and T. Foucault. 2012. Trading Fees and Efficiency in Limit Order Markets. Review of Financial Studies 25:3389–3421.

Cong, L. W., Y. Li, and N. Wang. 2021. Tokenomics: Dynamic Adoption and Valuation. Review of Financial Studies 34(3):1105–1155.

Cong, L. W., Y. Li, and N. Wang. 2022. Token-based platform finance. Journal of Financial Economics 144:972–991. URL https://www.sciencedirect.com/science/article/pii/S0304405X21004414.

Easley, D., N. M. Kiefer, M. O'Hara, and J. Paperman. 1996. Liquidity, information and infrequently traded stocks. Journal of Finance 51:1405–1436.

Easley, D., and M. O'Hara. 1987. Price, trade size, and information in securities markets. Journal of Financial Economics 19:69–90.

Easley, D., M. O'Hara, and S. Basu. 2019. From Mining to Markets: The Evolution of Bitcoin Transaction Fees. Journal of Financial Economics 134(1):91–109.

Foucault, T., O. Kadan, and E. Kandel. 2012. Liquidity Cycles and Make/Take Fees in Electronic Markets. Journal of Finance 68.

Foucault, T., R. Kozhan, and W. W. Tham. 2017. Toxic Arbitrage. Review of Financial Studies 30:1053–1094.

Glosten, L. R., and P. R. Milgrom. 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. Journal of financial economics 14:71–100.

Huberman, G., J. D. Leshno, and C. Moallemi. 2021. Monopoly without a Monopolist: An Economic Analysis of the Bitcoin Payment System. The Review of Economic Studies URL https://doi.org/10.1093/restud/rdab014.

John, K., L. Kogan, and F. Saleh. 2023. Smart Contracts and Decentralized Finance. Annual Review of Financial Economics Forthcoming.

John, K., M. O'Hara, and F. Saleh. 2022. Bitcoin and Beyond. Annual Review of Financial Economics 14.

Kozhan, R., and G. Viswanath-Natraj. 2021. Decentralized stablecoins and collateral risk. WBS Finance Group Research Paper .

Lehar, A., and C. Parlour. 2022. Decentralized Exchanges. Working Paper .

Lehar, A., C. A. Parlour, and M. Zoican. 2022. Liquidity Fragmentation on Decentralized Exchanges. Available at SSRN 4267429 .

Makarov, I., and A. Schoar. 2019. Trading and Arbitrage in Cryptocurrency Markets. Journal of Financial Economics 135(2):293–319.

Malinova, K., and A. Park. 2015. Subsidizing Liquidity: The Impact of Make/Take Fees on Market Quality. Journal of Finance 70.

Mayer, S. 2022. Token-Based Platforms and Speculators. Working Paper .

Milionis, J., C. C. Moallemi, T. Roughgarden, and A. L. Zhang. 2022. Automated market making and loss-versus-rebalancing. arXiv preprint arXiv:2208.06046 .

Park, A. 2021. The Conceptual Flaws of Constant Product Automated Market Making. University of Toronto Working Paper .

Saleh, F. 2021. Blockchain Without Waste: Proof-of-Stake. Review of Financial Studies 34(3):1156–1190.

Securities and Exchange Commission. 2018. Transaction Fee Pilot for NMS Stocks. Federal Register 83:26014–26031. URL https://www.federalregister.gov/documents/2018/06/07/2018-12084/transaction-fee-pilot-for-nms-stocks. Release No. 34-84875; File No. S7-05-18. RIN 3235-0761.

Appendices

A Supplementary Results

Lemma A.1.

Let $\delta : (0, f_C) \mapsto \mathbb{R}$ *denote any non-zero continuously differentiable function that satisfies:*

$$\Psi^\star_D(f_D, \delta(f_D)) = \beta \times \delta(f_D) \tag{A.1}$$

for all $f_D \in (0, f_C)$ *and for any* $\beta \in \mathbb{R}$ *where* $\Psi^\star_D$ *refers to* $\Psi^\star_D(f_D, \delta)$ *which is given by Equation (21). Then, the following result holds:*

$$\frac{\partial \Psi^\star_D}{\partial f_D} = -\delta(f_D) \times \frac{\partial P^\star_D(f_D, \delta)}{\partial \delta} \times \frac{d\delta}{df_D}$$

for all $f_D \in (0, f_C)$ *where* $\Psi^\star_D$ *and* $P^\star_D$ *are each evaluated at* $(f_D, \delta(f_D))$*.*

Proof. We begin by taking the total derivative in Equation (A.1) with respect to f_D which yields:

$$\frac{\partial \Psi^\star_D}{\partial f_D} + \frac{\partial \Psi^\star_D}{\partial \delta} \times \frac{d\delta}{df_D} = \beta \times \frac{d\delta}{df_D} \tag{A.2}$$

and further implies:

$$\frac{\partial \Psi^\star_D}{\partial f_D} = \left(\beta - \frac{\partial \Psi^\star_D}{\partial \delta}\right) \times \frac{d\delta}{df_D} \tag{A.3}$$

By explicit calculation, Equation (21) yields:

$$\frac{\partial \Psi^\star_D}{\partial \delta} = \left(P^\star_D(f_D, \delta(f_D)) + f_D \times V\right) + \delta(f_D) \times \frac{\partial P^\star_D}{\partial \delta} \tag{A.4}$$

whereas Equation (A.1) is equivalent to:

$$P^\star_D(f_D, \delta(f_D)) + f_D \times V = \beta \tag{A.5}$$

so that applying Equation (A.5) to Equation (A.4) and then applying the result to Equation (A.3) yields:

$$\frac{\partial \Psi_D^\star}{\partial f_D} = -\delta(f_D) \times \frac{\partial P_D^\star}{\partial \delta} \times \frac{d\delta}{df_D} \tag{A.6}$$

thereby completing the proof. □

B Proofs

B.1 Proof of Proposition 3.1

A liquidity trader with trade size $\delta \in \mathbb{R}$ trades with the DEX if and only if the cost of doing so is less than the cost of trading with the CEX. This is the case if and only if

$$P_D^\star(\delta) \cdot \delta + f_D \cdot |\delta| \cdot V \leqslant V \cdot \delta + f_C \cdot |\delta| \cdot V$$

Therefore, $\delta_-^\star(f_D) < 0$ is the trade size of a liquidity trader that wishes to sell ETH and is indifferent between trading at the DEX and CEX, given by:

$$\frac{I_{USD}}{I_{ETH} - \delta_-^\star(f_D)} \cdot \delta_-^\star(f_D) - f_D \cdot \delta_-^\star(f_D) \cdot V = V \cdot \delta_-^\star(f_D) - f_C \cdot \delta_-^\star(f_D) \cdot V$$

which after solving for $\delta_-^\star(f_D)$ using the fact that $I_{USD}^\star = V \cdot I_{ETH}^\star$ and $I_{ETH}^\star = \frac{I^\star(f_D)}{2 \cdot V}$ yields our expression for $\delta_-^\star(f_D)$.

Similarly, $\delta_+^\star(f_D) > 0$ is the trade size of a liquidity trader that wishes to buy ETH and is indifferent between trading at the DEX and CEX, given by:

$$\frac{I_{USD}}{I_{ETH} - \delta_+^\star(f_D)} \cdot \delta_+^\star(f_D) + f_D \cdot \delta_+^\star(f_D) \cdot V = V \cdot \delta_+^\star(f_D) + f_C \cdot \delta_+^\star(f_D) \cdot V$$

which again after substituting and rearranging and using the fact that $I_{USD}^\star = V \cdot I_{ETH}^\star$ and $I_{ETH}^\star = \frac{I^\star(f_D)}{2 \cdot V}$ yields our expression for $\delta_+^\star(f_D)$.

Finally, we note that given we have assumed $f_C \leqslant \gamma$, then all traders with trade size $\delta_j \notin \Delta^\star(f_D)$ optimally trade at the CEX, rather than utilizing their outside option, given that this yields them a strictly positive payoff. Similarly, we note that if the cost of trading at the DEX is weakly less than the cost of trading at the CEX for all traders with $\delta_j \in \Delta^\star(f_D)$ and all traders that trade at the CEX receive a weakly positive payoff, then all traders that trade at the DEX must also receive a weakly positive payoff and therefore prefer trading at the DEX to utilizing their outside option.

B.2 Proof of Proposition 3.2

Proof. To solve for the equilibrium return $r_D^\star(f_D)$ we start by rearranging (8) to obtain

$$r_D = \frac{2 \cdot V \cdot f_D}{G(r_D)} \cdot \mathbb{E}[\sum_{j \in \mathcal{D}} |\delta_j|]$$

Then, noting that $\mathcal{D} = \{j : \delta_j \in [\bar{\delta}_-, \bar{\delta}_+]\}$ and $N \sim Poisson(1)$ implies that the expected number of trades are $Pr(\delta_j \in [\delta_-^\star, \delta_+^\star])$ and therefore

$$\mathbb{E}[\sum_{j \in \mathcal{D}} |\delta_j|] = Pr(\delta_j \in [\delta_-^\star, \delta_+^\star]) \cdot \mathbb{E}[|\delta_j| \mid \delta_j \in [\delta_-^\star, \delta_+^\star]] = \frac{(\delta_+^\star)^2 + (\delta_-^\star)^2}{4}$$

Therefore, using the fact that

$$(\delta_+^\star)^2 + (\delta_-^\star)^2 = (\frac{(f_C - f_D)^2}{(1 + (f_C - f_D))^2} + \frac{(f_C - f_D)^2}{(1 - (f_C - f_D))^2}) \cdot (\frac{I^\star(f_D)}{2V})^2$$

then implies

$$r_D = \frac{2 \cdot V \cdot f_D}{G(r_D)} \cdot \frac{1}{4}(\frac{(f_C - f_D)^2}{(1 + (f_C - f_D))^2} + \frac{(f_C - f_D)^2}{(1 - (f_C - f_D))^2}) \cdot (\frac{I^\star(f_D)}{2V})^2$$

and using $I^\star(f_D) = G(r_D^\star) = (r_D^\star)^{\frac{1}{\theta}}$, then after rearranging we obtain

$$r_D^\star = \left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{\theta}{\theta-1}}$$

Finally, substituting $r_D^\star$ into $I^\star(f_D) = (r_D^\star)^{\frac{1}{\theta}}$ yields

$$I^\star(f_D) = \left(\frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{1}{\theta-1}}$$

Finally, note that (4) implies that inventory must be deposited in the ratio of 1 USD per $\frac{1}{V}$ ETH and given that each investor is born with a unit of USD capital then they must split that by providing $\frac{1}{2}$ USD and $\frac{1}{2V}$ ETH to the DEX. Therefore, $I^\star_{USD} = \frac{1}{2}I^\star(f_D)$ and $I^\star_{ETH} = \frac{1}{2V}I^\star(f_D)$. □

B.3 Proof of Proposition 4.1

Proof. First note that

$$T^\star(f_D) = \frac{G(r_D^\star)}{f_D} \cdot r_D^\star = \left(\frac{f_D^\alpha(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}\right)^{\frac{\theta+1}{\theta-1}}$$

where $\alpha = \frac{2}{\theta+1} < 1$. Next, denote by $g(f_D)$ the following function

$$g(f_D) := \frac{f_D^\alpha(f_C - f_D)^2(1 + (f_C - f_D)^2)}{((1 - (f_C - f_D)^2)^2}$$

Note that $g(f_D) > 0$ for $f_D \in (0, f_C)$ and $g(0) = g(f_C) = 0$. Therefore, proving the result only requires showing that $g(f_D)$ has a unique local maximum on $[0, f_C]$. In order to do so, we will show that $log(g(f_D))$ has a unique local maximum on $(0, f_C)$ which implies that $g(f_D)$ must have a unique local maximum on $(0, f_C)$. In particular, we will do this by showing that $log(g(f_D))$ is strictly concave (i.e. $\frac{\partial^2}{\partial f_D^2} log(g(f_D)) < 0$). First, note that

$$log(g(f_D)) = \alpha log(f_D) + 2log(f_C - f_D) + log(1 + (f_C - f_D)^2) - 2log(1 - (f_C - f_D)^2)$$

so that

$$\frac{\partial}{\partial f_D} log(g(f_D)) = \frac{\alpha}{f_D} - \frac{2}{f_C - f_D} - \frac{2(f_C - f_D)}{1 + (f_C - f_D)^2} - \frac{4(f_C - f_D)}{1 - (f_C - f_D)^2}$$

and therefore

$$\frac{\partial^2}{\partial f_D^2} log(g(f_D)) = -\frac{\alpha}{f_D^2} - \frac{2}{(f_C - f_D)^2} + \frac{2(1 - (f_C - f_D)^2)}{(1 + (f_C - f_D)^2)^2} + \frac{4(1 + (f_C - f_D)^2)}{(1 - (f_C - f_D)^2)^2}$$

Finally, note that $-\frac{\alpha}{f_D^2} \leqslant 0$ and $(f_C - f_D)^2 \leqslant \gamma^2$, thereby implying:

$$\frac{\partial^2}{\partial f_D^2} log(g(f_D)) \leqslant \sup_{z: z \in [0, \gamma^2]} \Big(-\frac{2}{z} + \frac{2(1 - z)}{(1 + z)^2} + \frac{4(1 + z)}{(1 - z)^2} \Big) < 0 \tag{A.7}$$

where the last inequality follows from direct verification by using $\gamma = 25\%$.

□

B.4 Proof of Proposition 4.2

Proof. In order to prove this result, we will prove that both $\delta_+^\star(f_D)$ and $-\delta_-^\star(f_D)$ each have a unique local maximum. In order to do so, we will prove that $log(\delta_+^\star(f_D))$ and $log(-\delta_-^\star(f_D))$ are concave and therefore each have a unique local maximum, implying that $\delta_+^\star(f_D)$ and $-\delta_-^\star(f_D)$ each have a unique local maximum. In order to do so, first note that

$$log(\delta_+^\star(f_D)) = log(\frac{f_C - f_D}{1 + f_C - f_D}) + log(I^\star(f_D)) - log(2V)$$

and

$$log(-\delta_-^\star(f_D)) = log(\frac{f_C - f_D}{1 - (f_C - f_D)}) + log(I^\star(f_D)) - log(2V)$$

Next, we note that

$$\frac{\partial}{\partial f_D} log(\frac{f_C - f_D}{1 + f_C - f_D}) = \frac{-1}{f_C - f_D} + \frac{1}{1 + f_C - f_D}$$

$$\frac{\partial^2}{\partial f_D^2} log(\frac{f_C - f_D}{1 + f_C - f_D}) = \frac{-1}{(f_C - f_D)^2} + \frac{1}{(1 + f_C - f_D)^2} < 0$$

$$\frac{\partial}{\partial f_D} log(\frac{f_C - f_D}{1 - (f_C - f_D)}) = \frac{-1}{f_C - f_D} - \frac{1}{1 - (f_C - f_D)}$$

$$\frac{\partial^2}{\partial f_D^2} log(\frac{f_C - f_D}{1 - (f_C - f_D)}) = \frac{-1}{(f_C - f_D)^2} + \frac{1}{(1 - (f_C - f_D))^2} < 0$$

where the last inequality holds whenever $f_C - f_D < .5$ which is guaranteed to hold given that we have assumed that $f_C < \gamma \leqslant .25$.

Next, using (14) we can see that

$$\frac{\partial}{\partial f_D} log(I^\star(f_D)) = (\frac{1}{\theta - 1})(\frac{1}{f_D} - \frac{2}{f_C - f_D} - \frac{2(f_C - f_D)}{1 + (f_C - f_D)^2} - \frac{4(f_C - f_D)}{1 - (f_C - f_D)^2})$$

and therefore

$$\frac{\partial^2}{\partial f_D^2} log(I^\star(f_D)) = (\frac{1}{\theta - 1})(-\frac{1}{f_D^2} - \frac{2}{(f_C - f_D)^2} + \frac{2(1 - (f_C - f_D)^2)}{(1 + (f_C - f_D)^2)^2} + \frac{4(1 + (f_C - f_D)^2)}{(1 - (f_C - f_D)^2)^2})$$

Finally, note that $-\frac{1}{f_D^2} \leqslant 0$ and $(f_C - f_D)^2 \leqslant \gamma^2$, thereby implying:

$$\frac{\partial^2}{\partial f_D^2} log(I^\star(f_D)) \leqslant \frac{1}{\theta - 1} \sup_{z : z \in [0, \gamma^2]} \Big(-\frac{2}{z} + \frac{2(1 - z)}{(1 + z)^2} + \frac{4(1 + z)}{(1 - z)^2} \Big) < 0$$

where the last inequality follows from direct verification by using $\gamma = 25\%$.

What we have shown is that there exists $\tilde{f}_+$ and $\tilde{f}_-$ such that $\delta_+^\star(f_D)$ is increasing in f_D for $f_D \in [0, \tilde{f}_+)$ and decreasing in f_D for $f_D \in (\tilde{f}_+, f_C]$ while $\delta_-^\star(f_C)$ is decreasing for $f_D \in [0, \tilde{f}_-)$ and increasing for $f_D \in (\tilde{f}_-, f_C]$.

In order to conclude the proof, we apply Lemma A.1 to $\delta_+^\star(f)$ for $\beta = V \times (1 + f_C)$ and apply Lemma A.1 to $\delta_-^\star(f)$ for $\beta = V \times (1 - f_C)$ which yields:

$$\frac{\partial \Psi_D^\star}{\partial f_D} = -\delta_+^\star(f_D) \frac{\partial P_D^\star(f_D, \delta)}{\partial \delta} \frac{d\delta_+^\star}{df_D}, \qquad \frac{\partial \Psi_D^\star}{\partial f_D} = -\delta_-^\star(f_D) \frac{\partial P_D^\star(f_D, \delta)}{\partial \delta} \frac{d\delta_-^\star}{df_D}$$

Finally, we note that $\frac{\partial P_D^\star(f_D,\delta)}{\partial\delta} > 0$ coupled with $\delta_+^\star(f_D) > 0$ when combined with the aforementioned result on the sign of $\frac{d\delta_+^\star}{df_D}$ implies that $\frac{\partial\Psi_D^\star}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta_+^\star(f_D))} < 0$ for all $f_D \in (0, \hat{f}_+)$ and $\frac{\partial\Psi_D^\star}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta_+^\star(f_D))} > 0$ for all $f_D \in (\hat{f}_+, f_C)$. Similarly, $\frac{\partial P_D^\star(f_D,\delta)}{\partial\delta} > 0$ coupled with $\delta_-^\star(f_D) < 0$ when combined with the aforementioned result on the sign of $\frac{d\delta_+^\star}{df_D}$ implies that $\frac{\partial\Psi_D^\star}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta_-^\star(f_D))} < 0$ for all $f_D \in (0, \hat{f}_-)$ and $\frac{\partial\Psi_D^\star}{\partial f_D}|_{(f_D,\delta)=(f_D,\delta_-^\star(f_D))} > 0$ for all $f_D \in (\hat{f}_-, f_C)$.

□

B.5 Proof of Proposition 4.3

Proof. We first note that

$$P_D^\star(\delta) = \frac{VI^\star(f)}{I^\star(f) - 2V\delta}$$

and therefore

$$\lambda^\star(f,\delta) = 2V^2 \cdot \frac{I^\star(f)}{(I^\star(f) - 2V\delta)^2}$$

Next, note that

$$\frac{d}{df}\lambda^\star(f,\delta) = -2V^2 \cdot \frac{I^\star(f) + 2V\delta}{(I^\star(f) - 2V\delta)^3} \cdot \frac{\partial I^\star(f)}{\partial f}$$

Finally, we note that $\frac{I^\star(f)+2V\delta}{(I^\star(f)-2V\delta)^3} > 0$ for all feasible trades as $\delta < I_{ETH}^\star$ implies $2V\delta < I^\star(f_D)$. Further, we have shown in the proof of Proposition 4.2 that

$$\frac{\partial^2}{\partial f^2} log(I^\star(f)) < 0$$

and therefore $I^\star(f)$ has a unique local maximum, which combined with the fact that $I^\star(f) \geqslant 0$ for all $f \in [0, f_C]$ and $I^\star(0) = I^\star(f_C) = 0$, implies that there exists $\tilde{f} \in (0, f_C)$ such that $\frac{\partial I^\star(f)}{\partial f} > 0$ whenever $f < \tilde{f}$ and $\frac{\partial I^\star(f)}{\partial f} < 0$ whenever $f > \tilde{f}$. Hence, $\frac{d}{df}\lambda^\star(f,\delta) < 0$ for all $f \in (0, \tilde{f})$ and $\frac{d}{df}\lambda^\star(f,\delta) > 0$ for all $f \in (\tilde{f}, f_C)$. □

B.6 Proof of Proposition 4.4

Proof. First, we note that

$$\lambda^{\star}(I,\delta) = 2V^2 \frac{I}{I - 2V\delta}$$

thus,

$$\frac{d\lambda^{\star}(I,\delta)}{dI} = -2V^2 \frac{I + 2V\delta}{(I - 2V\delta)^3} < 0$$

for all possible inventory levels I and feasible trades $\delta < I_{ETH} = \frac{1}{2V} I$. □

B.7 Proof of Proposition 4.5

Proof. We have shown in the proof of Proposition 4.2 that $\frac{\partial^2}{\partial f^2} log(I^{\star}(f)) < 0$ for all $f \in (0, f_C)$ and therefore there is a unique critical point of $I^{\star}(f)$ over the interval $(0, f_C)$. Combining this with the fact that $I^{\star}(f) > 0$ for all $f \in (0, f_C)$ and $I^{\star}(0) = I^{\star}(f_C) = 0$ then implies that there must exists $\tilde{f}$ such that $f \in (0, \tilde{f})$ implies $\frac{\partial I^{\star}}{\partial f_D} > 0$ for $f_D \in (0, \tilde{f})$ and $\frac{\partial I^{\star}}{\partial f_D} < 0$ for all $f_D \in (\tilde{f}, f_C)$. □

B.8 Proof of Proposition 4.6

Proof. In order to prove this result, we simply note that $r_D^{\star}(f_D) = I^{\star}(f_D)^{\theta}$. Therefore, $\frac{dr_D^{\star}}{df_D} = \theta(I^{\star}(f_D))^{\theta-1} \frac{dI^{\star}}{df_D}$ and we know from Proposition 4.5 that there exists $\tilde{f} \in (0, f_C)$ such that $\frac{dI^{\star}}{df_D} > 0$ when $f \in (0, \tilde{f})$ while $\frac{dI^{\star}}{df_D} < 0$ when $f \in (\tilde{f}, f_C)$. Therefore, given that $I^{\star}(f) > 0$ for all $f \in (0, f_C)$ it must be the case that $\frac{dr_D^{\star}}{df_D} > 0$ for all $f \in (0, \tilde{f})$ and $\frac{dr_D^{\star}}{df_D} < 0$ for all $f \in (\tilde{f}, f_C)$. □

B.9 Proof of Proposition 5.1

Proof. The first order condition for (25) is given by

$$(f_D - f_C) \cdot |\delta| + \frac{(I^{\star})^2}{(I^{\star} - 2V\mu\delta)^2} \delta - \delta = 0 \tag{A.8}$$

Solving (A.8) for μ and accounting for the fact that $\mu \leqslant 1$ yields

$$\mu^\star(\delta) = \begin{cases} \frac{I^\star}{2V\delta}\left(1 - \sqrt{\frac{1}{1-(f_C-f_D)}}\right) & \delta \in [-1, \tilde{\delta}_-] \\ 1 & \delta \in [\tilde{\delta}_-, 0] \\ 1 & \delta \in [0, \tilde{\delta}_+) \\ \frac{I^\star}{2V\delta}\left(1 - \sqrt{\frac{1}{1+(f_C-f_D)}}\right) & \delta \in [\tilde{\delta}_+, 1] \end{cases}$$

where

$$\tilde{\delta}_+ = \frac{I^\star}{2V}\left(1 - \sqrt{\frac{1}{1+(f_C-f_D)}}\right) \qquad \text{and} \qquad \tilde{\delta}_- = \frac{I^\star}{2V}\left(1 - \sqrt{\frac{1}{1-(f_C-f_D)}}\right)$$

are the values of $\delta > 0$ such that $\mu(\tilde{\delta}_+) = 1$ and $\delta < 0$ such that $\mu(\tilde{\delta}_-) = 1$, respectively. Therefore, the equilibrium DEX trade volume is given by

$$\tilde{T}^\star(f_D) = \int_0^{\tilde{\delta}_+} \frac{x}{2} dx - \int_{\tilde{\delta}_-}^0 \frac{x}{2} dx + \int_{\tilde{\delta}_+}^1 \frac{I^\star}{4V}\left(1 - \sqrt{\frac{1}{1+(f_C-f_D)}}\right) dx - \int_{-1}^{\tilde{\delta}_-} \frac{I^\star}{4V}\left(1 - \sqrt{\frac{1}{1-(f_C-f_D)}}\right) dx$$

$$= \frac{1}{4}(\tilde{\delta}_+^2 + \tilde{\delta}_-^2) + (\frac{1-\tilde{\delta}_+}{2})\tilde{\delta}_+ - (\frac{1+\tilde{\delta}_-}{2})\tilde{\delta}_- = \frac{1}{2}(\tilde{\delta}_+ \cdot (1 - \frac{\tilde{\delta}_+}{2}) - \tilde{\delta}_- \cdot (1 + \frac{\tilde{\delta}_-}{2}))$$

□

B.10 Proof of Proposition 5.2

Proof. Denote by

$$\alpha_+ := \left(1 - \sqrt{\frac{1}{1+(f_C-f_D)}}\right) \qquad \text{and} \qquad \alpha_- =: \left(1 - \sqrt{\frac{1}{1-(f_C-f_D)}}\right)$$

Then, $\tilde{\delta}_+ = \frac{I^\star}{2V} \cdot \alpha_+$ and $\tilde{\delta}_- = \frac{I^\star}{2V} \cdot \alpha_-$. Next, using the fact that $I^\star = G(r_D^\star) = (r_D^\star)^{\frac{1}{\theta}}$ and $r_D^\star = \frac{f_D \cdot 2V}{I^\star} \cdot T^\star(f_D)$ we can see, after substituting and simplifying, that $r_D^\star(f_D)$ is implicitly defined by the equation

$$\phi(r_D, f_D) := r_D - f_D(\frac{1}{2}(\alpha_+ - \alpha_-) - \frac{1}{8V} r_D^{\frac{1}{\theta}}(\alpha_+^2 + \alpha_-^2)) = 0$$

Next, invoking the implicit function theorem, we know that

$$\frac{\partial}{\partial f_D} r_D^\star(f_D) = -(\frac{\partial \phi(r_D, f_D)}{\partial r})^{-1} \cdot \frac{\partial \phi(r_D, f_D)}{\partial f} =$$

$$\frac{1}{A(r_D^\star(f_D), f_D)} \Big(\frac{1}{2}(\alpha_+ - \alpha_-) - \frac{1}{8V} r_D^\star(f_D)^{\frac{1}{\theta}}(\alpha_+^2 + \alpha_-^2)) + f_D \cdot \big(\frac{1}{2}(\frac{\partial}{\partial f_D}[\alpha_+ - \alpha_-] - \frac{1}{8V} r_D^\star(f_D)^{\frac{1}{\theta}}(\frac{\partial}{\partial f_D}[\alpha_+^2 + \alpha_-^2]))) \Big)$$

where $A(r_D, f_D) := 1 + f_D \cdot \frac{1}{\theta} \cdot r_D^{\frac{1}{\theta}-1}(\alpha_+^2 + \alpha_-^2) > 0$. Then, using the fact that $r_D^\star(0) = 0$, we note that

$$\frac{\partial}{\partial f_D} r_D^\star(0) = \frac{1}{2}(\alpha_+ - \alpha_-) > 0$$

Therefore, $r_D^\star(f_D)$ is increasing in f_D when $f_D = 0$ which implies by continuity that there exists $\hat{f}$ such that $\frac{\partial}{\partial f_D} r_D^\star(f_D) > 0$ whenever $f_D < \hat{f}$. Thus, $I^\star(f_D) = (r_D^\star(f_D))^{\frac{1}{\theta}}$ implies that $\frac{\partial}{\partial f_D} I^\star(f_D) > 0$ whenever $f_D < \hat{f}$.

Next, we note that

$$\frac{\partial}{\partial f_D} \tilde{T}^\star(f_D) = \frac{1}{2}((1 - \tilde{\delta}_+(f_D)) \cdot \frac{\partial}{\partial f_D} \tilde{\delta}_+(f_D) - (1 + \tilde{\delta}_-(f_D)) \cdot \frac{\partial}{\partial f_D} \tilde{\delta}_-(f_D))$$

While

$$\frac{\partial}{\partial f_D} \tilde{\delta}_+(f_D) = \frac{\alpha_+(f_D)}{2V} \frac{\partial}{\partial f_D} I^\star(f_D) + \frac{I^\star(f_D)}{2V} \frac{\partial}{\partial f_D} \alpha_+(f_D)$$

and

$$\frac{\partial}{\partial f_D} \tilde{\delta}_-(f_D) = \frac{\alpha_-(f_D)}{2V} \frac{\partial}{\partial f_D} I^\star(f_D) + \frac{I^\star(f_D)}{2V} \frac{\partial}{\partial f_D} \alpha_-(f_D)$$

Finally, using the fact that $I^\star(0) = (r_D^\star(0))^{\frac{1}{\theta}} = 0$, $\lim_{f \to 0^+} \frac{\partial}{\partial f_D} \alpha_+(f_D) < +\infty$, $\lim_{f \to 0^+} \frac{\partial}{\partial f_D} \alpha_-(f_D) <$

$+\infty$, and whenever $f_D < f_C$ then $\alpha_+(f_D) > 0$ while $\alpha_-(f_D) < 0$ implies that

$$\frac{\partial}{\partial f_D}\tilde{\delta}_+(f_D)\Big|_{f_D=0} = \frac{\alpha_+(f_D)}{2V}\frac{\partial}{\partial f_D}I^\star(f_D)\Big|_{f_D=0} > 0$$

and

$$\frac{\partial}{\partial f_D}\tilde{\delta}_-(f_D)\Big|_{f_D=0} = \frac{\alpha_-(f_D)}{2V}\frac{\partial}{\partial f_D}I^\star(f_D)\Big|_{f_D=0} < 0$$

and therefore by continuity there must exist $\check{f}$ such that $\frac{\partial}{\partial f_D}\tilde{\delta}_+(f_D) > 0$ and $\frac{\partial}{\partial f_D}\tilde{\delta}_-(f_D) < 0$ whenever $f_D < \check{f}$. Finally, $(1 - \tilde{\delta}_+(f_D)) > 0$ and $(1 + \tilde{\delta}_-(f_D)) > 0$ whenever $f_D < f_C$ implies that $\frac{\partial}{\partial f_D}\tilde{T}^\star(f_D) > 0$ whenever $f_D < \check{f}$ and we have proven the result. □

B.11 Proof of Proposition 5.3

Proof. The FOC for $f_C^\star(f_D)$ implies that for any f_D we have

$$(1 - \frac{(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2}{2}) - \frac{f_C}{2}\cdot(\frac{\partial}{\partial f_C}[(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2]) = 0 \quad \text{(A.9)}$$

We will first show that $f_C^\star(0) = \gamma$. In order to do so, note that

$$\frac{1}{2}\cdot\frac{\partial}{\partial f_C}[(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2] = \delta_+^\star(f_C, f_D)\frac{\partial}{\partial f_C}[\delta_+^\star(f_C, f_D)] + \delta_-^\star(f_C, f_D)\frac{\partial}{\partial f_C}[\delta_-^\star(f_C, f_D)]$$

$$\frac{\partial}{\partial f_C}\delta_+^\star(f_C, f_D) = \frac{1}{(1 + (f_C - f_D))^2}\frac{I^\star(f_C, f_D)}{2V} + \frac{f_C - f_D}{1 + (f_C - f_D)}\frac{1}{2V}\frac{\partial}{\partial f_C}[I^\star(f_C, f_D)]$$

while

$$\frac{\partial}{\partial f_C}\delta_-^\star(f_C, f_D) = -\left(\frac{1}{(1 - (f_C - f_D))^2}\frac{I^\star(f_C, f_D)}{2V} + \frac{f_C - f_D}{1 - (f_C - f_D)}\frac{1}{2V}\frac{\partial}{\partial f_C}[I^\star(f_C, f_D)]\right)$$

Further, noting that

$$\frac{\partial}{\partial f_C}I^\star(f_C, f_D) = \frac{1}{\theta - 1}\cdot(g(f_C, f_D))^{\frac{1}{\theta-1}-1}\cdot\frac{\partial}{\partial f_C}g(f_C, f_D)$$

where

$$g(f_C, f_D) = \frac{f_D(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2}$$

and therefore after rearranging

$$\frac{\partial}{\partial f_C} g(f_C, f_D) = \frac{f_D(f_C - f_D)(1 + 3(f_C - f_D)^2)}{4V(1 - (f_C - f_D)^2)^3}$$

Thus, again after rearranging we obtain

$$\frac{\partial}{\partial f_C} I^\star(f_C, f_D) =$$

$$\frac{1}{\theta - 1} f_D^{\frac{1}{\theta-1}} \left(\frac{(f_C - f_D)^2(1 + (f_C - f_D)^2)}{8V((1 - (f_C - f_D)^2)^2} \right)^{\frac{2-\theta}{\theta-1}} \left(\frac{(f_C - f_D)(((1 + (f_C - f_D)^2)^2 + (f_C - f_D)(1 - (f_C - f_D)^2))}{4V(1 - (f_C - f_D)^2)^3} \right)$$

Therefore, it can be seen that $\frac{\partial}{\partial f_C} I^\star(f_C, f_D)|_{f_D=0} = 0$ and $I^\star(f_C, 0) = 0$. Hence, $\frac{\partial}{\partial f_C} \delta_+^\star(f_C, f_D)|_{f_D=0} = 0 = \frac{\partial}{\partial f_C} \delta_-^\star(f_C, f_D)|_{f_D=0}$. In this case, the left hand side of the FOC evaluated at $f_D = 0$ is equal to 1 and therefore, the only solution to (26) is the corner solution $f_C^\star(0) = \gamma$.

In order to prove the main result, we note that $f_C^\star(0) = \gamma$ implies that there exists $\check{f}$ such that $f_C^\star(f_D) = \gamma$ for all $f_D < \check{f}$. Namely, by the continuity of the functions $\delta_+^\star(f_C, f_D)$ and $\delta_-^\star(f_C, f_D)$ and their derivatives and the fact that

$$(1 - \frac{(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2}{2}) - \frac{f_C}{2} \cdot (\frac{\partial}{\partial f_C}[(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2]) \Big|_{(f_C, f_D) = (\gamma, 0)} = 1 > 0$$

implies that there must exist $\check{f} > 0$ such that for any $f'_D < \check{f}$ we have

$$(1 - \frac{(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2}{2})) - \frac{f_C}{2} \cdot (\frac{\partial}{\partial f_C}[(\delta_+^\star(f_C, f_D))^2 + (\delta_-^\star(f_C, f_D))^2]) \Big|_{(f_C, f_D) = (\gamma, f'_D)} > 0$$

and therefore $f_C^\star(f'_D) = \gamma$. Hence, given that $f_C^\star(f_D)$ is constant on the interval $[0, \check{f})$ then it must be the case that equilibrium DEX trading volume $T^\star(f_C^\star(f_D), f_D)$ is increasing whenever $f_D \in [0, \min\{\check{f}, \tilde{f}\})$ where $\tilde{f}$ is the threshold from Proposition 4.1.

□