Kyle Meets Friedman: Transforming a Trading Model into a Consumption Model*

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August 27, 2024

*This paper was previously circulated under the title "To Dissimulate or Not to Dissimulate? Insider Trading When Anticipating Future Information." We thank Kerry Back, Snehal Banerjee, Scott Condie, Vicente Cuñat, Vyacheslav (Slava) Fos, Brett Green, Allaudeen Hameed, Chong Huang, Mariana Khapko, Nitish Kumar, Lars Lefgren, Kai Li, Tao Li, Xuewen Liu, Alexey Malakhov, Jordi Mondria, Andy Naranjo, Uday Rajan, Jay Ritter, Xianwen Shi, Mark H. Showalter, Yuehua Tang, Christian vom Lehn, Baolian Wang, Gaowang Wang, Jieran Wu, Zhiqiang Ye, Jianfeng Yu, Bart Zhou Yueshen, Hongda Zhong, Marius Zoican, Junyuan Zou, Pavel Zryumov, audiences at the 2024 China International Conference in Finance, 2024 City University of Hong Kong International Finance Conference, and 2024 CUFE International Summer Conference of Finance, and seminar participants at Brigham Young University, Jinan University, Nanjing University, Shandong University, Shenzhen University, Sun Yat-Sen University, University of Chinese Academy of Sciences, University of Arkansas, University of Florida, University of Macau, University of Washington, Xiamen University, and Zhejiang University for constructive comments that have significantly improved the paper.

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Abstract

We analyze a model of a monopolistic informed investor who receives private information sequentially and faces a post-trading disclosure requirement. We show that this trading model can be transformed into a fictitious consumption-saving model with a borrowing constraint. Hence, insights from the consumption-saving literature can be adapted for the trading model. For example, analogous to the insights from the permanent income hypothesis, the informed investor "saves" more of his current information when expecting less future information advantage ("saving for rainy days") or more uncertainty about it ("precautionary saving") and smooths his information "usage" over time ("consumption smoothing").

JEL Classification Numbers: C72, D82, G14.

Keywords: informed investor trading; Dissimulation; Mixed strategy; Information arrival.

1 Introduction

How asset prices distill investors' information is a central question in economics. Broadly speaking, an investor's information can be revealed in two ways. The first is through the market mechanism. For example, as in Kyle (1985), the informed investor's trades move asset prices and partially reveals his information. The second is that the informed investor's trade size, which is either disclosed due to regulation or detected by other investors, further reveals his information. In this paper, to capture both aspects of information revelation, we analyze a dynamic model of a monopolistic informed investor who receives private information sequentially and faces a post-trading disclosure requirement.

Our main result is to demonstrate that characterizing the equilibrium of this trading model can be reduced to, somewhat unexpectedly, solving a fictitious consumption-saving model. This mathematical equivalence suggests that we can borrow the ideas in the well-established consumption-saving literature to apply them in the trading model. We illustrate this by showing that various insights from the permanent income hypothesis (Friedman, 1957), can be adapted to offer new perspectives on the dynamic trading model.

Specifically, we analyze a model that includes both Kyle (1985) and Huddart et al. (2001) as special cases. As in Kyle (1985), we consider an N-period economy with one risky asset and one monopolistic risk neutral informed investor. The risky asset is a claim to an uncertain cash flow in the final period. Each period, the informed investor receives a private signal about the risky asset's liquidation value. The informed investor trades against noise traders and a risk-neutral market maker sets the price. After the transaction each period, the informed investor's trade is, potentially imperfectly, revealed. This could be due to the post-trade disclosure policy modeled in Huddart et al. (2001), where the informed investor perfectly disclose his trade size each period. This could also be due to regulatory filings (such as the 13f filings by mutual funds and hedge funds), which provide imperfect signals about the informed investor's transactions. Finally, as modeled in (Yang and Zhu (2020)), this could also reflect the fact that some investors (such as high frequency traders) may have the technology to partially detect the informed investor's trades.

Our first contribution is methodological. We show that characterizing the equilibrium in this dynamic trading model is mathematically equivalent to solving a fictitious consumption-saving model. In our baseline model, where the informed investor fully discloses his transaction after his trade each period, we reduce the characterization of the equilibrium to solving the informed investor's optimal information "usage" problem, which can be transformed into a fictitious consumption-savings problem of an agent with a borrowing constraint.

The transformation is obtained by simply relabelling variables. In the trading model, the informed investor receives a private signal each period and decides on how much information to "utilize" in the current period and how much to "dissimulate" for future use. By relabeling the informed investor's private signal as "income," the utilized information as "consumption," the dissimulated information as "savings," and the expected trading profit as the "utility from consumption," we transform the trading model into a consumption-saving model. One notable feature of the trading model is that the informed investor can dissimulate his current information for future use but cannot transfer his future information to utilize today. This asymmetry manifests itself as a borrowing constraint in the consumption-saving model: the agent can save his current wealth to consume in the future but cannot borrow against his future income to consume today.

This mathematical equivalence implies that the insights from the consumption-saving literature can be adapted to our dynamic trading model and hence shed new light on informed trading and its role in incorporating information into asset prices. To illustrate this, we consider three prominent ideas inspired by the permanent income hypothesis of Friedman (1957) and show how they can be adapted to our dynamic trading model. In the discussion below, we focus on the implications of those ideas on the informed investor's trading strategies. It is easy to see that the trading strategies determine how much of the private information is incorporated into asset prices.

The first idea is the notion of "saving for rainy days." When anticipating times of scarcity, the agent would consume less today to save more of his wealth for the future. This intuition manifests itself in the trading model as follows. In periods with abundant private information, when anticipating less private information in the future, the informed investor would save his current private information for future use. In contrast, when anticipating abundant private information in the future, the informed investor utilizes all his current information.

These results have direct implications on the informed investor's trading strategy. In the former case, similar to the result in Huddart et al. (2001), the informed investor adopts a mixed strategy to dissimulate his current private information. In the latter case, however, he adopts a pure strategy to fully utilize all his private information. These results extend and sharpen the insight in Huddart et al. (2001), who focus on the special case in which the informed investor possesses all his private information in the first period. Anticipating no more private information in the future, the informed investor always adopt a mixed strategy to dissimulate some of his current information. In our more generalized setup, the informed investor's strategy depends on his expectation of his future private information. If future information is sufficiently abundant, he would adopt a pure strategy to fully utilize his current private information.

The second, and related, idea is the notion of "consumption smoothing": the agent tries to minimize the fluctuation in his consumption over time. The counterpart of this insight in our dynamic trading model is "information-usage smoothing." We formally show that the informed investor's objective is to minimize the time variation in the amount of his private information revealed through trading and disclosure each period.

In the consumption-saving model, perfect consumption smoothing is not always feasible. The agent can save in periods of abundance but can only consume his current wealth in periods of scarcity because, as noted earlier, he cannot borrow against his future income. Parallel to this intuition, in our trading model, the informed investor would like to utilize the same amount of information each period (i.e., "walk down the demand curve" and have the same price impact each period). However, this is not always possible because he can transfer his current information to future periods but not the other way around. We show that, in the equilibrium of the dynamic trading model, the informed investor minimizes the time variation of his information usage over time, given the timing of his private information. Related, this result implies that, in equilibrium, the informed investor smooths his price impact by minimizing the variation of his price impact over time. In special cases, e.g. in the model of Huddart et al. (2001), the informed investor's price impact is a constant over time. More generally, however, despite the insider's effort to smooth his price impact, it varies over time due to the timing of the insider's private information.

The third idea explored in our analysis is the notion of "precautionary saving," which suggests that an agent would save more today if he anticipates more income uncertainty in the future. To analyze the implications of this insight on trading, we extend our baseline model in introduce uncertainty to the size of the informed investor's future information advantage. We show that, similar to the result on saving for rainy days in the baseline model, the investor would utilize less of his current information if he expects less information advantage in the future on average. Moreover, as suggested by the notion of precautionary saving, the informed investor would save more of his current private information for future use, if there is more uncertainty on how much private information he would receive in the future.

We extend our baseline model, where the disclosure fully reveals the informed investor's trades, to analyze the case of imperfect disclosure.[1] That is, after the transaction each period, the market maker observes a signal, which is the informed investor's trade size plus a noise. This extension accounts for some important scenarios, such as the periodic regulatory filings by mutual funds and hedge funds, which do not fully disclose their transactions. Moreover,

[1]We also consider three other extensions of our baseline model. In one extension, the intensity of noise trading varies over time. In another, the informed investor's private information is like a perishable good: it may become public knowledge in each period. Finally, we analyze the limiting case, in which the trading frequency approaches infinity and hence the model converges to the continuous-time limit. We show that, with minimal adjustments, the trading model in each case can be transformed into a consumption-saving model.

this extended model includes both Kyle (1985) and Huddart et al. (2001) as limiting cases. As the variance of the noise in the signal approaches zero, the disclosure becomes perfect and the extended setting converges to our baseline model, which is a generalization of the model in Huddart et al. (2001). In the other limiting case, as the variance of the noise approaches infinity, the disclosure reveals no information and the setting converges to a generalized version of Kyle (1985).

We characterize the equilibrium in this generalized model and show that when the noise in the disclosure is higher than a certain threshold, the informed investor always adopt a pure strategy. The intuition is that due to the high noise level in the disclosure, the informed investor does not need to add noise to his trade to hide his information. Note that the noise in the disclosure is infinity in Kyle (1985). Hence, the informed investor never needs to hide his information and always adopts a pure strategy.

Finally, to interpret the informed investor's mixed strategy, we consider an alternative setup whereby the informed investor commits to a strategy of adding noise to his demand and deliberately chooses the variance of the noise to optimize his trading profits. The rest of the setup remains identical to our baseline model. We find that the equilibrium in this variation model with commitment is identical to that in our baseline model. That is, the implications on whether to dissimulate and how much noise to add to demand are the same across the two models. Hence, the mixed strategy in our baseline model can be interpreted "as if" the informed investor actively chooses how much noise to add in each period.

Our paper adds to the literature on informed trading by corporate informed investors and institutional investors in financial markets. This literature is voluminous and so we discuss most related studies, organized according to the two important features in our setting: mixed strategies and sequential information arrivals. In terms of the former, our paper is most related to Huddart et al. (2001), who is the first study to demonstrate that, in a Kyle (1985) model, the informed investor plays a mixed strategy when his trade is mandated to be disclosed. Yang and Zhu (2020) investigate the behavior of an informed investor who leaks a signal about the demand to back-runners. The informed investor can choose between the pure and mixed strategies, and is more likely to choose the latter if the information leakage is more severe. Back and Baruch (2004) analyze a variant of Glosten and Milgrom (1985) model and show that an informed investor would adopt a mixed strategy by randomizing over orders to buy, sell, and wait. Our paper complements these studies in several ways. First, methodology wise, we transform the equilibrium characterization into a simple consumption-saving problem. Second, we extend and sharpen the results in Huddart et al. (2001) and characterize the condition for dissimulation in equilibrium. Third, we provide an interpretation of the mixed strategy played in a Kyle (1985) model and show that commitment has no value in the linear equilibrium.

The second feature—sequential information arrivals—is relevant to many settings in practice. The information acquisition process may result from the dynamics of informational events—such as IPOs (e.g., Welch, 1992; Lowry and Schwert, 2002), mergers (e.g. Ferreira and Laux, 2007) and acquisitions (e.g., Denis and Macias, 2013)—or the dynamics of research and learning activities (e.g. Banerjee and Breon-Drish, 2022; Johannes et al., 2014). Numerous studies examine the effects of sequential information acquisition (e.g. Bernhardt and Miao, 2004; Caldentey and Stacchetti, 2010; Chau and Vayanos, 2008; Foucault et al., 2016; Sastry and Thompson, 2019). Disclosure requirement and the ensuing mixed strategy distinguish our analysis from those studies.

2 Model

Our model is a generalization of Kyle (1985) and Huddart et al. (2001). In these two classic studies, the informed investor obtains all his private information about the asset's liquidation value in the initial period and receives no further private information afterwards. In contrast, our analysis focuses on the sequential arrival of private information.

The economy has one risky asset and lasts for N periods, denoted by $n = 1, ..., N$. The risky asset has a liquidation value at the final period N, which is denoted as F and has an ex ante distribution of $\mathcal{N}(0, \sigma_F^2)$ with $\sigma_F > 0$. The market is populated by an informed investor, a continuum of noise traders, and a market maker. Everyone is risk neutral. The informed investor submits a market order to trade x_n shares in period n. The market maker sets the asset price to break even. The time line of events in period n is summarized in Figure 1.

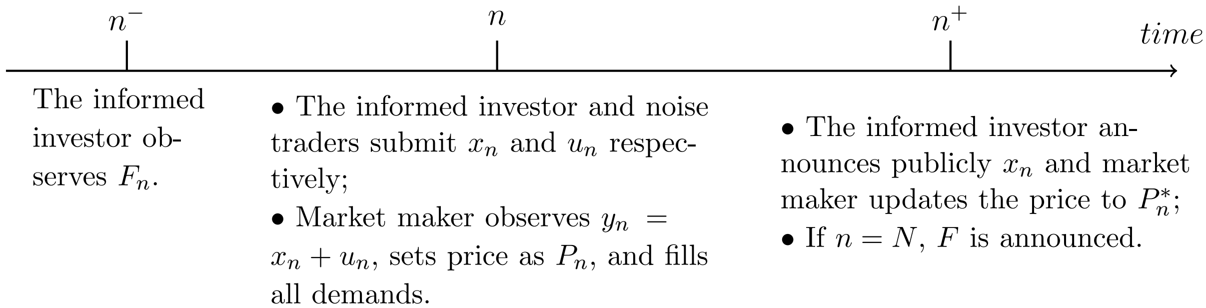


Figure 1. The timeline of the events in period n.

The informed investor observes private information about the asset's liquidation value and, critical to our analysis, his information arrives over time. To capture this sequential learning feature, we divide the asset's liquidation value F into N elements as follows:

$$F = \sum_{n=1}^{N} F_n,$$

where $F_n \sim \mathcal{N}(0, \sigma_{F_n}^2)$ with $\sigma_{F_n} \geq 0$ and is serially independent across n. One can think of the asset as N projects. F_i represents the earnings from project i, which is realized in period i. The assumption of independence is without loss of generality because if the earnings are correlated across projects, we can orthognalize and redefine them to ensure independence over time.

By construction, $\sigma_F^2 = \sum_{n=1}^{N} \sigma_{F_n}^2$. As shown by Figure 1, in each period n, for $n = 1, ..., N$, the informed investor observes F_n at time n^-, which is before the trading time of the period, as indicated by time n in Figure 1. Note that F_n is long-lived information in the sense that it affects the asset's final liquidation value and never becomes public before the final period.

Moreover, the model in Huddart et al. (2001) can be viewed as a special case of our model with $\sigma^2_{F_1} = \sigma^2_F$ and $\sigma^2_{F_n} = 0$ for $n > 1$, which implies that the informed investor obtains all his private information in the first period.

At time n, the trading time of period n, noise traders have an aggregate demand of u_n shares, with $u_n \sim \mathcal{N}(0, \sigma^2_u)$ (with $\sigma_u > 0$) and u_n is independent across n and from F_n. As standard in the literature, noise trading provides the randomness to hide the informed investor's trade from the market maker. Upon receiving the aggregate order flow from the informed investor and noise traders, $y_n = x_n + u_n$, the market maker sets the price P_n to his expectation of the liquidation value to execute the trade. As in Huddart et al. (2001), the informed investor is required to disclose his trading ex post. That is, after the transaction in period n but before the next trading period $n+1$ (denoted by n^+ in Figure 1), the informed investor publicly discloses his trade size x_n.

Remark. Under this assumption, disclosure perfectly reveals the informed investor's trade x_n. This assumption is made to simply the analysis. In Section ****, we relax this assumption so that after the disclosure, other investors observe $x_n + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$ and is independent from all other variables. This assumption captures the fact that disclosure is infrequent. For example, mutual funds and hedge funds only need to disclosure their positions each quarter. Naturally, those disclosures do not precisely reveal their trades. This assumption also represents the fact some investors, such as high-frequency traders, can estimate the trade size of informed investors (Yang and Zhu (2020)). This generalized setting includes both Kyle (1985) and HHL(2001) as special cases. As σ approaches 0, the disclosure perfectly reveals x_n and hence the setting converges to that in Huddart et al. (2001). Similarly, as σ approaches ∞, the disclosure does not reveal any information about x_n and hence the setting converges to that in Kyle (1985).

In response to the informed investor's disclosure, the market maker adjusts her break-even price from P_n to P^*_n. Specifically, in period n, the market maker's information set is $\mathcal{I}^M_n \equiv \{y_1, ..., y_n, x_1, ..., x_{n-1}\}$ at the time of trading and is $\mathcal{I}^M_{n+} \equiv \{y_1, ..., y_n, x_1, ..., x_n\}$ after the informed investor's disclosure of his trade x_n. At the time of transaction, the market maker sets the execution price to

$$P_n = E[F|\mathcal{I}_n^M]. \tag{1}$$

After the informed investor's disclosure, the market maker adjusts the asset price to

$$P_n^* = E[F|\mathcal{I}_{n+}^M]. \tag{2}$$

When computing prices P_n and P_n^* in equations (1) and (2), the market maker takes as given the informed investor's trading strategies.

The informed investor's information set in period n is $\mathcal{I}_n^I \equiv \{F_1, ..., F_n, P_1, ..., P_{n-1}, P_1^*, ..., P_{n-1}^*\}$. The informed investor maximizes his expected trading profits:

$$\max_{x_n,...,x_N} E\left[\sum_{j=n}^{N} \pi_j|\mathcal{I}_n^I\right], \tag{3}$$

where $\pi_j \equiv x_j(F - P_j)$ is his trading profit directly attributable to his period-j trade. In computing his optimal trade in (3), the informed investor takes the market maker's pricing rules as given. Following Kyle (1985), we define an equilibrium as follows:

Definition 1. An equilibrium is defined as trading strategies and pricing rules (x_n, P_n, P_n^*), for $n = 1, ..., N$, such that at period n: (a) the market maker sets prices according to (1) and (2), taking the informed investor's trading strategies as given; and (b) the informed investor's strategy $\{x_n,...x_N\}$ solves (3), taking the market maker's pricing rules as given.

3 Equilibrium

In Section 3.1, we follow Kyle (1985) and Huddart et al. (2001) to conjecture and verify a linear equilibrium. We then show that characterizing this equilibrium is mathematically equivalent to solving a consumption-saving model in Section 3.2.

3.1 Equilibrium Characterization

We follow Kyle (1985) and Huddart et al. (2001) and consider linear equilibria. That is, in period n, for $n = 1, ..., N$, the trading strategies and the pricing rules are given by

$$x_n = \beta_n(\sum_{i=1}^{n} F_i - P^*_{n-1}) + z_n, \tag{4}$$

$$P_n = P^*_{n-1} + \lambda_n y_n, \tag{5}$$

$$P^*_n = P^*_{n-1} + \gamma_n x_n, \tag{6}$$

where $z_n \sim \mathcal{N}(0, \sigma^2_{z_n})$, $P^*_0 = 0$, and the parameters $\{\beta_n,\ \lambda_n,\ \gamma_n,\ \sigma_{z_n}\}$ will be determined in equilibrium.

Intuitively, $\sum_{i=1}^{n} F_i - P^*_{n-1}$ is the difference between the informed investor's expected liquidation value computed based on his private information and the asset price determined based on the public information. The conjecture in equation (4) is that the informed investor's trade is linear in this difference. Moreover, as pointed out in Huddart et al. (2001), due to the disclosure requirement, the informed investor may play a mixed strategy, i.e., add noise to his trade to dissimulate his private information. Mathematically, in period n, the informed investor adopts a mixed strategy if $\sigma_{z_n} > 0$ and a pure strategy if $\sigma_{z_n} = 0$. The pricing function in (5) reflects that the market maker adjusts the execution price in period n based on the aggregate order flow y_n. After the informed investor's disclosure, as shown in (6), the market maker further adjusts the asset price based on the disclosed informed investor order flow x_n.

One key decision that the informed investor makes is how much information to "utilize" each period. Specifically, let k_n^2 be the variance of the stock price change in period n:

$$k_n^2 \equiv Var(P_n^* - P_{n-1}^*). \tag{7}$$

Empirically, k_n corresponds to return volatility. Nonetheless, in our analysis, we focus more on its interpretation from an information perspective. The variance of the price change is driven by the new information revealed by the informed investor's trade in period n. If the informed investor trades more aggressively in period n, then $P_n^* - P_{n-1}^*$ reveals more new information and its variance k_n^2 is higher. Hence, k_n^2 can be interpreted as the amount of private information utilized by the informed investor in period n. Moreover, the total amount of private information utilized by the informed investor during the first n periods is simply $\sum_{i=1}^{n} k_i^2$. This is because, in equilibrium, price process is set by the risk-neutral market maker and hence is a martingale and therefore price changes are independent over time.

We use Σ_n to denote the information advantage the informed investor has accumulated after his disclosure in period n. That is, Σ_n is the total amount of the private information that is possessed by the informed investor but not yet revealed after the disclosure of his trade at period n:

$$\Sigma_n \equiv Var\left(\sum_{i=1}^{n} F_i | P_1^*, ..., P_n^*\right). \tag{8}$$

This definition implies

$$\Sigma_n = \sum_{i=1}^{n} \sigma_{F_i}^2 - \sum_{i=1}^{n} k_i^2, \tag{9}$$

because the first term on the right hand side is the total amount of private information that the informed investor obtained in the first n periods while the second term, as noted earlier, is the total amount utilized.

The informed investor faces constraints on how much private information he can utilize:

$$k_n^2 \leq \Sigma_{n-1} + \sigma_{F_n}^2, \tag{10}$$

for $n = 1, ..., N$, with $\Sigma_0 \equiv 0$. Intuitively, Σ_{n-1} is the unused private information the informed investor has accumulated at the beginning of period n. He then observes private signal F_n (which has a variance of $\sigma_{F_n}^2$) and so his balance of unused information increases by $\sigma_{F_n}^2$. Hence, the total amount of private information available to the informed investor in period n is $\Sigma_{n-1} + \sigma_{F_n}^2$. Moreover, equation (9) implies that the dynamic of Σ_n is given by

$$\Sigma_n = \Sigma_{n-1} + \sigma_{F_n}^2 - k_n^2, \tag{11}$$

for $n = 1, ..., N$. Since the amount of private information revealed by the informed investor's trading and disclosure is k_n^2, the unrevealed private information amount at the end of period n, Σ_n, is given by (11). The following theorem characterizes the linear equilibrium.

Theorem 1. *There exists a unique linear equilibrium in which the informed investor's trading strategies and the market maker's pricing rules are given by equations (4)–(6) with parameters characterized as follows: for* $n = 1, ..., N$*,*

$$\beta_n = \frac{k_n \sigma_u}{\Sigma_n + k_n^2}, \tag{12}$$

$$\lambda_n = \frac{k_n}{2\sigma_u}, \tag{13}$$

$$\gamma_n = \frac{k_n}{\sigma_u}, \tag{14}$$

$$\sigma_{z_n}^2 = \frac{\Sigma_n}{\Sigma_n + k_n^2} \sigma_u^2, \tag{15}$$

where Σ_n *is given by (9) and* $\{k_1, \cdots, k_N\}$ *are the unique non-negative solution to the fol-*

lowing maximization problem:

$$\max_{k_1,\cdots,k_N} \sum_{i=1}^{N} k_i \tag{16}$$

$$\textit{subject to} \quad \sum_{i=1}^{n} k_i^2 \le \sum_{i=1}^{n} \sigma_{F_i}^2, \;\; \textit{for } n = 1, ..., N. \tag{17}$$

The above theorem shows that all equilibrium parameter values are pinned down by equations (12)–(15) once the values of $\{k_1, \cdots, k_N\}$ are determined. The values of $\{k_1, \cdots, k_N\}$ can be computed from the constrained maximization problem in (16) and (17).

The objective in (16) is essentially to maximize the sum of the informed investor's expected trading profits across N periods. Specifically, since the market maker is risk neutral and breaks even, the informed investor's expected profits in period n must be equal to the noise trader's expected loss in period n: $E[\pi_n] = \lambda_n \sigma_u^2$. Combined with the expression of λ_n in equation (13) in Theorem 1, we obtain $E[\pi_n] = k_n \sigma_u / 2$. Hence, the objective in (16) is equivalent to maximizing the informed investor's expected total trading profits.

Conditions in (17) are the informed investor's "information budget constraint." In each period n, the total amount of information the informed investor has utilized during periods 1 through n should be no more than the total amount of information he has received by then. These constraints are equivalent to those in (10) and (11). Specifically, in period n, the private information available to the informed investor is the information accumulated from the past Σ_{n-1} and the information obtained in the current period $\sigma_{F_n}^2$. If the informed investor chooses not to utilize all his private information ($k_n^2 < \Sigma_{n-1} + \sigma_{F_n}^2$), the unused information would be "saved" for future use. The essence of constraints in (10) and (11) is that the informed investor can transfer his private information across time, but in an "asymmetrical" way: he can "save" his private information for future use but cannot "borrow" his future private information to utilize today.

3.2 Equivalence to a Consumption-Saving Model

The previous discussion hints at the striking similarity between the informed investor's maximization problem in (16)-(17) and a dynamic consumption-saving problem. We show in this section that there is indeed an equivalence between the two problems. Specifically, we will define an N-period consumption-saving problem and show that it is mathematically equivalent to the informed investor's optimization problem defined by (16)-(17).

To avoid confusion in our discussion, we will refer to the agent in the consumption-saving problem as a "consumer," which is to contrast with the "informed investor" in our model in Section 2. Intuitively, for a given period n, the informed investor receiving his private information (which is measured by $\sigma^2_{F_n}$) corresponds naturally to a consumer receiving an "income." Similarly, the informed investor's information usage k_n^2 corresponds to the consumer's "consumption." As noted earlier, the informed investor can save his current private information for future use but cannot borrow his future private information to use today. This feature corresponds to a friction the consumer faces: he can save but cannot borrow. Guided by the intuition above, we can transform the maximization problem in (16)-(17) into a dynamic consumption-saving problem with a borrowing constraint.

Specifically, let Y_n, C_n, and S_n denote the consumer's income, consumption, and savings in period n, respectively, for $n = 1, ..., N$. The consumer cannot borrow but can save his income for future consumption:

$$C_n \leq S_{n-1} + Y_n, \tag{18}$$

$$S_n = S_{n-1} + Y_n - C_n, \tag{19}$$

with $S_0 = 0$. The constraint in (18) shows that the consumption in a given period cannot be more than the consumer's current income and savings. That is, the consumer cannot borrow against his future income to consume. Equation (19) shows that the unconsumed resource $S_{n-1} + Y_n - C_n$ becomes the savings for the next period. Note that this equation implies that the interest rate is zero.

The above description suggests that the consumer's income and consumption (Y_n and C_n) correspond to $\sigma^2_{F_n}$ and k^2_n in our trading-game model in Section 2. Comparison between equations (11) and (19) shows that the consumer's savings S_n corresponds to Σ_n in our trading game. Finally, the correspondence between k^2_n and C_n suggests that k_n corresponds to $\sqrt{C_n}$. Hence, the informed investor's objective function (16) corresponds to the consumer maximizing his utility from consumption over time, where his utility function is $u(C_n) = \sqrt{C_n}$. Therefore, we can transform the maximization problem in (16)-(17) into the following consumption-saving problem:

$$\max_{\{C_n,\cdots,C_N\}} \sum_{i=1}^{N} u(C_i), \tag{20}$$

subject to (18) and (19).

Table 1 summarizes the correspondences between the variables in the trading game and those in the consumption-saving problem and the following proposition establishes the mathematical equivalence between the two models.

Table 1. Transform the trading game into a consumption-saving problem.

	Trading game	Consumption-saving problem
Variable:		
	Information leakage k^2_n	Consumption C_n
	Expected profit $k_n\sigma_u/2$	Utility $\sqrt{C_n}$
	Information endowment $\sigma^2_{F_n}$	Income Y_n
	Unused information amount Σ_n	Saving S_n
Friction:		
	Asymmetric information transfer	Borrowing constraint
	$k^2_n \leq \Sigma_{n-1} + \sigma^2_{F_n}$	$C_n \leq S_{n-1} + Y_n$
	• If $k^2_n < \Sigma_{n-1} + \sigma^2_{F_n}$: "mixed"	• $C_n < S_{n-1} + Y_n$: "saving"
	• If $k^2_n = \Sigma_{n-1} + \sigma^2_{F_n}$: "pure"	• $C_n = S_{n-1} + Y_n$: "consuming all"

Theorem 2. *The maximization problem of (16) under constraints (10) and (11) is equivalent to the consumption-saving problem in (20) subject to (18) and (19), if we relabel* $\sigma^2_{F_n}$, k^2_n,

and Σ_n *as* Y_n*,* C_n*, and* S_n*, respectively.*

The above proposition establishes the mathematical equivalence between our trading game in Section 2 and a standard dynamic consumption-saving problem where the consumer has a constant relative risk aversion utility function with a relative risk aversion coefficient of 1/2 and face a borrowing constraint.

4 Kyle Meets Friedman

The transformation in Theorem 2 suggests that we can directly borrow the insights in the consumption-saving literature to guide our analysis of the trading model. The permanent income hypothesis of Friedman (1957), arguably the most important insight in the consumption-saving literature, highlights the role of expectations in shaping the consumption behavior. In this section, we consider three prominent ideas inspired by this hypothesis and show how they can be adapted to shed lights on our dynamic trading model.

4.1 Saving for Rainy Days

The idea of "saving for rainy days" suggests that when anticipating times of scarcity, the agent would consume less today to save more for the future. This intuition manifests itself in the trading game as follows. In periods with abundant private information, when anticipating less private information in the future, the informed investor would save his current private information for future use. In contrast, in periods of scarcity of private information, the informed investor utilizes more or even all his current information.

This intuition is formally illustrated in a two-period example. Let us consider the case of $N = 2$, that is, there are two rounds of trading in the economy and the informed investor receives his private signals F_1 and F_2 before the first and second trading periods, respectively. The following corollary characterizes the equilibrium for this case.

Corollary 1. *If* $N = 2$*, the equilibrium is characterized in the following two cases:*

Case 1: If $\sigma_{F_1}^2 > \sigma_{F_2}^2$, then the informed investor plays a mixed strategy in period 1 and a pure strategy in period 2, and the equilibrium variables are given by

$$\sigma_{z_1}^2 = \frac{\sigma_{F_1}^2 - \sigma_{F_2}^2}{2\sigma_{F_1}^2}\sigma_u^2, \quad \sigma_{z_2}^2 = 0,$$
$$\beta_1 = \frac{\sigma_F \sigma_u}{\sqrt{2}\sigma_{F_1}^2}, \quad \beta_2 = \frac{\sqrt{2}\sigma_u}{\sigma_F}, \quad k_1 = k_2 = \frac{\sigma_F}{\sqrt{2}},$$
$$\lambda_1 = \lambda_2 = \frac{\sigma_F}{2\sqrt{2}\sigma_u}, \quad \gamma_1 = \gamma_2 = \frac{\sigma_F}{\sqrt{2}\sigma_u}.$$

Case 2: If $\sigma_{F_1}^2 \leq \sigma_{F_2}^2$, then the informed investor plays a pure strategy in both periods, and the equilibrium variables are given by

$$\sigma_{z_1}^2 = \sigma_{z_2}^2 = 0,$$
$$\beta_i = \frac{\sigma_u}{\sigma_{F_i}}, \quad \lambda_i = \frac{\sigma_{F_i}}{2\sigma_u}, \quad \gamma_i = \frac{\sigma_{F_i}}{\sigma_u}, \quad k_i = \sigma_{F_i}, \quad \textit{for } i = 1, 2.$$

This corollary formally shows that the informed investor's information usage in period 1 depends on not only his current information but also his expectation of future information.

The informed investor plays a mixed strategy in period 1 if and only if he receives more private information in period 1 than second 2 (i.e., $\sigma_{F_1} > \sigma_{F_2}$). This can be intuitively understood from our consumption-saving analogy. Suppose a consumer's total income across two periods is \$1. He would like to allocate his wealth equally across two periods, i.e., $C_1 = C_2 = \$0.5$. However, whether this allocation is feasible depends on the timing of his incomes in the two periods because the consumer can save but not borrow.

If the consumer receives \$0.7 in period 1 and \$0.3 in period 2, then he can achieve the ideal allocation: He can consume \$0.5 out of the first period income \$0.7, and save the remaining \$0.2, so that he can also consume \$0.5 in period 2. If he receives \$0.3 in period 1 and \$0.7 in period 2, then he will be forced to consume his income each period (i.e., $C_1 = \$0.3$

and $C_2 = \$0.7$) because he cannot borrow to consume the ideal amount in period 1.

We can recast the above intuition into our trading game. Parallel to the example above, the informed investor is endowed with a total amount of information of 1 (i.e., $\sigma^2_{F_1} + \sigma^2_{F_2} = 1$). He would like to utilize the same amount of information each period, that is, $k_1^2 = k_2^2 = 0.5$. However, whether this ideal allocation is feasible depends on the timing of information arrival. If the informed investor receives more information in the first period, say, $\sigma^2_{F_1} = 0.7$ and $\sigma^2_{F_2} = 0.3$ (as in Case 1 of Proposition 1), then he can achieve his ideal allocation. He plays a mixed strategy and chooses $k_1^2 = 0.5$ in period 1. This leaves a balance 0.2 of unused information (i.e., $\Sigma_1 = \sigma^2_{F_1} - k_1^2 = 0.7 - 0.5 = 0.2$), so that he can choose $k_2^2 = 0.5$ (i.e., $k_2^2 = \Sigma_1 + \sigma^2_{F_2} = 0.2 + 0.3 = 0.5$) in the second period. By contrast, if the informed investor receives less information in the first period, for instance, $\sigma^2_{F_1} = 0.3$ and $\sigma^2_{F_2} = 0.7$ (as in Case 2 of Proposition 1), then the informed investor will use up his private information each period (i.e., $k_1^2 = 0.3$ and $k_2^2 = 0.7$). In this case, he plays a pure strategy in both periods.

The above results illustrate that analogous to the permanent income hypothesis, the informed investor's current information usage depends on not only his current information but also his future information. These results generalize and sharpen those in Huddart et al. (2001). The model in Huddart et al. (2001) belongs to Case 1 with $\sigma^2_{F_1} = \sigma^2_F$ and $\sigma^2_{F_2} = 0$. Our analysis in Case 1 shows that the dissimulation result in Huddart et al. (2001) holds more generally, i.e., as long as $\sigma^2_{F_1} > \sigma^2_{F_2}$. The dissimulation result, however, disappears in Case 2, where the informed investor receives less private information in the first period than in the second ($\sigma^2_{F_1} < \sigma^2_{F_2}$). Anticipating the arrival of more information in the second period, the informed investor does not dissimulate his information in the first period. Instead, he utilizes all his private information available at that time.

These results have direct implications on the informed investor's trading strategy. In the former case, similar to the result in Huddart et al. (2001), the informed investor adopts a mixed strategy to dissimulate his current private information. In the latter case, however, he adopts a pure strategy to fully utilize all his private information. These results extend and sharpen the insight in Huddart et al. (2001), who focus on the special case in which the informed investor possesses all his private information in the first period. With abundant private information, the informed investor always adopt a mixed strategy to dissimulate his information. In our more generalized setup, the informed investor's strategy depends on his expectation of his future private information. If future information is sufficiently abundant, he would adopt a pure strategy to fully utilize his current private information.

At least since Friedman (1957), it has been recognized that one's current consumption depends on his future expectations. Adapting this insight to our trading model, we obtain that the investor's information usage in period n, k_n, depends on not only his current information but also his expectation of future information. This has direct implications on the informed investor's trading strategies and the equilibrium asset prices.

Note that, as in Huddart et al. (2001), if the informed investor plays a pure strategy, disclosure would fully reveal his private information. This is illustrated in equation (4). If the informed investor adopts a pure strategy in period n, i.e., $z_n = 0$, then the disclosure of x_n reveals his information $\sum_{i=1}^{n} F_i - P_{n-1}^*$. Hence, if the informed investor does not want to utilize all his current information, he would adopt a mixed strategy: By adding noise to his trading order, the informed investor can dissimulate his private information for future use.

What determines the informed investor's trading strategy? The consumption-saving model described in Section 3.2 offers a clear answer. In times of abundance (i.e., $S_{n-1}+Y_n$ is high relative to future incomes), the consumer chooses to save his current resources for future consumption. This, as noted in Proposition 2, corresponds to saving his current information for future use in the trading model. If the informed investor has abundance of private information in period n (i.e., $\Sigma_{n-1} + \sigma_{F_n}^2$ is high relative to future private information), he would play a mixed strategy, i.e., save some private information for future use: $k_n^2 < \Sigma_{n-1} + \sigma_{F_n}^2$, which is equivalent to $\Sigma_n > 0$. If his private information is scarce in period n, however, the informed investor would play a pure strategy to utilize all his information: $k_n^2 = \Sigma_{n-1} + \sigma_{F_n}^2$, which is equivalent to $\Sigma_n = 0$. Therefore, as summarized in the following proposition, Σ_n is an indicator of whether the informed investor plays a pure or mixed strategy in equilibrium.

Proposition 1. *In equilibrium, the informed investor plays a mixed strategy in period n if and only if $\Sigma_n > 0$.*

4.2 Consumption Smoothing

Consumption smoothing is a key insight in the consumption-saving literature. Its counterpart in the trading model is that the informed investor would like to smooth his information usage over time. Indeed, in the two-period example in the previous subsection, the informed investor minimizes the difference between his information usage across the two periods. What is the notion of information-usage smoothing in an N-period model? We formalize it in the following proposition.

Proposition 2. *Denote $\overline{k} \equiv \sum_{i=1}^{N} k_i/N$ and $\overline{\lambda} \equiv \sum_{i=1}^{N} \lambda_i/N$. The informed investor's maximization problem in (16) and (17) is equivalent to either of the following two minimization problems:*

(1) Smoothing information usage over time:

$$\min_{\{k_1,\cdots,k_N\}\in\mathbb{R}^N_{\geq 0}} \sum_{i=1}^{N}(k_i-\overline{k})^2, \tag{21}$$

$$\text{subject to} \quad \sum_{i=1}^{n} k_i^2 \leq \sum_{i=1}^{n} \sigma_{F_i}^2, \ \ \text{for } n=1,...N-1, \tag{22}$$

$$\sum_{i=1}^{N} k_i^2 = \sum_{i=1}^{N} \sigma_{F_i}^2. \tag{23}$$

(2) Smoothing price impact over time:

$$\min_{\{\lambda_1,\cdots,\lambda_N\}\in\mathbb{R}^N_{\geq 0}} \sum_{i=1}^{N}(\lambda_i-\overline{\lambda})^2, \tag{24}$$

subject to (22) and (23).

Moreover, information usage k_t and price impact λ_t are weakly increasing over time:

$$k_t \leq k_{t+1} \text{ and } \quad \lambda_t \leq \lambda_{t+1} \text{ for } t = 1, ..., N-1. \tag{25}$$

The objective function (21) shows that the informed investor aims to minimize the time variation of his information usage across periods. Moreover, the objective function (24) shows that minimizing the variation of information usage is equivalent to minimizing the variation in price impact. Intuitively, the informed investor's private information usage is closely linked to his price impact, and indeed, these two are proportional to each other in our model (see (13)). So, smoothing information usage across periods is the same as smoothing price impact over time.

The proposition also shows that the informed investor's information usage and price impact is weakly increasing over time (equation (25)). This is a direct consequence from information smoothing in (21). Ideally, the informed investor would like to keep his information usage k_t constant over time. Hence, $k_t > k_{t+1}$ is never optimal because the informed investor can increase his expected trading profit by saving his information in period t to use in the next period (i.e., reducing k_t to increase k_{t+1}). On the other hand, because the informed investor cannot transfer future private information to utilize today, $k_t < k_{t+1}$ can be sustained in equilibrium if the informed investor's information budget constraint is binding (i.e., he utilizes all available private information in period t). As noted in (13), since the price impact and information usage are closely related, the informed investor using information at a weakly increasing rate implies that the price impact is also weakly increasing, $\lambda_t \leq \lambda_{t+1}$. These results are in contrast to those in previous studies, where the price impact is usually either a constant (e.g., Kyle (1985), Huddart et al. (2001)) or tends to decrease over time (e.g. Caldentey and Stacchetti (2010)).[2]

This proposition generalizes the insight in Kyle (1985) and Huddart et al. (2001), where

[2]One notable exception is Collin-Dufresne and Fos (2016), where the price impact tends to increase over time due to a liquidity-timing option.

the informed investor utilizes his private information at a constant rate (i.e., minimizes the time variation of his information usage to zero). In our model, the total amount of the private information is σ_F^2. Hence, the best possible scenario is to utilize σ_F^2/N each period. This scenario, however, is not always feasible, as illustrated in the two-period example in Proposition 1. When is perfect information-usage smoothing possible? We characterise the general condition for perfect information smoothing in the following corollary.

Corollary 2. *The necessary and sufficient condition for perfect information-usage smoothing (i.e., $k_n^2 = \sigma_F^2/N$ for $n = 1, \cdots, N$) is*

$$\sum_{i=1}^{n} \sigma_{F_i}^2 \geq \frac{n}{N}\sigma_F^2, \text{ for } n = 1, \cdots, N. \tag{26}$$

Under condition (26), the equilibrium in period n has the following properties:

$$\lambda_n = \frac{\sigma_F}{2\sqrt{N}\sigma_u}, \tag{27}$$

$$E[\pi_n] = \frac{\sigma_F \sigma_u}{2\sqrt{N}}, \tag{28}$$

$$U_n = (1 - n/N)\sigma_F^2, \tag{29}$$

where U_n is the uncertainty of the liquidation value conditional on asset prices till period n:

$$U_n \equiv Var\left(F | P_1^*, ..., P_n^*\right).$$

If the inequalities in (26) hold strictly for $n \leq N - 1$, the informed investor adopts a mixed strategy in all but the last period.

Condition (26) is such that the informed investor can always "afford" to utilize σ_F^2/N private information each period. Specifically, if sufficient private information arrives early, the informed investor always has no less than σ_F^2/N unused information available in each period. Hence, he achieves perfect information smoothing by utilizing σ_F^2/N private information each period. Consequently, his price impact and expected trading profit are also constants across periods, as shown in equations (27) and (28), respectively. Since the informed investor's private information is revealed at a constant rate, as shown in equation (29), the stock price uncertainty decreases linearly over time. If the inequalities in (26) hold strictly for $n \leq N-1$, they guarantee that the informed investor always has *more* than σ_F^2/N private information in each period. To utilize σ_F^2/N private information each period, the informed investor needs to dissimulate his private information (i.e., adopt a mixed strategy) in all but the last period.

It is interesting to compare the above results with those in Huddart et al. (2001), where the informed investor receives all his private information in the first period: $\sigma_{F_1}^2 = \sigma_F^2$ and $\sigma_{F_i}^2 = 0$ for $i = 2, ...N$. This is a special case of (26). In equilibrium, the informed investor adopts a mixed strategy and utilizes the same amount of private information each period. Proposition 2 shows that these results hold more generally under the conditions in (26).

To further illustrate the implications in Proposition 2, we consider the following two cases. In Case 1, the informed investor's information arrives at a decreasing rate, that is

$$\sigma_{F_n}^2 > \sigma_{F_{n+1}}^2 \text{ for } n = 1, ..., N-1. \tag{30}$$

In Case 2, the informed investor's information arrives at an increasing rate, that is

$$\sigma_{F_n}^2 < \sigma_{F_{n+1}}^2 \text{ for } n = 1, ..., N-1. \tag{31}$$

These two cases are a generalized version of the two cases in the two-period example in Propostion 1. Condition (30) in Case 1 is a special case of (26). Hence, as shown in Proposition 2, the informed investor adopts a mixed strategy in all but the last period and perfectly smooths his information usage over time $k_n^2 = \sigma_F^2/N$, for $n = 1, ..., N$. Perfect smoothing is not feasible in Case 2. Since the private information arrives at an increasing rate, the informed investor does not possess enough private information in early rounds to utilize σ_F^2/N information each period. The equilibrium in this case is summarized in the following corollary.

Corollary 3. *Under the conditions in (31), the informed investor adopts a pure strategy in every period and the equilibrium in period n, for $n = 1, ..., N$, has the following properties:*

$$k_n = \sigma_{F_n}, \tag{32}$$

$$\lambda_n = \frac{\sigma_{F_n}}{2\sigma_u}, \tag{33}$$

$$E[\pi_n] = \frac{\sigma_{F_n}\sigma_u}{2}, \tag{34}$$

$$U_n > (1 - n/N)\sigma_F^2, \;\; n \neq N. \tag{35}$$

Anticipating more private information in the future, as shown in (32), the informed investor utilizes all his private information (i.e., adopts a pure strategy) each period. It has been noted in the literature that a monopolistic informed investor has the incentive to minimize the price impact by either breaking down his order into small ones (Kyle, 1985) or by adding noise to his order (HHL, 2001) to "go down" the market maker's demand curve. Proposition 3 shows that the anticipation of future private information expedites the informed investor's usage of his private information. It generalizes the results in the two-period example in Proposition 1 and shows that when private information arrives at an increasing rate, the informed investor chooses to fully utilize his private information each period. Moreover, since the informed investor utilizes information at an increasing rate, his price impact and expected trading profits also increase over time, as shown in equations (33) and (34). Finally, relative to the equilibrium in Case 1, the informed investor utilizes less private information and hence the stock price informativeness is lower (i.e., U_n is higher) in all but the final periods (see (29) and (35)).

To further illustrate the equilibrium, we analyze a numerical example of Cases 1 and 2. Specifically, we set $N = 10$, $\sigma_F^2 = 1$, and $\sigma_u^2 = 0.1$. The informed investor's private information arrives at a linearly decreasing rate in Case 1:

$$\sigma_{F_n}^2 = \frac{2(N-n+1)}{N(N+1)}\sigma_F^2, \tag{36}$$

and at a linearly increasing rate in Case 2:

$$\sigma_{F_n}^2 = \frac{2n}{N(N+1)}\sigma_F^2. \tag{37}$$

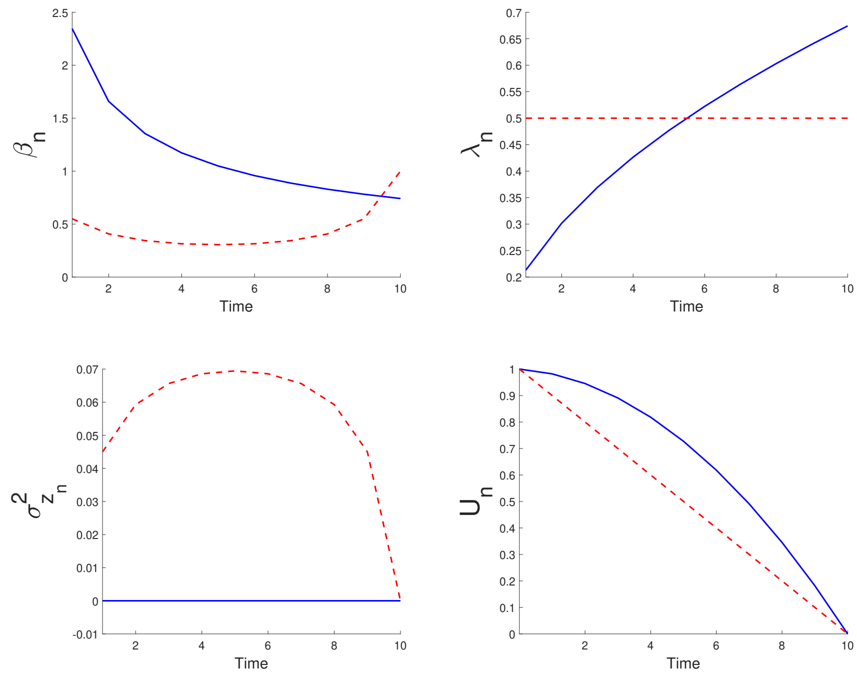
The equilibria in these two cases are summarized in Figure 2. The upper left panel plots the trading intensity β_n against the trading period n. The dashed line and solid line represents Cases 1 and 2 , respectively. In all but the final period, the informed investor trade less aggressively in Case 1 than in Case 2. This is because, in Case 1, the informed investor anticipates less private information in later periods and hence trades less aggressively to save his information for future use. In contrast, when anticipating more private information in Case 2, the informed investor would exploit his current information more aggressively in early periods.

The upper right panel plots the price impact. As shown in Proposition 2, in Case 1 (illustrated by the dashed line), the informed investor utilizes the same amount of information each period, leading to a constant price impact. The solid line shows that the price impact increases over time in Case 2. This is because the informed investor's information arrives at an increasing rate and, as shown in Proposition 3, the informed investor utilizes all his information each period.

The lower left panel reports how the informed investor dissimulates his private information. The dashed line shows that in Case 1, the case with a decreasing information arrival rate, the informed investor adopts a mixed strategy (i.e., $\sigma_{z_n}^2 > 0$) in all but the last period. In contrast, as shown by the solid line, the informed investor always adopts a pure strategy (i.e., $\sigma_{z_n}^2 = 0$) in Case 2.

Finally, the lower right panel plots U_n, the uncertainty about the liquidation value con-

Figure 2. Equilibrium under Monotonic Information Arrivals
This figure plots the trading intensity β_n, price impact λ_n, the noise in the informed investor's demand $\sigma^2_{z_n}$, and price uncertainty U_n respectively, for the case with a decreasing information arrival rate as in equation (36) (dashed line) and the case with an increasing information arrival rate as in equation (37) (solid line). Parameter values: $\sigma^2_F = 1, \sigma^2_u = 0.1$, and $N = 10$.



ditional on asset price history till period n, against time n. The dashed line shows that in Case 1, the informed investor utilizes the same amount of information each period and hence the uncertainty decreases linearly. In Case 2, where the informed investor's possesses less private information in earlier periods. Although all private information is revealed each period, the uncertainty still decreases more slowly than in Case 1 (i.e., the solid line is above the dashed line).

4.3 Precautionary Savings

Precautionary saving is a key insight in the consumption-saving literature. It suggests that a consumer would reduce his consumption today if he anticipates more income uncertainty in the future. The counterpart of this idea in our trading model is that the informed investor would utilize less of his current information if he anticipate more uncertainty in his information advantage in the future. To analyze this idea, we need to introduce uncertainty to the informed investor's future information advantage.

To simplify the analysis, we consider the two-period model that is analyzed in Proposition 1. The only modification is the uncertainty in the informed investor's advantage in second period:

$$\sigma_{F_2}^2 = \begin{cases} \overline{\sigma}_{F_2}^2 + \Delta, & \text{with probability } \frac{1}{2}, \\ \overline{\sigma}_{F_2}^2 - \Delta, & \text{with probability } \frac{1}{2}. \end{cases} \tag{38}$$

That is, in period 1, there is uncertainty in the informed investor's advantage in period 2, which is either $\overline{\sigma}_{F_2}^2 + \Delta$ or $\overline{\sigma}_{F_2}^2 + \Delta$, with equal likelihood. This uncertainty is resolve after period 1 before the informed investor receives his signal for the second period.

To best illustrate the notion of precautionary saving, we focus on the case $\sigma_{F_1}^2 > \overline{\sigma}_{F_2}^2$.[3] We characterize the equilibrium of this modified economy in the appendix and summarize the implications on precautionary savings in the following proposition.

Proposition 3. *In the equilibrium of the economy described in this subsection and in the case $\sigma_{F_1}^2 > \overline{\sigma}_{F_2}^2$, the informed investor's information usage in period 1 is increasing is the expectation of his second period information: $\frac{\partial k_1^2}{\partial \overline{\sigma}_{F_2}^2} > 0$, but is decreasing in the uncertainty $\frac{\partial k_1^2}{\partial \Delta} > 0$.*

5 Extensions

In this section, we consider three extensions. Section 5.1 examines the case in which the intensity of noise trading varies over time. Section 5.2 analyzes the case in which the informed investor's private information may be leaked to the public in each period. Finally, in Section 5.3, we explore the limiting case, in which the trading frequency approaches infinity and

[3]This corresponds to Case 1 in Proposition 1, where the informed investor's decision is an interior solution. In contrast, the informed investor's choice is a corner solution in Case 2. That is, he always utilizes all his information in period 1, making the analysis of savings trivial.

hence the model converges to the continuous-time limit. We show that, in each case, the trading game can be transformed into a consumption-saving problem. Hence, the analysis of each extension can be easily accommodated by adjusting the consumption-saving problem. Finally, in Section 5.4, we extend the baseline model to consider the case of imperfect disclosure. That is, after the transaction each period, the market maker observes an imperfect signal about the informed investor's trade size.

5.1 Time-Varying Noise Trading

In this extension, the variance of noise trading is time-varying. Specifically, in period n, noise traders have an aggregate demand of u_n shares, with $u_n \sim \mathcal{N}(0, \sigma_{u_n}^2)$, with $\sigma_{u_n} > 0$, and u_n is independent across n and from F_n. The following proposition characterizes the equilibrium in this case.

Proposition 4. *1) In the equilibrium of the economy with time-varying noise trading, the informed investor's trading strategies and the market maker's pricing rules are given by equations (4)–(6) with parameters characterized by equations (A.22)–(A.25) in the appendix, and* $\{k_1, \cdots, k_N\}$ *are the unique non-negative solution to the following maximization problem:*

$$\max_{k_1, \cdots, k_N} \sum_{i=1}^{N} k_i \sigma_{u_i} \tag{39}$$

subject to (17).

2) The maximization problem defined in (39) and (17) is equivalent to

$$\min_{\{k_1, \cdots, k_N\} \in \mathbb{R}_{\geq 0}^N} \sum_{i=1}^{N} \left[\omega_i (k_i' - \overline{k'})^2 \right], \tag{40}$$

subject to (22) and (23), where $\omega_i \equiv \sigma_{u_i}^2 / \sum_{j=1}^{N} \sigma_{u_j}^2$, $k_i' \equiv k_i / \sigma_{u_i}$, *and* $\overline{k'} \equiv \sum_{i=1}^{N} \omega_i k_i'$. *The* *maximization problem defined in (39) and (17) is also equivalent to*

$$\min_{\{\lambda_1,\cdots,\lambda_N\}\in\mathbb{R}^N_{\geq 0}} \sum_{i=1}^{N} \left[\omega_i(\lambda_i-\overline{\lambda})^2\right], \tag{41}$$

subject to (22) and (23), where $\overline{\lambda} \equiv \sum_{i=1}^{N}\omega_i\lambda_i$.

3) The maximization problem defined in (39) and (17) is equivalent to the following consumption-saving problem

$$\max_{\{C_n,\cdots,C_N\}} \sum_{i=1}^{N} u(C_i/p_i), \tag{42}$$

subject to (18) and (19), where $p_i \equiv 1/\sigma^2_{u_i}$ *is the price level in period* i*, if we relabel* $\sigma^2_{F_i}$, k_i^2*, and* Σ_i *as the* ***nominal*** *income, consumption, and saving,* Y_i, C_i*, and* S_i*, respectively.*

The above proposition shows three main results. First, as in Theorem 1, all equilibrium quantities can be written as functions of the informed investor's information usage k_i, for $i = 1, ..., N$, which are determined by a maximization problem. The objective function of this maximization problem, (39), is a generalized version of its counterpart in the baseline model (equation (16)). As noted in the baseline model, the informed investor's expected trading profits from his trading in period i is given by $k_i\sigma_{u_i}$. Hence, the objective in (39) has the same economic meaning as in the baseline case: maximizing the informed investor's expected trading profits.

Second, as in the baseline case, the informed investor tries to smooth his information usage and price impact over time. As shown in (40), the informed investor tries to minimize the time variation in k_i', which is the informed investor's normalized information usage k_i/σ_{u_i}. The normalization accounts for the fact that the informed investor's trading is more profitable when there is more noise trading. Moreover, in both the objective function (40) and the definition of $\overline{k'}$, the observation in period i is weighted by ω_i, noise trading variance in period i divided by the total noise trading variance across N periods. In this generalize case, "perfect smoothing" is achieved if k_i' is a constant over time. That is, if the informed investor's information usage in a given period is proportional to the noise trading size in that period. As is the case in the baseline model in Section 4.2, perfect smoothing is feasible if sufficient private information arrives in early periods. Similarly, the objective in (41) shows that the informed investor tries to minimize the time variation in price impact across time. These results are a generalized version of those in the baseline model. If we set $\sigma_{u_i} = \sigma_u$ for $i = 1, ...N$, the two minimization problems in (40) and (41) become those in the baseline model ((21) and (24)).

Finally, as in the baseline case, the informed investor's information usage problem can be transformed into a consumption-saving problem. The only modification is that the income consumption and savings are nominal variables with a price level $1/\sigma_{u_i}^2$ to account for the fact that the informed investor's trading is more profitable when there is more noise trading.

5.2 Potential Information Leakage

In this extension, we introduce the possibility of information leakage into the baseline model in Section 2. Specifically, before trading takes place each period, the fundamental information F may become public with a probability $1 - q \in [0, 1]$. The rest of the model is identical to that in the baseline model. After the information leakage, the informed investor has no incentive to trade, and the model becomes degenerate. Hence, we will focus on the equilibrium when the information leakage has not yet occurred. We use the same set of variables as in the baseline model in Section 2 to denote their counterparts in the current setting under the condition that the information leakage has not occurred yet. For example, we use k_n^2 and Σ_n to denote the informed investor's information usage and unused information in period n if the information leakage has not yet occurred. The following proposition characterises the equilibrium along the path where the information leakage does not occur.

Proposition 5. *1) In the equilibrium of the economy with information leakage, conditional on the information leakage not occurring, the informed investor's trading strategies and the* *market maker's pricing rules are given by equations (4)–(6) with parameters characterized by equations (12)–(15), and* $\{k_1, \cdots, k_N\}$ *are the unique non-negative solution to the following maximization problem:*

$$\max_{k_1,\cdots,k_N} \sum_{i=1}^{N} q^{i-1} k_i \tag{43}$$

subject to (17).

2) The maximization problem defined in (43) and (17) is equivalent to

$$\min_{\{k_1,\cdots,k_N\}\in\mathbb{R}^N_{\geq 0}} \sum_{i=1}^{N} (k_i - q^{i-1}\overline{k})^2, \tag{44}$$

subject to (22) and (23), where $\overline{k} \equiv \sum_{i=1}^{N} q^{i-1} k_i / N$. *It is also equivalent to*

$$\min_{\{\lambda_1,\cdots,\lambda_N\}\in\mathbb{R}^N_{\geq 0}} \sum_{i=1}^{N} (\lambda_i - q^{i-1}\overline{\lambda})^2, \tag{45}$$

subject to (22) and (23), where $\overline{\lambda} \equiv \sum_{i=1}^{N} q^{i-1} \lambda_i / N$.

3) The maximization problem defined in (39) and (17) is equivalent to the following consumption-saving problem:

$$\max_{\{C_n,\cdots,C_N\}} \sum_{i=1}^{N} q^{i-1} u(C_i), \tag{46}$$

subject to (18) and (19), if we relabel $\sigma^2_{F_n}$, k_n^2, *and* Σ_n *as* Y_n, C_n, *and* S_n, *respectively.*

The above proposition shows that the equilibrium in this case is similar to that in the baseline model. One notable change is the objective function (43), whereby information usage in period n is discounted by q^{n-1}. This is because that, each period, the trading game continues to the next period with a probability q. Once the games stops (i.e., the information revelation occurs), the informed investor can no longer benefit from trading on his information. Hence, the benefit from future information usage is discounted by q each period.

Moreover, as in the baseline case, the informed investor tries to smooth his information usage and price impact. The modifications in the minimization problems (44) and (45) account for the fact that, each period, the trading game continues with a probability q. In fact, the objectives in (44) and (45) suggest that the informed investor achieves "perfect information smoothing" and "perfect price-impact smoothing" if k_n/q^{n-1} is a constant for all n.

Finally, also similar to the result in the baseline case, the informed investor's information usage problem can be transformed into a consumption-saving problem. As shown in (46), the only adjustment relative to the baseline result in that the utility from the consumption in period n is discounted by q^{n-1}.

5.3 Continuous-Time Limit

To study the equilibrium in the continuous-time limit, we normalize the total duration of the N periods in our baseline model as 1 and assume that the length of each period is h. Hence, the nth period refers to the time interval $((n-1)h, nh]$ and the n-th trade occurs at the end of the period, i.e., at the moment $t = nh$. In the limit, as N approaches ∞, the length of each period h approaches 0 and hence our model converges to a continuous-time limit. This treatment is standard in the literature (e.g., Kyle, 1985; Vayanos, 1999).

Let $u(n)$ denote the cumulative noise trading until the end of the n-th period. That is, $\Delta u(n) \equiv u(n) - u(n-1)$ is the size of the noise trading during the n-th period and its distribution is $\Delta u(n) \sim \mathcal{N}(0, \sigma_u^2 h)$. As h approaches 0, the process $u(n)$ approaches its continuous-time limit $u(t)$ such that

$$du(t) = \sigma_u(t) dB_u(t), \tag{47}$$

where $B_u(t)$ is a standard Brownian motion.

The stock's fundamental value in the final period, $F(1)$, follows $\mathcal{N}(0, \sigma_F^2)$. It consists of N components: $F(1) = \sum_{n=1}^{N} \Delta F(nh)$, where $\Delta F(nh) \equiv F(nh) - F((n-1)h) \sim \mathcal{N}(0, \sigma_{F_n}^2 h)$ for $n = 1, ..., N$ and $F(0) = 0$. The informed investor observes $\Delta F(nh)$ in period n. As h approaches 0, we obtain

$$dF(t) = \sigma_F(t) dB_F(t), \tag{48}$$

where $B_F(t)$ is a standard Brownian motion that is independent of $B_u(t)$ with $\int_0^1 \sigma_F^2(t)dt = \sigma_F^2$.[4] The linear equilibrium conjectured in equations (4)–(6) converges to

$$dx(t) = \beta_t (F(t + dt) - P^*(t))dt + \sigma_z(t) dB_z(t), \tag{49}$$

$$P(t + dt) = P^*(t) + \lambda(t)(dx(t) + du(t)), \tag{50}$$

$$dP^*(t) = \gamma(t) dx(t), \tag{51}$$

where $B_z(t)$ is a standard Brownian motion that is independent of $B_u(t)$ and $B_F(t)$, $P^*(0) = 0$, and $\{\beta(t), \lambda(t), \gamma(t), \sigma_z(t)\}$ are determined in equilibrium.

Similar to the definitions of information usage and unrevealed information in the baseline model (i.e., equation (7) and (8)), we define their continuous-time counterparts as

$$k^2(t)dt \equiv Var(dP^*(t)), \tag{52}$$

$$\Sigma(t) \equiv Var[F(t) | \{P^*(s), \text{ for } 0 \leq s \leq t\}]. \tag{53}$$

During $(t, t + dt]$, the amount of private information obtained by the informed investor is

[4]For simplicity, we adopt the diffusion specification in (48). That is, the informed investor's information acquisition is "smooth" over time. It is straightforward to generalize the model to allow for the situation where the informed investor acquires a "bulk" of information at certain times. This can be accommodated by assuming that $F(t)$ follows a jump diffusion process.

$\sigma_F^2(t)dt$ and the amount revealed by his trading and disclosure is $k^2(t)dt$, leading to

$$d\Sigma(t) = (\sigma_F^2(t) - k^2(t))dt. \tag{54}$$

Finally, the budget constraints (18) and (19) of the consumption-saving problem become

$$C(t)dt \leq S(t) + Y(t)dt, \tag{55}$$

$$dS(t) = (Y(t) - C(t))dt, \tag{56}$$

for $0 \leq t \leq 1$. The equilibrium in the limit case is characterized by the following proposition.

Proposition 6. *As the trading period length h approaches 0, the equilibrium in Theorem 1 converges to the following: For $0 \leq t \leq 1$, $x(t)$, $P(t)$ and $P^*(t)$ are given by (49)–(51), and if $\Sigma(t) = 0$, then*

$$\beta(t) = \frac{\sigma_u}{k(t)}, \quad \lambda(t) = \frac{k(t)}{2\sigma_u}, \quad \gamma(t) = \frac{k(t)}{\sigma_u}, \quad \sigma_z(t) = 0; \tag{57}$$

if $\Sigma(t) > 0$, then

$$\beta(t) = \frac{k(t)\sigma_u}{\Sigma(t)}, \quad \lambda(t) = \frac{k(t)}{2\sigma_u}, \quad \gamma(t) = \frac{k(t)}{\sigma_u}, \quad \sigma_z(t) = \sigma_u, \tag{58}$$

where $\Sigma(t)$ is given by (54) and $k(t)$ is determined by the following maximization problem

$$\max_{k(t)\geq 0} \int_0^1 k(t)dt \tag{59}$$

$$\textit{subject to} \quad \int_0^t k^2(s)ds \leq \int_0^t \sigma_F^2(s)ds, \ \textit{for } t \in [0,1]. \tag{60}$$

The maximization problem (59) is equivalent to the following consumption-saving problem

$$\max_{C(t)\geq 0} \int_0^1 u(C(t))dt, \tag{61}$$

subject to (55) and (56), if we relabel $\sigma_F^2(t)$, $k^2(t)$, and $\Sigma(t)$ as $Y(t)$, $C(t)$, and $S(t)$, respectively.

The above proposition shows that the equilibrium in the continuous-time limit is similar to that in Theorem 1. For example, the informed investor's demand function and the market maker's price rule have similar functional forms as those in Theorem 1. The proposition also shows that the stock price achieves the strong-form efficiency when the informed investor adopts a pure strategy. As noted in Section **??**, the post-disclosure stock price fully reveals the informed investor's private information when she adopts a pure strategy. Moreover, in this limiting case, the pre- and post-disclosure prices are infinitely close to each other, that is, $\lim_{h\to 0} P(t) - P^*(t) = 0$ almost surely for $0 \leq t \leq 1$. Hence, the stock price is strong-form efficient. This result is reminiscent to those in Chau and Vayanos (2008) and Foucault et al. (2016), where in the limiting case as the trading frequency approaches infinity, the stock price becomes efficient while the informed investor still earns trading profits. Parallel to the result in Section 3, the informed investor's maximization problem can be transformed into a continuous-time consumption-saving problem in (61). Finally, we show in the appendix that the equilibrium in the continuous time model is the same as the equilibrium in Proposition 6, the limit of the discrete-time equilibrium as the trading frequency approaches infinity.

5.4 Partial Disclosure

In the analysis so far, the disclosure perfectly reveals the insider's trade x_n. More generally, however, there are reasons why the disclosure may be imperfect. For example, financial institutions such as hedge funds and mutual funds are only required to report their stock holdings at the end of each quarter. Hence, the disclosure only reveals their trades imperfectly. Alternatively, as noted in Yang and Zhu (2020), some investors such as high frequency traders can partially infer informed investors' trades ex post. To capture this feature, we modify our model such that after the trade in period n, the market maker observes a signal

d_n

$$d_n = x_n + \epsilon_n,$$

where $\epsilon_n \sim \mathcal{N}(0, \sigma_\epsilon^2)$, ϵ_n is serially independent and independent of all other variables. The rest of the model remains the same as described in Section 2.

This formulation not only captures the scenarios mentioned above, but also generalizes our model to include some classic models as special cases. For example, as σ_ϵ^2 approaches zero, the disclosure perfectly reveals the informed investor's trade. Hence, our model converges to a generalized version of the setting in Huddart et al. (2001). In the other limiting case, as σ_ϵ^2 approaches ∞, the disclosure does not contain any information and our model converges to a generalized version of Kyle (1985).

Proposition 7. *In the N-periods model, prices are*

$$P_n = E[F|P_1^*, \cdots, P_{n-1}^*, x_n + u_n] = P_{n-1}^* + \lambda_n(x_n + u_n), \tag{62}$$

$$P_n^* = E[F|P_1^*, \cdots, P_{n-1}^*, x_n + u_n, x_n + \epsilon_n] = P_{n-1}^* + \lambda'_n(x_n + u_n) + \gamma_n(x_n + \epsilon_n). \tag{63}$$

Insider strategy has the form

$$x_n = \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n, \quad \textit{with } z_n \sim \mathcal{N}(0, \sigma_{z_n}^2). \tag{64}$$

Insider profits are recursive as

$$E[\sum_{i=n+1}^{N} \pi_i | F_1, \cdots, F_{n+1}, P_1^*, \cdots, P_n^*] = \alpha_n(\sum_{i=1}^{n+1} F_i - P_n^*)^2 + \delta_n. \tag{65}$$

Coefficients satisfy

$$\lambda_n = \frac{\beta_n(\Sigma_{n-1} + \sigma^2_{F_n})}{\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u}, \tag{66}$$

$$\lambda'_n = \lambda_n - \frac{\gamma_n(\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n})}{\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u}, \tag{67}$$

$$\Sigma_n = \frac{1}{\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u}[(\sigma^2_u + \sigma^2_{z_n})(\Sigma_{n-1} + \sigma^2_{F_n}) - \frac{(\beta_n(\Sigma_{n-1} + \sigma^2_{F_n})\sigma^2_u)^2}{(\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n})\sigma^2_u + \sigma^2_\epsilon(\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u)}], \tag{68}$$

$$\gamma_n = \frac{\beta_n(\Sigma_{n-1} + \sigma^2_{F_n})\sigma^2_u}{(\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n})\sigma^2_u + \sigma^2_\epsilon(\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u)}, \tag{69}$$

$$\alpha_{n-1} = \alpha_n(1 - (\lambda'_n + \gamma_n)\beta_n)^2 + (1 - \lambda_n\beta_n)\beta_n, \tag{70}$$

$$\delta_{n-1} = \delta_n - \lambda_n\sigma^2_{z_n} + \alpha_n(\sigma^2_{F_{n+1}} + {\lambda'_n}^2\sigma^2_u + {\gamma_n}^2\sigma^2_\epsilon + (\lambda'_n + \gamma_n)^2\sigma^2_{z_n}). \tag{71}$$

When insider chooses pure strategy, $\sigma^2_{z_n} = 0$ *and*

$$\beta_n = \frac{1 - 2\alpha_n(\lambda'_n + \gamma_n)}{2\lambda_n - 2\alpha_n(\lambda'_n + \gamma_n)^2}, \tag{72}$$

with SOC

$$2\lambda_n - 2\alpha_n(\lambda'_n + \gamma_n)^2 > 0. \tag{73}$$

When insider chooses mixed strategy, $\sigma^2_{z_n} > 0$ *and*

$$1 - 2\alpha_n(\lambda'_n + \gamma_n) = 0, \tag{74}$$

$$\lambda_n - \alpha_n(\lambda'_n + \gamma_n)^2 = 0. \tag{75}$$

6 Interpretation of the Mixed Strategy

When describing the mixed strategy played by an informed trader in Kyle-type models, researchers such as Huddart et al. (2001) and Yang and Zhu (2020) often loosely interpret it as the informed trader adding noise through randomization. This interpretation has the flavor that the trader consciously randomizes by actively choosing the amount of noise in his strategy.[5] Although this interpretation is intuitive and appealing, it is well recognized in the game theory literature that this interpretation— dubbed as a "naive" interpretation of "mixed strategies as objects of choices" by Osborne and Rubinstein (1994, p. 37)—is not entirely satisfactory.[6] When the informed investor plays a mixed strategy in equilibrium, although he does introduce a noise component into his order, it is implemented as a passive action, as opposed to a deliberate choice. The informed investor is just indifferent across all orders given the market maker's pricing rules, and he is not actively choosing the size of the noise. The value of σ_{z_n} is pinned down by the equilibrium conditions, in particular, by the market maker's equilibrium behavior.

To formally accommodate the usual and intuitive interpretation of the informed investor deliberately randomizing, we consider an alternative game in which the informed investor can commit to a linear trading strategy in each period as specified in equation (4) and then chooses its parameters $\{\beta_n, \sigma_{z_n}\}_{n=1}^{N}$ at the beginning of the economy, say, in period 0 before any trading occurs.[7] The commitment is common knowledge in the game. The rest of the

[5]For instance, when defining dissimulation as the mixed strategy, HHL (2001, p. 666) state that "(t)he strategy balances immediate profits from informed trades against the reduction in future profits following trade disclosure and, hence, revelation of some of the informed investor's information. Our results show the optimality of adding a random noise component to informed trades, thereby diminishing the market maker's ability to draw inferences from the public record."

[6]When discussing mixed strategies, Rubinstein (1991, p. 912-913) wrote: "The concept of mixed strategy has often come under heavy fire. To quote Aumann (1987a): 'Mixed strategy equilibria have always been intuitively problematic.'" The literature has suggested ways of interpreting mixed strategies based on purification, beliefs, large populations, and evolution (see the discussions in Osborne and Rubinstein (1994) and Oechssler (1997)).

[7]Under this specification, the informed investor chooses all trading parameters simultaneously before observing any information. Alternatively, we can also assume that the informed investor chooses those parameters sequentially after observing information each period. For instance, in the two-period economy, the informed investor chooses $\{\beta_1, \sigma_{z_1}\}$ in period 1 after observing F_1 and chooses $\{\beta_2, \sigma_{z_2}\}$ in period 2 after

model remains the same as the baseline model in Section 2. In our context, such a committed trading strategy can be interpreted as a predetermined trading plan that specifies in advance the trading rule according to an algorithm. The equilibrium in this variation game is such that the informed investor chooses β_n and σ_{z_n} to maximize his expected total trading profit over N periods and the market maker takes commitment (4) as given and sets asset prices according to his expected liquidation value of the risky asset.

Proposition 8. *The equilibrium in the variation game with commitment is identical to that characterized in Theorem 1.*

The above proposition shows that the mixed strategy analyzed in Section **??** can be thought of as the outcome of an optimization problem where the informed investor chooses the optimal amount of noise in his demand to dissimulate his private information, which therefore formalizes the idea of "mixed strategies as objects of choice."

Note that, in the variation game with commitment, in the worst case scenario, the informed investor can commit to the equilibrium trading strategy and hence earn the same expected profits as the informed investor in the baseline model. Hence, this commitment should have a non-negative value, and the equivalence between the equilibrium in this variation game and that in our baseline model implies that the commitment does not have value. This result is consistent with the recent paper by Bernhardt and Boulatov (2023), who show that commitment has no value in a one-period Kyle model.[8] We analyze a multi-period setting with mixed strategies, and use the finding to interpret the mixed strategies in our baseline model as predetermined trading plans implemented by algorithms.

observing $\{P_1^*, F_1, F_2\}$. Our results remain the same under this alternative assumption.

[8]Bernhardt and Boulatov (2023) also show that in games in which shocks are not normally distributed and so the equilibrium is nonlinear, commitment does have value. Moreover, they consider a Stackelberg setting in which the parameters chosen by the informed investor are observable to the market maker. Our result in Proposition 8 holds independent of whether the parameters of the informed investor's strategy are observable or not.

7 Conclusion

We analyze a dynamic model of a monopolistic informed investor who receives private information on an ongoing basis and is subject to a post-trading disclosure requirement each period. We show that solving the equilibrium of this trading game is equivalent to solving a fictitious consumption-saving problem. Hence, we can adopt the existing methods in that literature, such as dynamic programming, to construct the equilibrium of our trading game.

Analogous to the "consumption-smoothing" intuition in the consumption-saving literature, the informed investor in our trading game "smooths" his information usage over time given the dynamic constraints imposed by the sequential arrival of his private information. If the informed investor expects a reduction in his information advantage in the future, consistent with the insight in the existing literature, he would dissimulate his current private information through mixed strategies. Conversely, if the informed investor expects more information advantage in the future, he would adopt a pure strategy, which reveals all his private information after disclosure. We also consider various extensions of the trading game in our baseline model and find that solving each extension is equivalent to solving a generalized version of the fictitious consumption-saving problem in the baseline model.

Finally, we show that the mixed strategy in our model can be interpreted as the informed investor deliberately dissimulating his private information—i.e., actively choosing the size of the noise in his trading order to conceal his private information—via predetermined trading plans implemented with algorithms. This result also suggests that commitment has no value in boosting the informed investor's profits in our model.

Appendix: Proofs

Proof of Theorem 1. The proof is by backward induction. We first claim that prior to the $(n+1)^{th}$ trade, the expected future profits have the following quadratic form in the linear equilibrium:

$$E(\sum_{i=n+1}^{N} \pi_i | F_1, ..., F_{n+1}, P_1^*, ..., P_n^*) = \alpha_n (\sum_{i=1}^{n+1} F_i - P_n^*)^2 + \delta_n, \tag{A.1}$$

where α_n and δ_n are constants with $\alpha_N = \delta_N = 0$. Then with the linear pricing functions (5) and (6) (or more generally, $P_n = P_{n-1}^* + \lambda_n y_n + f(x_1, ..., x_{n-1})$ and $P_n^* = P_{n-1}^* + \gamma_n x_n + g(x_1, ..., x_{n-1})$ where f and g are measurable functions and turn to be zero, as in Kyle's proof), moving backward by one step yields

$$\begin{aligned} &E(\sum_{i=n}^{N} \pi_i | F_1, ..., F_n, P_1^*, ..., P_{n-1}^*) \\ &= E[x_n(F - P_n) + \alpha_n (\sum_{i=1}^{n+1} F_i - P_n^*)^2 + \delta_n | F_1, ..., F_n, P_1^*, ..., P_{n-1}^*] \\ &= x_n (\sum_{i=1}^{n} F_i - P_{n-1}^* - \lambda_n x_n) + \alpha_n (\sum_{i=1}^{n} F_i - P_{n-1}^* - \gamma_n x_n)^2 + \delta_n + \alpha_n \sigma_{F_{n+1}}^2. \end{aligned} \tag{A.2}$$

Before proceeding with the maximization problem, we examine the semi-strong efficiency condition to get that,

$$\begin{aligned} P_n &= E(F | P_1^*, ..., P_{n-1}^*, x_n + u_n) \\ &= P_{n-1}^* + E[\sum_{i=1}^{n} F_i - P_{n-1}^* | \beta_n (\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n + u_n] \\ &= P_{n-1}^* + \lambda_n (x_n + u_n) \end{aligned}$$

with

$$\lambda_n = \frac{\beta_n(\Sigma_{n-1} + \sigma^2_{F_n})}{\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n} + \sigma^2_u}. \tag{A.3}$$

Analogously, $p^*_n = p^*_{n-1} + \gamma_n x_n$ with

$$\gamma_n = \frac{\beta_n(\Sigma_{n-1} + \sigma^2_{F_n})}{\beta_n^2(\Sigma_{n-1} + \sigma^2_{F_n}) + \sigma^2_{z_n}}. \tag{A.4}$$

In deriving (A.3) and (A.4), we have used the relationship $E(\sum_{i=1}^{n} F_i - P^*_{n-1})^2 = \Sigma_{n-1} + \sigma^2_{F_n}$ resulting from the independence between F_n and $\{F_1, ..., F_{n-1}, P^*_{n-1}\}$. In the following, results of Theorem 1 would be verified separately for *Case (i)* in which the informed investor employs a pure strategy and *Case (ii)* in which the informed investor employs a mixed strategy for the n^{th} trade.

Case (i). In the pure strategy case, $\sigma^2_{z_n} = 0$. From the formula for the informed investor's trading strategy (4) and the market maker's pricing rule (2), we have

$$P^*_n = \sum_{i=1}^{n} F_i, \quad \Sigma_n = 0. \tag{A.5}$$

Consequently,

$$\sum_{i=1}^{n} F_i - P^*_{n-1} = P^*_n - P^*_{n-1} = \gamma_n x_n = \gamma_n \beta_n (\sum_{i=1}^{n} F_i - P^*_{n-1})$$

from which, we obtain

$$\gamma_n = \frac{1}{\beta_n}. \tag{A.6}$$

In this case, we have

$$k_n^2 \equiv Var(P_n^* - P_{n-1}^*) = \Sigma_{n-1} + \sigma_{F_n}^2, \tag{A.7}$$

which, combined with (A.5) ensures that $\Sigma_n = \Sigma_{n-1} + \sigma_{F_n}^2 - k_n^2 = 0$. This means that (9) holds for n once it holds for $n-1$. In other words, pure strategy can ensure that (9) holds if the mixed strategy also ensures it (which will be shown to be the case shortly in Case (ii)).

From (A.5), the second term in (A.2) is zero, since $\sum_{i=1}^{n} F_i - P_{n-1}^* - \gamma_n x_n = \sum_{i=1}^{n} F_i - P_n^* = 0$, and thus the first-order condition (FOC) yields $x_n = \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*)$ with

$$\beta_n = \frac{1}{2\lambda_n}. \tag{A.8}$$

The second-order condition (SOC) requires $\lambda_n \geq 0$ which is equivalent to $k_n \geq 0$ in equilibrium. With $\sigma_{z_n}^2 = 0$, from (A.3), (A.6), (A.7), and (A.8), we have

$$\begin{aligned}
\lambda_n &= \frac{\sqrt{\Sigma_{n-1} + \sigma_{F_n}^2}}{2\sigma_u} = \frac{k_n}{2\sigma_u}, \\
\beta_n &= \frac{\sigma_u}{\sqrt{\Sigma_{n-1} + \sigma_{F_n}^2}} = \frac{\sigma_u}{k_n}, \\
\gamma_n &= \frac{k_n}{\sigma_u}.
\end{aligned}$$

Finally, from these expressions, we can compute

$$E[\pi_n] = \beta_n(1 - \lambda_n\beta_n)(\Sigma_{n-1} + \sigma_{F_n}^2) = \frac{k_n}{2}\sigma_u.$$

Case (ii). In the mixed strategy case, $\sigma_{z_n}^2 > 0$. Note that we discuss this case only for

$n < N$. The FOC of (A.2) gives

$$2(-\lambda_n + \alpha_n \gamma_n^2)x_n + (1 - 2\alpha_n\gamma_n)(\sum_{i=1}^{n} F_i - P_{n-1}^*) = 0.$$

Since this holds for all realizations of z_n in x_n, it requires

$$-\lambda_n + \alpha_n \gamma_n^2 = 0, \quad \text{(A.9)}$$

$$1 - 2\alpha_n\gamma_n = 0, \quad \text{(A.10)}$$

from which, we obtain

$$\gamma_n = 2\lambda_n. \quad \text{(A.11)}$$

From (A.3), (A.4) and (A.11),

$$\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2 = \sigma_u^2, \quad \text{(A.12)}$$

$$\lambda_n = \frac{\beta_n(\Sigma_{n-1} + \sigma_{F_n}^2)}{2\sigma_u^2}. \quad \text{(A.13)}$$

In this case, we have

$$k_n^2 = \gamma_n^2 Var(x_n) = \gamma_n^2[\beta_n^2(\Sigma_{n-1} + \sigma_{F_n}^2) + \sigma_{z_n}^2] = \gamma_n^2\sigma_u^2$$

from which

$$\gamma_n = \frac{k_n}{\sigma_u}. \quad \text{(A.14)}$$

We have used the fact $\gamma_n \geq 0$ which is equivalent to $k_n \geq 0$ (note that if $k_n < 0$, then $\gamma_n = -k_n/\sigma_u$ and replacing k_n with $-k_n$ would not change anything). Indeed $\alpha_n \geq 0$ since otherwise, for some realized variables, the informed investor would get negative profits which would be dominated by not trading. So, from (A.10), $\gamma_n \geq 0$.

From (A.11) and (A.14),

$$\lambda_n = \frac{k_n}{2\sigma_u}. \tag{A.15}$$

Substituting (A.15) in (A.13) yields

$$\beta_n = \frac{k_n \sigma_u}{\Sigma_{n-1} + \sigma_{F_n}^2}. \tag{A.16}$$

Then substituting (A.16) in (A.12) can deliver the volatility of the random component in the order flow:

$$\sigma_{z_n}^2 = (1 - \frac{k_n^2}{\Sigma_{n-1} + \sigma_{F_n}^2})\sigma_u^2. \tag{A.17}$$

We need to verify that $\sigma_{F_n}^2$ is positive in this case, i.e., $k_n^2 < \Sigma_{n-1} + \sigma_{F_n}^2$. Indeed,

$$\begin{aligned} k_n^2 &= Var(P_n^* - P_{n-1}^*) \\ &< Var[(\sum_{i=1}^{n} F_i - P_n^*) + (P_n^* - P_{n-1}^*)] \\ &= Var(\sum_{i=1}^{n-1} F_i - P_{n-1}^*) + \sigma_{F_n}^2 \\ &= \Sigma_{n-1} + \sigma_{F_n}^2, \end{aligned}$$

where the inequality follows from the fact that $(\sum_{i=1}^{n} F_i - P_n^*)$ is independent of $P_n^* - P_{n-1}^*$ and that $Var(\sum_{i=1}^{n} F_i - P_n^*) > 0$ when $\sigma_{z_n}^2 > 0$ (since $Var(\sum_{i=1}^{n} F_i - P_n^*) = Var(\sum_{i=1}^{n} F_i - P_{n-1}^* | \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n)$, positive when z_n is nondegenerate).

Now, from the projection theorem of normal variables, together with equations (A.12) and (A.17), we obtain

$$\begin{aligned}\Sigma_n &= Var(\sum_{i=1}^{n} F_i | P_1^*, ..., P_n^*, x_n) \\ &= Var[\sum_{i=1}^{n} F_i - P_{n-1}^* | \beta_n(\sum_{i=1}^{n} F_i - P_{n-1}^*) + z_n] \\ &= \frac{(\Sigma_{n-1} + \sigma_{F_n}^2)\sigma_{z_n}^2}{\sigma_u^2} = \Sigma_{n-1} + \sigma_{F_n}^2 - k_n^2,\end{aligned}$$

which verifies equation (9) for the mixed strategy case.

Furthermore, from (A.15), (A.16), and (A.17),

$$E[\pi_n] = \beta_n(1 - \lambda_n\beta_n)(\Sigma_{n-1} + \sigma_{F_n}^2) - \lambda_n\sigma_{z_n}^2 = \frac{k_n}{2}\sigma_u.$$

In this case, note that the informed investor strategy parameter β_n in equation (12) is well defined, since k_n is uniquely determined which would be shown later.

Finally, for both cases (i) and (ii), conjecture (A.1) can be justified by backward induction argument since when $n = N$, $\alpha_N = \delta_N = 0$ and when n is replaced by $n - 1$, it still holds, with recursions

$$\alpha_{n-1} = \beta_n(1 - \lambda_n\beta_n) + \alpha_n(1 - \gamma_n\beta_n)^2, \;\; \delta_{n-1} = \delta_n - \lambda_n\sigma_{z_n}^2 + \alpha_n(\sigma_{F_{n+1}}^2 + \gamma_n^2\sigma_{z_n}^2). \quad \text{(A.18)}$$

In conclusion, all results given by equations (4)–(9) hold for both cases.

We now show how k_n is determined. First, from the pricing rule (1), what the informed investor obtains is what noise traders lose, that is

$$E[\pi_n] = \lambda_n\sigma_u^2 = \frac{k_n}{2}\sigma_u.$$

Hence, the maximization objective of informed investor's life-time profits in expectation can be written in reduced form as

$$\max_{\{k_1,\cdots,k_N\}\in\mathbb{R}^N_{\geq 0}} k_1 + \cdots + k_N \tag{A.19}$$

with budgets (22). The solution is unique from the convex optimization theory.

□

Proof of Proposition 2. The correspondence is obvious. □

Proof of Proposition 1. The fact that $\Sigma_n > 0$ $(= 0)$ corresponds to the mixed (pure) strategy have been shown in the proof of Theorem 1. □

Proof of Proposition 1. We solve for the equilibrium dynamically. Let $N = 2$. In period 2, with unused information scale Σ_1 and the information endowment $\sigma^2_{F_2}$, the informed investor's problem is:

$$\max_{k_2^2 \leq \Sigma_1 + \sigma^2_{F_2}} k_2. \tag{A.20}$$

Solving this problem, we obtain $k_2^* = V_2(\Sigma_1) = \sqrt{\Sigma_1 + \sigma^2_{F_2}}$. Hence, from (11), $\Sigma_2 = \Sigma_1 + \sigma^2_{F_2} - k_2^2 = 0$. According to Proposition 1, $\sigma^2_{z_2} = 0$.

Now consider period 1. With information endowment $\sigma^2_{F_1}$, since $\Sigma_1 = \sigma^2_{F_1} - k_1^2$, the informed investor's problem becomes

$$\max_{k_1^2 \leq \sigma^2_{F_1}} k_1 + \sqrt{\sigma^2_{F_1} + \sigma^2_{F_2} - k_1^2}. \tag{A.21}$$

The optimal solution has two cases:

Case 1. If $\sigma_{F_1} > \sigma_{F_2}$, the optimum is $k_1 = \sigma_F/\sqrt{2}$. In this case, $\Sigma_1 = \sigma^2_{F_1} - k_1^2 = (\sigma^2_{F_1} - \sigma^2_{F_2})/2 > 0$, which according to Proposition 1 means $\sigma^2_{z_1} > 0$. Specifically, from (15), $\sigma^2_{z_1} = \frac{\sigma^2_{F_1} - \sigma^2_{F_2}}{2\sigma^2_{F_1}}\sigma^2_u$.

Case 2. If $\sigma_{F_1} \leq \sigma_{F_2}$, the optimum is $k_1 = \sigma_{F_1}$. In this case, $\Sigma_1 = \sigma^2_{F_1} - k_1^2 = 0$. Hence

$\sigma^2_{z_1} = 0$ according to Proposition 1. Results about β, λ, and γ are direct from (12), (13), and (14) respectively. □

Proof of Proposition 2. From the maximization problem defined by (16) and (22), the final constraint is an equality $\sum_{i=1}^{N} k_i^2 = \sum_{i=1}^{N} \sigma^2_{F_i}$. With this equality, we can show that

$$(k_1 - \overline{k})^2 + ... + (k_N - \overline{k})^2 = \sum_{i=1}^{N} k_i^2 - N\overline{k}^2 = \sigma^2_F - \frac{(k_1 + ... + k_N)^2}{N},$$

from which we can observe that the maximization problem defined by (16) and (22) is equivalent to the minimization problem (21) subject to (22) and (23). Since $\lambda_n = k_n/(2\sigma_u)$, problem in Part (2) is equivalent to problem in Part (1). □

Proof of Proposition 2. The *necessity* is obvious. Now for *sufficiency*, from Theorem 1 , we only need to show that $k_n^2 = \sigma_F^2/N$ is feasible for all $n \leq N$. Firstly, $k_1^2 = \sigma_F^2/N$ is feasible since the information available satisfies $\sigma^2_{F_1} \geq \sigma^2_F/N$. In general, if strategies $k^2_{t-1} = \sigma^2_F/N,\ t \leq n$ are all feasible and have been taken by the informed investor, then the feasible space for k_n^2 is $[0, \sigma^2_{F_n} + \Sigma_{n-1}]$ with $\Sigma_{n-1} = \sum_{i=1}^{n-1}(\sigma^2_{F_i} - \frac{\sigma^2_F}{N})$. The condition $\frac{n}{N}\sigma^2_F \leq \sum_{i=1}^{n} \sigma^2_{F_i}$ is equivalent to $\frac{\sigma^2_F}{N} \leq \sigma^2_{F_n} + \Sigma_{n-1}$ which precisely establishes the feasibility of $k_n^2 = \frac{\sigma^2_F}{N}$.

If (26) holds, in equilibrium $k_n = \frac{\sigma_F}{\sqrt{N}}$. Equations (27) and (28) follow directly. By definition, $U_n = \Sigma_n + Var(F_{n+1} + \cdots + F_N) = (1 - n/N)\sigma^2_F$. If (26) holds strictly, with $k_n = \frac{\sigma_F}{\sqrt{N}}$, $\Sigma_n = \sum_{i=1}^{n}(\sigma^2_{F_i} - \sigma^2_F/N) > 0$ for $n \leq N - 1$ and $\Sigma_N = 0$. From Proposition 2, informed investor adopts mixed strategies before the last period and pure strategy in the last period. □

Proof of Proposition 3. Suppose otherwise, if the informed investor does not always play pure strategies, then let us consider the first mixed one. Formally, denote

$$n_0 = inf\{n, \sigma^2_{z_n} > 0\}.$$

Then from Theorem 1 , $k_n^2 = \sigma_{F_n}^2$ and $\Sigma_n = 0$ for $n \leq n_0 - 1$ (if $n_0 = 1$, denote $k_0^2 = \sigma_{F_0}^2 = \Sigma_0 = 0$). Moreover, $k_{n_0}^2 < \sigma_{F_0}^2$ since $\sigma_{z_{n_0}}^2 > 0$ and $\Sigma_{n_0-1} = 0$. Thus there must exist some $n_1 > n_0$, such that $k_{n_1}^2 > \sigma_{F_{n_1}}^2$ to ensure that $\sum_{n=n_0}^{N} k_n^2 = \sum_{n=n_0}^{N} \sigma_{F_n}^2$. Now claim that in this case, if k_{n_0} and k_{n_1} are replaced by $\sqrt{k_{n_0}^2 + \epsilon}$ and $\sqrt{k_{n_1}^2 - \epsilon}$ respectively, with ϵ positive and small enough, and with other k_n unchanged, then $\sum_{n=n_0}^{N} k_n$ can be larger. In fact, since $k_{n_1}^2 > \sigma_{F_{n_1}}^2 \geq \sigma_{F_{n_0}}^2 > k_{n_0}^2$, we can let $\epsilon \in (0, k_{n_1}^2 - k_{n_0}^2)$. Then it is direct to show that

$$\sqrt{k_{n_0}^2 + \epsilon} + \sqrt{k_{n_1}^2 - \epsilon} > k_{n_0} + k_{n_1}.$$

This contradicts the maximization objective (16). Hence, informed investor always adopts pure strategies. From Proposition 1 and (11), $k_n^2 = \sigma_{F_n}^2$ always hold. Equations (32)-(34) follow directly. From $\Sigma_n = 0$ and that $\sigma_{F_n}^2$ increases with n, $U_n = \sigma_{F_{n+1}}^2 + \cdots + \sigma_{F_N}^2 > (1 - n/N)\sigma_F^2,\ n \neq N$. □

Proof of Proposition 4. It is easy to check that Theorem 1 still holds when we replace σ_u^2 with $\sigma_{u_n}^2$ for all expressions. Thus (4)- (6) hold with with parameters:

$$\beta_n = \frac{k_n \sigma_{u_n}}{\Sigma_n + k_n^2}, \tag{A.22}$$

$$\lambda_n = \frac{k_n}{2\sigma_{u_n}}, \tag{A.23}$$

$$\gamma_n = \frac{k_n}{\sigma_{u_n}}, \tag{A.24}$$

$$\sigma_{z_n}^2 = \frac{\Sigma_n}{\Sigma_n + k_n^2}\sigma_{u_n}^2, \tag{A.25}$$

where Σ_n is characterized by (8),(9) and (11).

Further, from

$$E\pi_n = \lambda_n \sigma_{u_n}^2 = \frac{k_n \sigma_{u_n}}{2},$$

the informed investor's aim prior to the nth trade becomes the maximization problem (39) subject to (17).

From the maximization aim (39), we also have the equality $\sum_{i=1}^{N} k_i^2 = \sigma_F^2$. With this equality, a calculation shows directly

$$\sum_{i=1}^{N}(k_i - \frac{\sigma_{u_i}(k_1\sigma_{u_1} + \cdots + k_N\sigma_{u_N})}{{\sigma_{u_1}}^2 + \cdots + {\sigma_{u_N}}^2})^2 = \sigma_F^2 - \frac{(k_1\sigma_{u_1} + \cdots + k_N\sigma_{u_N})^2}{{\sigma_{u_1}}^2 + \cdots + {\sigma_{u_N}}^2}.$$

The LHS can be transformed to the form (with a constant coefficient)

$$\sum_{i=1}^{N}\left[\omega_i(k_i' - \overline{k'})^2\right],$$

where $\omega_i \equiv \sigma_{u_i}^2 / \sum_{j=1}^{N} \sigma_{u_j}^2$, $k_i' \equiv k_i/\sigma_{u_i}$, and $\overline{k'} \equiv \sum_{i=1}^{N} \omega_i k_i'$. Then the maximization problem (39) subject to (17) has the same solution as the minimization problem (40) subject to (22) and (23). With (A.23), (41) is direct. Part (3) is obvious.

□

Proof of Proposition 5. Along the path where information leakage never happens (but still with the probability it happens), the equilibrium can be solved for by the proof of Theorem 1 with only parameters α_n and δ_n replaced by $q\alpha_n$ and $q\delta_n$ respectively in (A.2) and following relationships. Thus, Theorem 1 still holds for this path. Since

$$E\pi_n = q^{n-1}\lambda_n\sigma_u^2 = q^{n-1}k_n\sigma_u/2,$$

the informed investor's aim is (43), subject to (17. Other results can be shown by the same techniques used in the proof of Proposition 4. □

Proof of Proposition 6 and continuous time equilibrium. We discuss equilibrium in two different cases, as we have done in the proof of Theorem 1.

Mixed strategy case. The partially revealing strategy x_t is a mixed one, as

$$dx_t = \beta_t(F(t+dt) - P^*(t))dt + dz_t,$$

where z_t is a Brownian Motion independent of B^F and B^u. When $\Sigma(t) > 0$, the limit results $\beta_t, \lambda_t, \gamma_t, \sigma_{z_t}$ can follow directly from Theorem 1.

Pure strategy case. $\Sigma(t) = 0$. Note that the limit of k_n and σ_u in the discrete time framework correspond to $k_t\sqrt{dt}$ and $\sigma_u(t)\sqrt{dt}$ respectively. Then from expressions (12)-(15), in the limit,

$$\beta(t) = \frac{\sigma_u}{k(t)}, \quad \lambda(t) = \frac{k(t)}{2\sigma_u}, \quad \gamma_t = \frac{k(t)}{\sigma_u}, \quad \sigma_z(t) = 0.$$

Mixed strategy case. $\sigma(t) > 0$. Then from expressions (12)-(15), in the limit,

$$\beta(t) = \frac{k(t)\sigma_u}{\Sigma(t)}, \quad \lambda(t) = \frac{k(t)}{2\sigma_u}, \quad \gamma(t) = \frac{k(t)}{\sigma_u}, \quad \sigma_z(t) = \sigma_u.$$

Other results are natural generalizations of those of discrete time version.

We now show the continuous time equilibrium. $F(t), x(t), P(t), P^*(t)$ and Σ_t are defined the same as (48)-(51) and (53) respectively. The informed investor strategy $x = x(F(t), t)$ aims to maximize the life-time profits as

$$E\int_0^1 (F - P(t+dt))dx = E\int_0^1 (F - P^*(t) - \lambda dx)dx = \int_0^1 \lambda_t \sigma_u^2(t)dt. \tag{A.26}$$

Similarly, we discuss the equilibrium by two cases.

The mixed strategy case. Consider an interval with mixed strategies where $\Sigma(s) > 0$, $s \in [t, t+dt]$. From the projection theory,

$$P(t+dt) - P_t^* = E[F(t+dt) - P_t^*|dx(t) + du(t)] = \lambda_t(dx(t) + du(t))$$

with

$$\lambda_t = \frac{\beta(t)\Sigma_t}{\sigma_z^2(t) + \sigma_u^2(t)}. \tag{A.27}$$

From *Kalman filtering theory* (refer to Kallianpur (2013, p. 269)),

$$\frac{d\Sigma_t}{dt} = \sigma_f^2(t) - \frac{(\beta(t)\Sigma(t))^2}{\sigma_z^2(t)}. \tag{A.28}$$

So the life-time profits (A.26) can be written as

$$\int_0^1 \lambda_t \sigma_u^2(t)dt = \int_0^1 \frac{\beta(t)\Sigma_t\sigma_u^2(t)}{\sigma_z^2(t) + \sigma_u^2(t)}dt = \int_0^1 (\sigma_f^2 - \frac{d\Sigma_t}{dt})^{1/2} \frac{\sigma_z(t)\sigma_u^2(t)}{\sigma_z^2(t) + \sigma_u^2(t)}dt, \tag{A.29}$$

from which the optimal noisiness is

$$\sigma_z(t) = \sigma_u(t). \tag{A.30}$$

Substituting (A.30) and (54) in (A.29), we can get the life-time profits over intervals Ω where informed investor takes mixed strategy, as

$$\int_\Omega \frac{k_t\sigma_u(t)}{2}dt. \tag{A.31}$$

The pure strategy case. Consider an interval with pure strategies where $\Sigma(s) = 0$, $s \in [t, t+dt]$. In this case, the future profits $V_{t+dt}(0)$ would be irrelevant to the trading in this interval, as long as it keeps pure. From dynamic programming theory, x_t only needs to maximize the instantaneous profits, as

$$\max_{x_t}(F(t+dt) - P^*(t) - \lambda_t dx_t)dx_t \tag{A.32}$$

which yields

$$dx_t = \beta(t)(F(t+dt) - P^*(t))$$

with

$$\beta_t = \frac{1}{2\lambda_t}. \tag{A.33}$$

From the projection theory,

$$\lambda_t = \frac{\beta(t)\sigma_f^2(t)}{\beta_t^2\sigma_f^2(t) + \sigma_u^2(t)}. \tag{A.34}$$

Combining (A.34) and (A.33),

$$\lambda_t = \frac{\sigma_f(t)}{2\sigma_u(t)} = \frac{\sigma_k(t)}{2\sigma_u(t)}. \tag{A.35}$$

In sum, the life-time profits in this case can also have the form (A.31) with Ω replaced as Ω^c. Thus, the expected life-time profits have the expression

$$\int_0^1 \frac{k_t\sigma_u(t)}{2}dt.$$

Other results are similar to those in the previous steps. □

Proof of Proposition 8. In this proof, we consider the 2-periods model, and the N-periods cases are similar.

Recall that before all trading begins, the informed investor commits to the following strategies in period-1 and 2, respectively,

$$x_i = \beta_i(F - P^*_{i-1}) + z_i, \;\; i = 1, 2, \tag{A.36}$$

The noise z_i is normally distributed and independent of all other variables. Trading parameters β_i and $Var(z_i)(\equiv \sigma_{z_i}^2))$ are decision variables chosen by the informed investor at the beginning of the economy. The informed investor's aim is to maximize the life-time profits in ex ante expectation:

$$\max_{\{\beta_i, \sigma_{z_i}^2\}_{i=1,2}} E[\pi_1 + \pi_2], \tag{A.37}$$

where $\pi_i = (F - P_i)(\beta_i(F - P_i^*) + z_i),$ for $i = 1, 2$.

With (A.36), the market maker knows that market orders are normally distributed with zero-mean and hence from projection theorem, they set pricing functions as

$$P_i = P_{i-1}^* + \lambda_i(x_i + u_i),\ i = 1, 2,\ and\ P_1^* = P_0^* + \gamma_1 x_1 \tag{A.38}$$

with

$$\lambda_i = \frac{\beta_i(\Sigma_{i-1} + \sigma_{F_i}^2)}{\beta_i^2(\Sigma_{i-1} + \sigma_{F_i}^2) + \sigma_{z_i}^2 + \sigma_u^2},\ i = 1, 2,\ and\ \gamma_1 = \frac{\beta_1(\Sigma_0 + \sigma_{F_1}^2)}{\beta_1^2(\Sigma_0 + \sigma_{F_1}^2) + \sigma_{z_1}^2}. \tag{A.39}$$

Now, we compute the profits in (A.37). With the committed trading strategies (A.36) and corresponding pricing functions (A.38), we can compute

$$E[\pi]_1 = E[(F - P_1)(\beta_1(F - P_0^*) + z_1)] = (1 - \lambda_1\beta_1)\beta_1\sigma_{F_1}^2 - \lambda_1\sigma_{z_1}^2, \tag{A.40}$$

and

$$E[\pi_2] = E[(F - P_2)(\beta_2(F - P_1^*) + z_2)] = (1 - \lambda_2\beta_2)\beta_2 E(F - P_1^*)^2 - \lambda_2\sigma_{z_2}^2, \tag{A.41}$$

where

$$E(F - P_1^*)^2 = \sigma_{F_2}^2 + E(F_1 - P_1^*)^2 = \sigma_{F_2}^2 + (1 - \gamma_1\beta_1)^2\sigma_{F_1}^2 + \gamma_1^2\sigma_{z_1}^2. \tag{A.42}$$

With (A.40), (A.41), and (A.42), we can express the informed investor's problem (A.37) as

$$\max_{\{\beta_i,\sigma_{z_i}^2\}_{i=1,2}} (1-\lambda_1\beta_1)\beta_1\sigma_{F_1}^2 - \lambda_1\sigma_{z_1}^2 + (1-\lambda_2\beta_2)\beta_2[\sigma_{F_2}^2 + (1-\gamma_1\beta_1)^2\sigma_{F_1}^2 + \gamma_1^2\sigma_{z_1}^2] - \lambda_2\sigma_{z_2}^2$$

The maximization about $\sigma_{z_2}^2$ yields, $\sigma_{z_2}^2 = 0$ under the condition $\lambda_2 > 0$. The FOC about β_2 yields $\beta_2 = 1/(2\lambda_2)$. With these results, from (A.39),

$$\beta_2 = \frac{\sigma_u}{\sqrt{\Sigma_1 + \sigma_{F_2}^2}} \quad \textit{and} \quad \lambda_2 = \frac{\sqrt{\Sigma_1 + \sigma_{F_2}^2}}{2\sigma_u}. \tag{A.43}$$

The FOC about $\sigma_{z_1}^2$ yields

$$-\lambda_1 + \frac{\gamma_1^2\sigma_u}{2\sqrt{\Sigma_1 + \sigma_{F_2}^2}} = 0. \tag{A.44}$$

Note that we focus on the case $\sigma_{z_1}^2 > 0$ here (so FOC works). Another case $\sigma_{z_1}^2 = 0$ can be given similarly. With (A.44), the FOC about β_1 gives

$$1 - \frac{\gamma_1\sigma_u}{\sqrt{\Sigma_1 + \sigma_{F_2}^2}} = 0. \tag{A.45}$$

All SOCs are satisfied. (A.44) and (A.45) together give the same result as in the main setup

$$\gamma_1 = 2\lambda_1.$$

These are key steps. Other results including the determination of k_n are the same as those in Theorem 1 and Proposition 1.

□

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