The Prudential Toolkit with Shadow Banking

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Abstract

Several countries now require banks or money market funds to impose state-contingent costs on short-term creditors to absorb financial stress. We study these requirements as part of the broader prudential toolkit in a model with five key ingredients: banks may face an aggregate stress state with high withdrawals; a fire-sale externality motivates a mix of non-contingent and state-contingent regulation; banks may use shadow technologies to circumvent regulation; parameters of the shadow technologies may be private information; and bailouts may occur. We characterize the optimal policy for various combinations of these ingredients and demonstrate that the threat of shadow activities constrains state-contingent regulation more than non-contingent regulation, especially when imperfect information and limited commitment coexist. The planner triggers shadow activities with positive probability under imperfect information, and shadow activities that deplete resources in the stress state elicit larger bailouts under limited commitment, rendering the requirement of state-contingent costs a weak instrument.

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JEL classifications: D62, E61, G01, G21, G28

1 Introduction

Financial crises involve high social costs so banks are heavily regulated. Some regulations introduce state-contingency into the returns of short-term creditors while others do not. Examples of non-contingent regulation include liquidity and capital requirements, which aim to increase the resiliency of banks to outflows and losses so that depositors and other debt-holders are unaffected. Examples of state-contingent regulation include dynamic liquidity fees at institutional prime money market funds when redemption volume is high (U.S.) and bail-ins of debt instruments at banks when capitalization is low (Europe). In both cases, state-contingent regulation introduces explicit variability into investor payoffs to decrease outflows or absorb losses that would otherwise be large.

A major impediment to designing effective regulation is the information gap between banks and regulators. Regulators only see what banks report, and banks can restructure activities within the discretion allowed by accounting standards to provide reports that circumvent the corrective aim of regulation. Bank-sponsored shadow banking is a form of regulatory circumvention. The literature on shadow banking has boomed since the global financial crisis, focusing on activities that are meant to circumvent largely non-contingent regulation; see Acharya et al. (2013) for a prominent example on capital requirements and Hachem and Song (2021) for liquidity requirements. However, little is known about shadow activities that are meant to circumvent state-contingent regulation, including how these activities might impact the effectiveness of non-contingent regulation and how they might affect the size of government bailouts.

In this paper, we study the optimal mix of state-contingent and non-contingent regulation when banks may find it profitable to engage in regulatory circumvention. We characterize this mix as a function of the cost parameters of the shadow activities available to banks, the planner's information about these parameters, and the planner's ability to commit to a bailout policy. This allows us to speak to the optimal mix of regulation for a wide range of environments, from the most to least favorable for the planner. We show that state-contingent regulation is more likely to be constrained by the threat of shadow activities; the range of cost parameters over which the planner cannot implement the unconstrained optimum is larger than for non-contingent regulation. If the planner has imperfect information, then shadow activities are triggered with positive probability and we show that the circumvention of state-contingent regulation interacts more aggressively with the size of the bailout in the absence of commitment, resulting in endogenously asymmetric welfare losses from uncertainty about (or overestimation of) the cost parameters of the state-contingent shadow technologies available to banks.

We start with a benchmark model of bank liquidity management that motivates the use of both state-contingent and non-contingent regulation. Banks in the model obtain funding that can be withdrawn either early or late in the tradition of Diamond and Dybvig (1983). If an aggregate stress state occurs, all banks experience significant early withdrawals, which may require them to raise cash from outside investors by selling assets at an endogenously-determined market price. In anticipation of this, banks make liquidity choices before the realization of the stress state. The key ingredient in our benchmark formulation is that there are two liquidity choices: an ex-ante liquidity ratio and a haircut on early withdrawals in the stress state. A bank determines the haircut it will apply as part of the contract offered to depositors (or short-term creditors more generally) and allocates its funding between cash and a long-term investment project before aggregate uncertainty is resolved. Banks make both decisions taking as given the market price in the stress state, giving rise to a classic pecuniary externality. An individual bank neglects that selling fewer projects in the stress state raises the sale price of these projects, allowing other banks to cover a given cash shortfall with fewer project sales. Accordingly, each bank holds less cash and applies a lower haircut than would be socially optimal.

In response to the pecuniary externality, the planner introduces a floor on the ex-ante liquidity ratio (non-contingent regulation) along with a floor on the haircut that must be applied in the stress state (state-contingent regulation). Which regulation is used more—in the sense of contributing more to a decline in the cash shortfall in the stress state—depends on the probability and severity of the stress state. We show that the planner will use state-contingent regulation more than non-contingent regulation when the stress state is severe but unlikely; relying more on non-contingent regulation would imply idle cash with high probability and lead to a welfare gain from imposing marginally higher costs on depositors to forego marginally fewer projects. This underscores that state-contingent and non-contingent regulation are not perfect substitutes.

We then add three ingredients to the benchmark model. The first ingredient is shadow activities that banks can use to relax binding regulation. With two regulations, we consider two shadow activities. The first allows banks to invest in more projects without affecting the liquidity ratio on their balance sheet. The second allows banks to impose less state-contingency on depositors without affecting the contracted haircut. While both activities are engaged before the resolution of uncertainty, only the cost of the state-contingent action is incurred in the stress state, augmenting the amount of liquidity that the bank needs to generate through project sales. The cost of the non-contingent action is instead incurred as part of the portfolio allocation decision, detracting from the amount of funding invested in the bank's project.

Shadow activities are privately beneficial to banks in the presence of binding regulation but socially wasteful, so the planner never finds it optimal to trigger them, leading to implementation constraints on the design of regulation. These constraints require that the marginal benefit of each shadow activity to banks does not exceed the marginal cost. We show that the lowest cost parameter at which the planner can implement the unconstrained optimum without violating the implementation constraint for state-contingent regulation exceeds the lowest cost parameter at which he can do so without violating the one for non-contingent regulation. In other words, state-contingent regulation is more likely to be constrained by the threat of shadow activities than non-contingent regulation. This reflects that the marginal benefits of the two shadow activities to banks are the same while the marginal cost of the state-contingent activity is only incurred in the stress state and at a price that neglects the externality.

If both regulatory instruments are constrained by shadow activities, then both must be set below their unconstrained solutions. If only one regulatory instrument is constrained, then the planner will compensate when the externality is sufficiently strong by setting the other regulation above its unconstrained solution. In this case, we show that the welfare loss from changing one shadow activity from prohibitively expensive to free, conditional on the other shadow activity remaining prohibitively expensive, is the same for both activities if the stress state probability is such that the two regulations would be used equally in the absence of shadow activities. At this particular probability, then, the total welfare loss from each shadow activity is symmetric, which serves as a useful baseline against which to compare the effects of additional ingredients.

The second ingredient is a bailout instrument for the planner. A bailout in the stress state diverts funds from the production of a socially valuable public good but decreases the amount of liquidity that banks need to raise through project sales. By increasing the sale price of projects, this decreases the incentives of banks to hold liquidity and apply haircuts, which is the classic moral hazard concern associated with bailouts. We show that the threat of shadow activities leads the planner to use the bailout instrument for bailout costs at which he would not otherwise. We also show that the planner's ability to commit to a bailout size matters for optimal policy in the model if and only if shadow activities are a threat. This is because the size of the bailout affects the planner's implementation constraints through the moral hazard concern when banks can engage in shadow activities. Without commitment, the threat of one shadow activity now motivates both regulations being set below their unconstrained solutions. A bailout instrument without commitment also weakly lowers welfare compared to the case where the planner does not have this instrument. In contrast, a planner who can commit to a bailout size will always do at least as well with the bailout instrument. With or without commitment though, there is again symmetry in the total welfare loss from changing one shadow activity from prohibitively expensive to free, conditional on the other shadow activity remaining prohibitively expensive, when both regulations would be used equally in the absence of shadow activities.

The third ingredient is imperfect information about the cost parameters of the shadow technologies available to banks. We consider a planner that is uncertain about these costs and chooses the constrained optimal policy recognizing his uncertainty. We also consider a naive planner that ignores these costs altogether. The key departure in both cases is that shadow activities may now occur as part of the regulated equilibrium that the planner implements. We show that emergence of shadow activities in equilibrium increases the size of the bailout when the planner lacks commitment and that this interaction is more pronounced for state-contingent regulation than for non-contingent regulation, resulting in asymmetric welfare losses even if the two regulations would be used equally in the absence of shadow activities.

Without commitment, the bailout is larger when state-contingent regulation is circumvented in equilibrium because this circumvention cost directly lowers the sale price of projects in the stress state. The circumvention cost for non-contingent regulation instead directly lowers project output, which increases the share that banks have to sell in the stress state at a given price but not the price itself. This difference is immaterial when the planner knows the cost parameters because regulation is designed to not trigger shadow activities; the welfare losses from the threat of shadow activities then only stem from the implementation constraints they introduce. With imperfect information, however, even the constrained optimal regulation can trigger shadow activities, which sets off the differential effects on bailouts in the absence of commitment.

With imperfect information and lack of commitment, the constrained optimal planner achieves lower welfare when he is uncertain about the cost to banks of circumventing state-contingent regulation than when he is uncertain about the cost to banks of circumventing non-contingent regulation. The naive planner also generates a larger welfare loss from naively using state-contingent regulation than naively using non-contingent regulation, as well as an amplification of welfare losses when both regulations are being circumvented. In particular, when the circumvention of one regulation elicits a bailout in the stress state, anticipation of this bailout increases the extent to which the other regulation binds on banks, increasing their incentive to also circumvent the other regulation. This elicits an even larger bailout, which feeds back into the incentive to circumvent the first regulation, and so on.

While our paper is primarily theoretical, we close with some empirical evidence that ties our study of state-contingent regulation and its circumvention to reality. Compared to non-contingent regulation whose circumvention has been well-documented in the literature that followed the global financial crisis, much less is known about the circumvention of state-contingent regulation. The setting we explore for this purpose is the issuance of contingent convertible bonds by European banks and the provision of credit lines as a potential form of insurance to investors against conversion. We find evidence that banks provide more credit lines when they issue more of these bonds, with price movements suggesting that the lines decrease the degree of state-contingency in the bonds. Thus, the threat of shadow activities extends to state-contingent regulation and should be closely studied in light of our theoretical results.

Related Literature Our paper is related to three strands of literature. First is the literature on pecuniary externalities. Lorenzoni (2008) shows that because atomistic agents ignore the general equilibrium price effects of their choices, over-borrowing occurs and justifies regulation. Our paper contains a pecuniary externality operating through fire sales. See Shleifer and Vishny (1992) and Kiyotaki and Moore (1997) for early theoretical work on fire sales and Davila and Korinek (2018) for a more recent treatment. Evidence of fire sales is presented in Pulvino (1998), Aguiar and Gopinath (2005), and Campbell et al. (2011), among others. Chernenko and Sunderam (2020) also provide evidence of fire sale externalities that justify policy intervention. A pecuniary externality that leads to fire sales motivates regulation in our model. Importantly, however, we are interested in two types of regulation, as well as bailouts and regulatory circumvention in this environment.

Our paper also relates to the literature on bailouts and bail-ins. A large literature examines the benefits and costs of bank bailouts when lack of government commitment regarding ex post bailout provision creates moral hazard by influencing banks' ex ante risk-taking. Key contributions include Gorton and Huang (2004), Diamond and Rajan (2012), Farhi and Tirole (2012), Keister (2016), Chari and Kehoe (2016), Bianchi (2016), and Davila and Walther (2020). While complete elimination of bailouts is generally found to be inefficient, the moral hazard problem associated with bailout provision justifies regulatory interventions to limit the extent to which bailouts are used. One approach to limiting bailouts is the requirement of bail-ins by bank creditors, which can take various forms such as suspension of convertibility, issuance of contingent convertible bonds, or imposition of redemption fees. See Bernard et al. (2022), Keister and Mitkov (2023), X. Huang and Keister (2024), and Walther and White (2020) for recent contributions. Voluntary bail-ins are found to be inefficiently small, necessitating regulatory measures to force a minimum level of loss absorption by creditors. This aligns with the state-contingent regulation in our model. We contribute to the literature here by introducing the possibility of regulatory circumvention through shadow activities and characterizing the optimal mix of state-contingent and non-contingent regulation as well as bailout provision in this environment.

Finally, our paper relates to the literature on shadow banking. For detailed overviews of shadow banking activities in the U.S., Euro Area, and China, see Pozsar et al. (2010), Bakk-Simon et al. (2012), and Hachem (2018), respectively. Shadow banking is often sponsored by traditional banks to loosen financial constraints, most prominently regulation. Theoretical work finds that shadow banking may increase efficiency in good times but also heightens financial fragility and exposure to aggregate tail risks through interconnections with regulated banks, e.g., Gennaioli et al. (2013), Moreira and Savov (2017), and Ordonez (2018). Accordingly, it may be optimal to relax bank regulation when shadow activities are available; see Plantin (2015) and J. Huang (2018) for contributions in this vein. Our model distinguishes between two types of regulation and hence two types of shadow activities. While shadow activities that circumvent non-contingent regulation can increase returns outside of the stress state, both types of shadow activities exacerbate the stress state and are welfare-deteriorating. Our contribution is the characterization of the optimal regulatory response to the threat of shadow activities, differentiating between state-contingent and non-contingent regulation. Existing work on shadow banking has focused on the latter. We find that state-contingent regulation is more constrained by the threat of shadow activity, especially when limited information and limited commitment of the planner coexist.

The rest of the paper proceeds as follows: Section 2 introduces the benchmark model that motivates a mix of state-contingent and non-contingent regulation; Section 3 studies optimal policy when banks can take shadow actions to circumvent regulation and the planner has full information about the cost functions of these actions; Section 4 adds a bailout instrument to the planner's toolkit and studies the effect on optimal regulation with and without commitment; Section 5 presents the results when the planner is imperfectly informed about shadow activities; Section 6 connects our modeling to features of the data and presents evidence consistent with shadow activities around state-contingent regulation; Section 7 concludes. All proofs are collected in Appendix A.

2 Benchmark Model

In this section, we present a benchmark model of bank liquidity management that motivates corrective regulation. The model is a parsimonious version of existing models with pecuniary externalities. The key ingredient in our formulation is that there are two liquidity choices, only one of which is state-contingent. This leads to a mix of state-contingent and non-contingent regulation that we build on in the rest of the paper.

2.1 Environment

There are three dates, $t \in \{0, 1, 2\}$, and a unit mass of identical, price-taking banks. Each bank has funding normalized to one at $t = 0$ and allocates its funding between liquid and illiquid assets. The allocation to liquid assets (cash) is denoted by $\lambda \in [0, 1]$. We model the illiquid asset as a project that returns $f(1 - \lambda)$ at $t = 2$, where $f(0) = 0$, $f'(\cdot) \geq 1$, and $f''(\cdot) < 0$. Projects run from $t = 0$ to $t = 2$ and cannot be liquidated in between. However, at $t = 1$, a bank can sell a claim to share $s \in [0, 1]$ of its project's future return at a price q per share, which gives the bank cash $qsf(1 - \lambda)$ at $t = 1$ and requires a payment of $sf(1 - \lambda)$ at $t = 2$. In other words, the bank sells a one-period bond with face value $sf(1 - \lambda)$ and yield $1/q - 1$.

Bank funding at $t = 0$ takes the form of deposits that can be withdrawn at $t = 1$ or $t = 2$. We focus on aggregate rather than idiosyncratic bank risk. With probability $p \in (0, 1)$, an aggregate stress state occurs at $t = 1$ in which a fraction $\theta \in (0, 1)$ of depositors withdraw from each bank; the remaining fraction $1 - \theta$ withdraw at $t = 2$. With probability $1 - p$, there are zero withdrawals at $t = 1$; everyone withdraws at $t = 2$.

Banks can impose a haircut $h \in [0, 1]$ on deposit withdrawals at $t = 1$ if the stress state is realized, reducing the payment to depositors from θ to $\theta(1 - h)$. The size of the haircut is decided at $t = 0$ and repaid to depositors at $t = 2$. That is, depositors are made whole but with a delay. The repayment is simply to keep the accounting tidy; it can be dropped with only notational effects.

Whereas λ bolsters the bank's liquidity position ex ante, h bolsters it ex post. The cost of higher λ is foregone project returns. The cost of higher h is variability in depositor payoffs, which risk averse depositors do not like. Formally, the cost to the bank of imposing a haircut on depositors is denoted by $\Phi(p, \theta, h)$, where $\Phi(\cdot)$ is zero if any of its arguments are zero and increasing in each argument. This cost can be interpreted as additional compensation, over and above making the affected depositors whole, that the bank has to provide at $t = 2$ regardless of whether the stress state materializes at $t = 1$ to entice depositors to participate at $t = 0$; for this reason, the size of the haircut is decided at $t = 0$. To simplify the exposition, we use the functional form $\Phi(p, \theta, h) = p\theta\phi(h)$, where $\phi(0) = 0$, $\phi'(\cdot) \geq 0$, and $\phi''(\cdot) > 0$.

The timeline of events is summarized in Figure 1. The ingredients we have introduced so far appear in black text; ingredients we will introduce later in the paper—namely, regulation, regulatory circumvention, and bailouts—are marked in gray.

We now turn to the determination of the price q. Conditional on its choices of λ and h at $t = 0$, each bank faces a cash shortfall equal to $\max\{\theta(1 - h) - \lambda, 0\}$ at $t = 1$ if the stress state occurs. To cover this shortfall, each bank sells share s of future project returns to satisfy

$$qsf(1 - \lambda) = \max\{\theta(1 - h) - \lambda, 0\} \tag{1}$$

Shares are sold to short-horizon outside investors who have deep pockets and a return technology $g(\cdot)$ from $t = 1$ to $t = 2$. Thus, to give banks cash $qsf(1 - \lambda)$ at $t = 1$, investors need a return of

$g\left(qsf\left(1-\lambda\right)\right)$ at $t=2$. This pins down q as the solution to

$$g\left(qsf\left(1-\lambda\right)\right)=sf\left(1-\lambda\right) \tag{2}$$

The function $g\left(\cdot\right)$ has the properties $g\left(0\right)=0$, $g'\left(\cdot\right)\geq 1$, and $g''\left(\cdot\right)>0$. These properties imply that the elasticity of $g(x)$ with respect to its argument x satisfies $\varepsilon_g\left(x\right)\equiv\frac{xg'(x)}{g(x)}\geq 1$ with equality if and only if $x=0$. In other words, the marginal cost $g'\left(x\right)$ of project sales $x>0$ exceeds the average cost $\frac{g(x)}{x}$ and hence it becomes more costly to raise cash via project sales as the cash shortfall, $x=\max\left\{\theta\left(1-h\right)-\lambda,0\right\}$, increases.

Eqs. (1) and (2) implicitly define the price q as a function of the bank choice variables λ and h, namely,

$$q=\frac{\theta\left(1-h\right)-\lambda}{g\left(\theta\left(1-h\right)-\lambda\right)}\text{ if }\theta\left(1-h\right)>\lambda \tag{3}$$

However, with a unit mass of banks, each individual bank takes q as given when making its choices. Partial differentiation of Eq. (3) yields

$$\frac{\partial q}{\partial\lambda}=\frac{1}{\theta}\frac{\partial q}{\partial h}=\frac{1}{g\left(\theta\left(1-h\right)-\lambda\right)}\left(\frac{\theta\left(1-h\right)-\lambda}{g\left(\theta\left(1-h\right)-\lambda\right)}g'\left(\theta\left(1-h\right)-\lambda\right)-1\right)>0 \tag{4}$$

where the inequality follows from the elasticity of $g\left(\cdot\right)$. The effect of λ and h on q is not internalized by the banks, giving rise to a pecuniary externality which we formalize below.

2.2 Decentralized Equilibrium

Suppose $\theta\left(1-h\right)>\lambda$ so that project sales occur at $t=1$; we verify this conjecture later. Then the expected profit of a representative bank is

$$\Pi\left(\lambda,h;q\right)\equiv\left(1-p\right)\left[\lambda+f\left(1-\lambda\right)-1\right]+p\left[\left(1-s\right)f\left(1-\lambda\right)-\left(1-\theta\right)-\theta h\right]-p\theta\phi\left(h\right)$$

where s is given by Eq. (1). In the non-stress state, which occurs with probability $1-p$, the bank experiences no withdrawals at $t=1$ so it carries the full value of its liquid assets λ to $t=2$, earns the full return $f\left(1-\lambda\right)$ from its project, and pays out all depositors. In the stress state, which occurs with probability p, the bank only earns fraction $\left(1-s\right)$ of the return from its project at

$t = 2$, having sold share s to outside investors to cover deposit withdrawals at $t = 1$ in excess of liquid assets after applying the haircut. The bank has to repay the haircut h to the affected depositors at $t = 2$ and also pay out the remaining depositors. The last term in the bank's expected profit is the cost of imposing a haircut on risk averse depositors, i.e., $\Phi(\cdot)$.

The bank chooses λ and h to maximize $\Pi(\lambda, h; q)$ taking as given the price q. The first order condition for an interior liquidity ratio λ is

$$\underbrace{\frac{p}{q}}_{MB_\lambda^{private}} = \underbrace{f'(1-\lambda) - (1-p)}_{MC_\lambda} \tag{5}$$

The left-hand side of Eq. (5) is the private marginal benefit of holding an additional unit of liquid assets, namely that the bank has to sell a smaller share of its project in the stress state. By avoiding the marginal sale, the bank avoids giving up a rate of return $1/q$ to outside investors in a state of the world that occurs with probability p. The right-hand side of Eq. (5) is the marginal cost of holding an additional unit of liquid assets, namely the foregone marginal return $f'(\cdot)$ from investing more funding into the project. If the stress state is not realized, which occurs with probability $1-p$, then the foregone return is net of the marginal return of the liquid assets carried into $t = 2$.

The first order condition for an interior haircut h is

$$\underbrace{\frac{p}{q}}_{MB_h^{private}} = \underbrace{p\left(1 + \phi'(h)\right)}_{MC_h} \tag{6}$$

The left-hand side of Eq. (6) is the private marginal benefit of imposing a higher haircut to raise an additional unit of liquidity in the stress state. It is the same as the private marginal benefit in Eq. (5), namely that the bank has to sell less of its project in the stress state. The right-hand side of Eq. (6) is the marginal cost of imposing a higher haircut to raise an additional unit of liquidity in the stress state, namely the repayment of the haircut to affected depositors at $t = 2$ and the additional compensation $\phi'(\cdot)$.

Ex ante cash holdings λ and ex post haircuts h have different marginal costs. However, both serve the same purpose—decreasing the need for project sales in the stress state—so they naturally have the same marginal benefit. Accordingly, their marginal costs must be equalized in equilibrium.

Definition 1 *The decentralized equilibrium is a triple* $\{\lambda^*, h^*, q^*\}$ *that solves Eqs. (5) and (6) with* q *given by Eq. (3).*

The lemmas that follow characterize some properties of the decentralized equilibrium. First, we establish that banks do not cover all of their liquidity needs in the stress state through project sales if the return required by outside investors makes doing so sufficiently costly:

Lemma 1 *If* $\frac{g(\theta)}{\theta} > 1 + \phi'(0)$*, then there is no decentralized equilibrium where* $\lambda^* = h^* = 0$*.*

Next, we establish that banks hold liquidity ex ante in addition to imposing haircuts ex post unless the marginal returns to projects are very high (in which case only haircuts are used) or very low (only ex ante liquidity holdings are used) relative to the marginal costs of haircuts:

Lemma 2 *If* $f'(1) = 1 + p\phi'(0)$*, then* $\lambda^* > 0$ *and* $h^* > 0$*. Higher* $f'(1)$ *weakly expands the parameter space where* $\lambda^* = 0$*, while lower* $f'(1)$ *weakly expands the parameter space where* $h^* = 0$*.*

Finally, we establish that the liquidity choices of banks leave them in need of project sales at discounted prices in the stress state:

Proposition 1 *(Project sales in the decentralized equilibrium). If* $g'(0) \geq 1$*, then* $q^* < 1$ *in any equilibrium where* $\theta(1 - h^*) > \lambda^*$*. If also* $g'(0) \leq 1 + \phi'(0)$*, then an equilibrium with* $h^* > 0$ *has* $\theta(1 - h^*) > \lambda^*$*.*

Consider as an example $\phi'(0) = 0$. Then $f'(1) = 1$ and $g'(0) = 1$ imply that the decentralized equilibrium has $\lambda^* > 0$, $h^* > 0$, $\theta(1 - h^*) > \lambda^*$, and $q^* < 1$. In other words, both forms of liquidity are used but project sales still occur in the stress state, verifying the conjecture at the beginning of this section. We carry these properties throughout the rest of the paper. Without loss of generality, the reader can simply assume $\phi'(0) = 0$, $f'(1) = 1$, and $g'(0) = 1$, bearing in mind that these are sufficient rather than necessary conditions.

2.3 Planner's Solution and Optimal Regulatory Mix

The planner chooses λ and h to maximize $\Pi(\lambda, h; q)$ taking into account the effect of these choices on the price q in Eq. (3).[1] The planner's first order conditions for λ and h are

$$\underbrace{\left(1+\frac{\theta(1-h)-\lambda}{q}\frac{\partial q}{\partial \lambda}\right)\frac{p}{q}}_{MB_{\lambda}^{social}} = \underbrace{f'(1-\lambda)-(1-p)}_{MC_{\lambda}} \tag{7}$$

and

$$\underbrace{\left(1+\frac{\theta(1-h)-\lambda}{q}\frac{1}{\theta}\frac{\partial q}{\partial h}\right)\frac{p}{q}}_{MB_{h}^{social}} = \underbrace{p\left(1+\phi'(h)\right)}_{MC_{h}} \tag{8}$$

respectively. Comparing Eqs. (7) and (8) to the first order conditions of the representative bank in Eqs. (5) and (6), the only difference is that the planner internalizes the effects of λ and h on q, as indicated by the terms $\frac{\partial q}{\partial \lambda}$ and $\frac{\partial q}{\partial h}$. Accordingly, the social marginal benefit of either holding more liquid assets at $t=0$ or imposing a higher haircut in the stress state at $t=1$ exceeds the private marginal benefit; there is no difference between the private and social marginal costs. The social marginal benefits of λ and h are the same for the same reason that the private marginal benefits were the same above, so the planner's solution will also equalize the marginal costs.

Definition 2 *The planner's benchmark solution is a triple* $\left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$ *that solves Eqs. (7) and (8) with* q *given by Eq. (3).*

The equalization of marginal costs is common for the planner and the decentralized banks, as is the determination of the price q. The key difference is that the planner takes into account the marginal cost of project sales when assessing the benefits of liquidity (ex ante and ex post) whereas the decentralized banks only take into account the average cost by taking q as given. The elasticity $\varepsilon_g(\cdot)$ thus governs the strength of the pecuniary externality.

The following proposition formalizes the difference between the planner's solution and the decentralized equilibrium. In particular, the planner chooses higher ex ante liquidity holdings as well

[1] We give the planner the same objective function as the banks to isolate the role of the pecuniary externality. Recall that this objective function incorporates the utility of depositors through the term $\Phi(\cdot)$. Formally including the payoffs of outside investors does not change the results. Suppose outside investors have an endowment e and an output function $G(\cdot)$. Deep pockets corresponds to $e >> 0$. With probability p, these investors only produce $G(e-x)$ because they fund the cash shortfall x of the banks. They must then receive compensation $g(x) \equiv G(e) - G(e-x)$ for doing so. Their expected payoff is then $(1-p)G(e) + p(G(e-x) + g(x))$ which simplifies to $G(e)$, a constant.

as higher ex post haircuts than the banks, which follows intuitively from the fact that the social marginal benefit of both choices exceeds the private marginal benefit:

Proposition 2 *(Inefficiency of the decentralized equilibrium). The decentralized equilibrium involves inefficiently low choices of liquid assets ($\lambda^* < \widehat{\lambda}$) and haircuts ($h^* < \widehat{h}$) by banks.*

A corollary of Proposition 2 is that the decentralized equilibrium experiences a higher cash shortfall than the planner's solution in the stress state. It then follows from Eqs. (1) and (3) that the bonds sold by banks to outside investors in the stress state will have a higher yield $1/q - 1$ and a higher face value $sf(\cdot)$ than those sold under the planner's solution.

The optimal regulation is straightforward; the planner simply introduces regulatory floors on both choice variables in the bank optimization problem to lift them to the socially optimal level:

Proposition 3 *(Optimal regulation absent regulatory circumvention). The planner can implement $\{\lambda^*, h^*, q^*\} = \left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$ by imposing regulations $\lambda \geq \alpha$ and $h \geq \beta$ on the decentralized problem, where $\alpha = \widehat{\lambda}$ and $\beta = \widehat{h}$.*

The regulation $\lambda \geq \alpha$ applies at $t = 0$, regardless of the state of the world that is realized at $t = 1$. It is therefore a non-contingent regulation. In contrast, the regulation $h \geq \beta$ only applies in the stress state at $t = 1$, increasing the variability of depositor payoffs across states of the world. Accordingly, it is a state-contingent regulation.

Lemma 3 *A higher probability p of the stress state increases $\widehat{\lambda}$ and $\widehat{q}$ and decreases $\widehat{h}$. A higher withdrawal fraction θ in the stress state increases $\widehat{\lambda}$ and $\widehat{h}$ and decreases $\widehat{q}$.*

All else constant, a higher probability p of the stress state increases the net benefit of using cash to bolster bank liquidity positions ex ante. Accordingly, the planner's choice of $\widehat{\lambda}$ rises, which increases the price $\widehat{q}$ in the stress state and decreases the need for haircuts $\widehat{h}$ to bolster liquidity positions ex post. A corollary is that the planner will impose a high haircut in the stress state rather than keep liquid assets idle with high probability when the stress state is unlikely. In contrast, higher withdrawals θ in the stress state directly lower the price and increase the net benefit of using both cash and haircuts to bolster bank liquidity positions. It then follows from Proposition 3 that higher θ increases both the non-contingent regulation α and the state-contingent regulation β while higher p increases α but decreases β.[2]

While the planner will use both regulations, which one will be used more? The following lemma provides an intuitive condition:

Lemma 4 *Consider* $f'''(\cdot) = \phi'''(\cdot) = 0$. *Then* $\widehat{h} - h^* > \widehat{\lambda} - \lambda^*$ *if and only if* $p\phi'' < -f''$, *and* $\theta\left(\widehat{h} - h^*\right) > \widehat{\lambda} - \lambda^*$ *if and only if* $p\phi'' < -\theta f''$.

Here we are considering two complementary interpretations of a regulation being used more; the first is if it leads to a bigger change in the regulated variable relative to the decentralized solution, and the second is if it contributes more to a decline in the cash shortfall in the stress state. Non-contingent regulation leads to idle liquid assets if the stress state is not realized. This is more costly when the return function for investment projects is more concave. State-contingent regulation only targets liquidity in the stress state and hence avoids the cost of idle resources. If the stress state is rare and the marginal cost of haircuts does not increase as rapidly as the marginal cost of not investing in projects, i.e., the first condition in Lemma 4, then the planner prefers to use state-contingent regulation, pushing ex post haircuts further above h^* than he pushes ex ante liquidity ratios above λ^*. A stronger condition in the same direction, i.e., the second condition in Lemma 4, implies that the planner will also rely more on haircuts than liquidity ratios to reduce the cash shortfall in the stress state. While Lemma 4 makes some simplifying assumptions on higher-order derivatives to get a closed-form condition, the intuition behind this condition applies more generally and underscores that state-contingent and non-contingent regulation are not perfect substitutes.

3 Model with Regulatory Circumvention

The analysis so far has assumed that banks cannot take any actions to circumvent regulation. We now relax this assumption and explore how optimal regulation (Proposition 3) changes.

[2]Note that increasing or decreasing regulation is not the same as tightening or loosening regulation. The latter depends on how much the planner's solution changes relative to the decentralized one, or equivalently how the Lagrange multiplier on the regulatory constraint in the bank's problem changes, which depends on higher order derivatives of the functional forms. For example, if $g''(\cdot)$ is sufficiently large, then increasing α and β in response to higher θ is equivalent to tightening both regulations. We do not impose such conditions here.

3.1 Shadow Activities

We allow for two shadow activities that can be used by banks in response to regulation at $t = 0$. These activities represent changes in how business operations are structured or reported within the discretion allowed by accounting standards without changing the economic nature of the operations.

First, the representative bank can take an action $\omega_\lambda \geq 0$ at a cost $\kappa_\lambda \omega_\lambda \lambda$ to relax its non-contingent regulatory constraint to $(1 + \omega_\lambda)\lambda \geq \alpha$. The cost of this action is increasing in the amount by which the bank increases its accounting liquidity ratio, $(1 + \omega_\lambda)\lambda$, relative to its economic liquidity ratio, λ, with $\kappa_\lambda > 0$. An example of the action ω_λ is moving funding into an off-balance-sheet vehicle that is outside the regulatory perimeter. The cost to do so is a monetary incentive to depositors to make this move. See Hachem and Song (2021) for a model with this form of regulatory circumvention.

Second, the bank can take a separate action $\omega_h \geq 0$ at a cost $\theta \kappa_h \omega_h h$ to relax its state-contingent regulatory constraint to $(1 + \omega_h) h \geq \beta$. The cost of this action is increasing in the amount by which the bank increases its contracted haircut, $(1 + \omega_h) h$, relative to its economic haircut, h, with $\kappa_h > 0$. An example of the action ω_h is the provision of insurance to depositors (or short-term creditors more generally) against a haircut, for instance, by issuing a credit line that can be taken down in the stress state and only fully recognized on the bank's balance sheet when taken down. The cost to do so is a potential capital charge. See Section 6.1 for a formalization of this example.

While both actions can be taken at $t = 0$, only the cost of the state-contingent action ω_h is incurred in the stress state. We model the cost $\kappa_\lambda \omega_\lambda \lambda$ as paid at $t = 0$. It thus detracts from the amount of funding invested in the bank's project.[3] The cost $\theta \kappa_h \omega_h h$ is instead paid at $t = 1$ if the stress state is realized. It thus augments the amount of liquidity that the bank needs to generate through project sales. Formally, the expected profit of a representative bank becomes

$$\begin{aligned}\widetilde{\Pi}(\lambda, h, \omega_\lambda, \omega_h; q) \equiv\ & (1-p)\left[\lambda + f(1 - \lambda - \kappa_\lambda \omega_\lambda \lambda) - 1\right] \\ & + p\left[(1-s) f(1 - \lambda - \kappa_\lambda \omega_\lambda \lambda) - (1-\theta) - \theta h\right] - p\theta\phi(h)\end{aligned}$$

[3] In the exact modeling of Hachem and Song (2021), the monetary incentive to depositors is payable at $t = 2$. Moving the payment of $\kappa_\lambda \omega_\lambda \lambda$ from $t = 0$ to $t = 2$ would only amplify our later results about the planner being more constrained by the threat of state-contingent shadow activities.

where s is now given by

$$qsf\left(1-\lambda-\kappa_{\lambda}\omega_{\lambda}\lambda\right)=\theta\left(1-h+\kappa_{h}\omega_{h}h\right)-\lambda \tag{9}$$

and the required return of outside investors pins down the price q as

$$q=\frac{\theta\left(1-h+\kappa_{h}\omega_{h}h\right)-\lambda}{g\left(\theta\left(1-h+\kappa_{h}\omega_{h}h\right)-\lambda\right)} \tag{10}$$

Notice from Eqs. (9) and (10) that neither the yield $1/q-1$ nor the face value $sf(\cdot)$ of the bonds sold by banks to outside investors in the stress state are affected by the cost $\kappa_{\lambda}\omega_{\lambda}\lambda$; an increase in this cost only increases the share s of projects that banks relinquish to offset the decline in project output and maintain a constant face value. In contrast, an increase in the cost $\theta\kappa_{h}\omega_{h}h$ will increase both the yield and the face value of the bonds, where the increase in the face value is achieved through an increase in s.

The bank now chooses λ, h, ω_{λ}, and ω_{h} to maximize $\widetilde{\Pi}\left(\lambda,h,\omega_{\lambda},\omega_{h};q\right)$ subject to the constraints $\left(1+\omega_{\lambda}\right)\lambda\geq\alpha$ and $\left(1+\omega_{h}\right)h\geq\beta$ taking as given the price q. The solution to this problem, together with Eq. (10), is the regulated equilibrium conditional on regulations α and β.

3.2 Constrained Optimal Regulation

The planner chooses the non-contingent regulation α and the state-contingent regulation β to select the regulated equilibrium that achieves the highest value of $\widetilde{\Pi}(\cdot)$ taking into account the effect of bank choices on the price q. The resulting α and β constitute the constrained optimal regulation.

Proposition 4 *(Planner never triggers shadow activities). There is no shadow activity at the constrained optimal regulation.*

The intuition for Proposition 4 comes from the fact that shadow activities are socially wasteful; they use up resources that could otherwise be invested or used to increase the sale price of projects. Accordingly, the planner never finds it optimal to set regulation that triggers shadow activities. The planner's choices of α and β are therefore subject to implementation constraints that ensure shadow activities are not profitable for banks. These constraints require that the private marginal benefits of shadow activities do not exceed their marginal costs.

Formally, the implementation constraint that ensures banks find it optimal to choose $\omega_\lambda = 0$, i.e., non-contingent regulation is not circumvented, is

$$\underbrace{\overbrace{\left[f'(1-\lambda)-(1-p)\right]}^{MC_\lambda} - \overbrace{\frac{p}{q}}^{MB_\lambda^{private}}}_{MB_{\omega_\lambda}} \leq \underbrace{\kappa_\lambda f'(1-\lambda)}_{MC_{\omega_\lambda}} \text{ evaluated at } \{\lambda, h\} = \{\alpha, \beta\}$$

If the left-hand side of this constraint is positive, then a bank would like to choose $\lambda < \alpha$, incentivizing the action $\omega_\lambda > 0$. The right-hand side of the constraint must be sufficiently large that this action is too costly, in terms of foregone project returns, to be profitable.

The implementation constraint that ensures banks find it optimal to choose $\omega_h = 0$, i.e., state-contingent regulation is not circumvented, is

$$\underbrace{\overbrace{p\left(1+\phi'(h)\right)}^{MC_h} - \overbrace{\frac{p}{q}}^{MB_h^{private}}}_{MB_{\omega_h}} \leq \underbrace{\frac{p\kappa_h}{q}}_{MC_{\omega_h}} \text{ evaluated at } \{\lambda, h\} = \{\alpha, \beta\}$$

If the left-hand side of this constraint is positive, then a bank would like to choose $h < \beta$, incentivizing the action $\omega_h > 0$. The right-hand side of the constraint must be sufficiently large that this action is too costly, in terms of required project sales in the stress state, to be profitable.

In the limiting case of $\kappa_\lambda \to \infty$ and $\kappa_h \to \infty$, the implementation constraints are always slack so the planner can implement the optimal regulation because shadow activities are prohibitively expensive for banks to use. The logic extends to sufficiently high costs, as shown next:

Proposition 5 *(Cost thresholds for binding implementation constraints). There are finite positive thresholds $\overline{\kappa}_\lambda \in (0,\infty)$ and $\overline{\kappa}_h \in (0,\infty)$ such that the planner can implement $\{\widehat{\lambda}, \widehat{h}, \widehat{q}\}$ if and only if $\kappa_\lambda \geq \overline{\kappa}_\lambda$ and $\kappa_h \geq \overline{\kappa}_h$. Moreover, $\overline{\kappa}_h > \overline{\kappa}_\lambda$.*

The cost threshold $\overline{\kappa}_\lambda$ makes the implementation constraint for non-contingent regulation hold with equality at $\{\alpha, \beta\} = \{\widehat{\lambda}, \widehat{h}\}$; it would be violated for lower κ_λ and slack otherwise. Similarly, the cost threshold $\overline{\kappa}_h$ makes the implementation constraint for state-contingent regulation hold with equality at $\{\alpha, \beta\} = \{\widehat{\lambda}, \widehat{h}\}$; it would be violated for lower κ_h and slack otherwise.

Importantly, Proposition 5 establishes $\overline{\kappa}_h > \overline{\kappa}_\lambda$. This indicates that the optimal regulation

$\{\widehat{\lambda}, \widehat{h}\}$ from Proposition 3 is more likely to violate the implementation constraint for state-contingent regulation than the one for non-contingent regulation at equal cost parameters. To understand this result, recall from Eqs. (7) and (8) that $\{\widehat{\lambda}, \widehat{h}\}$ equates the marginal costs of λ and h to their social marginal benefits. Substituting these social marginal benefits into the implementation constraints for $\{\alpha, \beta\} = \{\widehat{\lambda}, \widehat{h}\}$, it follows that each cost threshold equalizes the private marginal cost of the relevant shadow action with the wedge between the social and private marginal benefits of the regulation being circumvented. Recalling that λ and h have the same marginal benefits and thus the same wedges, it also follows that the thresholds $\overline{\kappa}_\lambda$ and $\overline{\kappa}_h$ equate the private marginal costs of the two shadow actions, i.e., $\overline{\kappa}_\lambda f'(1 - \widehat{\lambda}) = \frac{p\overline{\kappa}_h}{\widehat{q}}$. The cost of ω_h is only paid in the stress state while the cost of ω_λ is paid regardless of the state of the world. Moreover, the resources to pay the cost of ω_h are raised at the price q which neglects the externality associated with project sales. Accordingly, it will take a higher κ_h to make banks indifferent towards circumventing state-contingent regulation as compared to the κ_λ that makes them indifferent towards circumventing non-contingent regulation, i.e., $\overline{\kappa}_h > \overline{\kappa}_\lambda$.

The exact magnitude of $\overline{\kappa}_h$ is pinned down by the strength of the pecuniary externality, as captured by the elasticity $\varepsilon_g(\cdot)$ evaluated at the cash shortfall under the optimal regulation. Mathematically, $\overline{\kappa}_h = \varepsilon_g\left(\theta\left(1 - \widehat{h}\right) - \widehat{\lambda}\right) - 1$. If the elasticity is very close to 1, then the marginal cost of project sales in the stress state only slightly exceeds the average cost. Accordingly, the externality is small and the optimal regulation minimally binding on the decentralized equilibrium. The marginal benefit to banks of circumventing minimal regulation is negligible and hence $\overline{\kappa}_h$ is very close to zero, meaning the planner can implement the optimal regulation at almost all circumvention cost parameters κ_h. A larger elasticity implies a larger externality and thus more aggressive regulation in the absence of shadow activities. The circumvention benefits to banks are then higher, as is $\overline{\kappa}_h$. The magnitude of $\overline{\kappa}_h$ moves one-for-one with the elasticity because the cost of circumventing state-contingent regulation is paid in the stress state at the price affected by the externality. The magnitude of $\overline{\kappa}_\lambda$ depends on the same elasticity but does not move one-for-one with it.

In the rest of this section, we consider circumvention costs κ_λ and κ_h such that at least one of the planner's implementation constraints binds. The planner must therefore move away from the optimal regulation $\{\widehat{\lambda}, \widehat{h}\}$ to ensure that inefficient shadow activity is not triggered.

Proposition 6 *(Regulatory complementarity when only one shadow activity is a threat). Suppose* $\varepsilon_g'(\cdot) > 0$*. If* $\kappa_\lambda > \overline{\kappa}_\lambda$*, then the planner chooses* $\alpha > \widehat{\lambda}$ *and* $\beta < \widehat{h}$ *as* κ_h *is perturbed below* $\overline{\kappa}_h$*. If* $\kappa_h > \overline{\kappa}_h$*, then the planner chooses* $\alpha < \widehat{\lambda}$ *and* $\beta > \widehat{h}$ *as* κ_λ *is perturbed below* $\overline{\kappa}_\lambda$*.*

Intuitively, a tighter implementation constraint on one regulation decreases the extent to which that regulation can be used. This decreases the project sale price q which in turn increases the marginal benefit of the other regulation. If this other regulation is costly for banks to circumvent, the planner will then set it above the level he would have chosen in the absence of shadow activities to compensate for the inability to set the constrained regulation at the level he would like; see also Davila and Walther (2024) when perfectly and imperfectly regulated decisions are complements and the imperfectly regulated decision is associated with a negative externality. The condition $\varepsilon_g'(\cdot) > 0$ on the elasticity of the return function of outside investors in Proposition 6 connects to their taxonomy by delivering the relevant degree of complementarity between bank decisions, namely that the strength of the externality is increasing in the cash shortfall to which both decisions contribute.

Naturally though, the resulting complementarity between regulatory instruments, where one regulation is used more aggressively in response to a weakening of the other, disappears if both instruments are weakened by the threat of shadow actions, as established next:

Proposition 7 *(Limits to regulatory complementarity when both shadow activities are a threat). If both* κ_λ *and* κ_h *are low, then the planner sets* $\alpha < \widehat{\lambda}$ *and* $\beta < \widehat{h}$*. In the limiting case of* $\kappa_\lambda = \kappa_h = 0$*, he cannot do better than the decentralized equilibrium, i.e.,* $\alpha = \lambda^*$ *and* $\beta = h^*$*.*

Both implementation constraints bind when both shadow activities are sufficiently inexpensive for banks to use. In the limiting case of costless shadow activities, the entire effect of any regulation $\alpha > \lambda^*$ and/or $\beta > h^*$ is absorbed by the shadow activities $\omega_\lambda > 0$ and/or $\omega_h > 0$. Thus, the planner cannot do better than the decentralized equilibrium.

Figure 2 illustrates the constrained optimal levels of α and β as functions of κ_λ and κ_h over an entire grid of these cost parameters. As a benchmark, we set all other parameters so that the planner would use non-contingent and state-contingent regulation to the same extent in the absence of shadow activities, i.e., $\widehat{\lambda} - \lambda^* = \theta\left(\widehat{h} - h^*\right)$, which corresponds to $p\phi'' = -\theta f''$ in Lemma 4.

There are four regions defined by whether or not each implementation constraint is binding on the planner. These regions are demarcated in panel (a), with the planner's choices of α and β plotted in panels (c) and (d) respectively. Panel (b) plots the social welfare achieved under the constrained optimal regulation as compared to the optimal regulation studied in Section 2.3 (denoted $\widehat{\Pi}$) and the decentralized equilibrium studied in Section 2.2 (denoted Π^*).

When κ_λ and κ_h are both high, neither implementation constraint is binding, so the planner sets $\alpha = \widehat{\lambda}$ and $\beta = \widehat{h}$ as per Proposition 5. When κ_λ is high but κ_h is low, only the second implementation constraint (on β to ensure $\omega_h = 0$) is binding, so the planner sets $\alpha > \widehat{\lambda}$ and $\beta < \widehat{h}$, consistent with Proposition 6. Notice that for very low κ_h, the planner sets $\beta < h^*$, meaning he accepts lower haircuts than would prevail in the decentralized equilibrium. While this speaks to the weakness of the β instrument when κ_h is very low, the β chosen by the planner as part of the constrained optimal policy still constrains the banks; absent it, the privately optimal choice of h would fall even further below h^* as the tighter regulation on λ increases the price q and decreases the private marginal benefit of imposing haircuts in the stress state. When κ_λ is low but κ_h is high, only the first implementation constraint (on α to ensure $\omega_\lambda = 0$) is binding, so the planner sets $\alpha < \widehat{\lambda}$ and $\beta > \widehat{h}$, again consistent with Proposition 6. For very low κ_λ, the planner sets $\alpha < \lambda^*$, which has the same intuition as $\beta < h^*$ in the previous case. Finally, when both κ_λ and κ_h are low, both implementation constraints are binding, so the planner sets $\alpha < \widehat{\lambda}$ and $\beta < \widehat{h}$ as per Proposition 7. Within this region, the planner may set either $\alpha < \lambda^*$ or $\beta < h^*$ but never both.

For any combination of $\kappa_\lambda \in (0, \overline{\kappa}_\lambda)$ and $\kappa_h \in (0, \overline{\kappa}_h)$, the constrained optimal regulation achieves welfare strictly higher than the decentralized equilibrium and strictly lower than the optimal regulation. The welfare surface plotted in panel (b) also exhibits the same overall decline as κ_h falls towards zero and as κ_λ falls towards zero when keeping the cost of the other shadow activity prohibitively high. This property reflects the assumption of $p\phi'' = -\theta f''$ in the figure, as formalized next:

Lemma 5 *Suppose $\varepsilon_g'(\cdot) > 0$ and $f'''(\cdot) = \phi'''(\cdot) = g'''(\cdot) = 0$. Welfare is the same under the constrained optimal regulation for $\kappa_\lambda = 0$ and $\kappa_h \to \infty$ as for $\kappa_\lambda \to \infty$ and $\kappa_h = 0$ if $p\phi'' = -\theta f''$.*

If instead $p\phi'' < -\theta f''$, then the marginal cost of haircuts increases less rapidly than the marginal cost of holding liquid assets. Accordingly, the planner would prefer to use state-contingent regulation in the absence of shadow activities to not leave resources idle in the non-stress state; recall Lemma 4. The inability to use such regulation as κ_h declines and triggers the relevant implementation constraint would then be costlier, especially when the stress state is rare, as tightening the non-contingent regulation to fully compensate would leave resources idle with high probability. As a result, the welfare surface in panel (b) would exhibit an overall larger decline as κ_h falls towards zero than it would as κ_λ falls towards zero. The opposite would occur if $p\phi'' > -\theta f''$.

The thresholds $\overline{\kappa}_\lambda$ and $\overline{\kappa}_h$ help anchor the four regions in Figure 2. The following lemma explores how these thresholds depend on key parameters in the model:

Lemma 6 *If $\varepsilon_g'(\cdot) > 0$, then $\frac{d\overline{\kappa}_h}{dp} < 0$, $\frac{d(\overline{\kappa}_h/\overline{\kappa}_\lambda)}{dp} < 0$, $\frac{d\overline{\kappa}_h}{d\theta} > 0$, and $\frac{d\overline{\kappa}_\lambda}{d\theta} > 0$.*

As the probability p of the stress state falls, the threshold $\overline{\kappa}_h$ increases both in absolute terms and relative to the threshold $\overline{\kappa}_\lambda$. Together with $\overline{\kappa}_h > \overline{\kappa}_\lambda$ from Proposition 5, this suggests that the implementation constraint on the planner's choice of state-contingent regulation will bind for an even larger area of the parameter space than the implementation constraint for non-contingent regulation. The threat of the shadow activity ω_h is therefore especially constraining relative to the threat of the shadow activity ω_λ when the stress state is rare. This reflects that the planner would like to rely more heavily on state-contingent regulation when the stress state is rare (to avoid idle liquidity in the non-stress state) but cannot do so without triggering actions that will require additional liquidity should the stress state materialize.

As the severity θ of the stress state rises, both $\overline{\kappa}_h$ and $\overline{\kappa}_\lambda$ increase. Recall from Lemma 3 that the planner would like to increase both ex ante liquidity ratios and ex post haircuts if the stress state will be more severe. The comparative statics in Lemma 6 indicate that both increases are larger than what banks would choose themselves in response to higher θ, hence the private marginal benefit to a bank of both shadow activities increases. The cost parameter at which the bank no longer finds it profitable to undertake the shadow action then also increases.

4 Regulation and Bailouts

We now extend the set of instruments available to the planner. In addition to setting regulations α and β, the planner can provide a bailout $b \geq 0$ in the stress state. The size of the bailout is chosen with regulation at $t = 0$ if the planner can commit to the choice of b; otherwise, the size of the bailout is chosen at $t = 1$, after the realization of the stress state. This section explores how, if at all, the addition of a bailout instrument affects the planner's choice of regulation with and without the possibility of regulatory circumvention, i.e., conditional on the cost parameters κ_λ and κ_h.

4.1 Bailout Instrument

A bailout $b > 0$ will decrease the amount of liquidity that banks need to raise through project sales in the stress state. In particular, the share s of future project returns sold by banks to outside investors now only needs to satisfy

$$qsf\left(1-\lambda-\kappa_\lambda\omega_\lambda\lambda\right)=\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b$$

which implies a sale price of

$$q=\frac{\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b}{g\left(\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b\right)} \tag{11}$$

instead of Eq. (10). The optimization problem of the representative bank is otherwise unchanged and takes as given the size of the bailout b.

All else constant, a bailout increases the price q which in turn decreases the incentives of banks to hold liquidity at $t = 0$ and apply haircuts in the stress state at $t = 1$, as formalized next:

Lemma 7 *Introducing a bailout $b > 0$ into the decentralized equilibrium would decrease λ^* and h^*.*

The social cost of the bailout is that it diverts funds from the production of a public good which is valued at $\nu > 0$ per unit of the good. Accordingly, the planner now chooses the non-contingent regulation α, the state-contingent regulation β, and the bailout b to select the regulated equilibrium that achieves the highest value of $\widetilde{\Pi}(\cdot) - p\nu b$ taking into account the effect of bank choices on q.

We first establish how the existence of a bailout instrument affects the optimal regulation in the benchmark model of Section 2:

Proposition 8 *(Optimal regulation with bailout instrument). In the absence of shadow activities ($\kappa_\lambda \to \infty$ and $\kappa_h \to \infty$), the planner's ability to commit to a bailout size at $t = 0$ is irrelevant for* *the optimal policy. The planner always chooses* $\alpha = \widehat{\lambda}$*,* $\beta = \widehat{h}$*, and* $b = 0$ *if* $\nu \geq \overline{\nu}_0$ *and* $\alpha < \widehat{\lambda}$*,* $\beta < \widehat{h}$*, and* $b > 0$ *if* $\nu < \overline{\nu}_0$*, where* $\overline{\nu}_0 \in (0, \infty)$*.*

A bailout lowers the marginal benefits of regulation, so when providing a bailout is not too costly in terms of foregone production of the public good, the planner prefers to provide it and loosen both regulations. Loosening the non-contingent regulation frees up resources for investment in projects, while loosening the state-contingent regulation reduces the cost imposed on depositors in the stress state. Furthermore, the optimal policy does not depend on commitment when $\kappa_\lambda \to \infty$ and $\kappa_h \to \infty$; the planner faces the same trade-off between raising the sale price q and producing less of the public good whether he chooses b at $t = 0$ or $t = 1$.

4.2 Constrained Optimal Policy

We now return to the environment with $\kappa_\lambda < \infty$ and $\kappa_h < \infty$, where banks may engage in shadow activities and the planner's choices are therefore subject to implementation constraints. The following proposition establishes several results about the effect of introducing the bailout instrument into this environment, including that the constrained optimal policy depends on commitment:

Proposition 9 *(Constrained optimal regulation with bailout instrument). The planner chooses* $\{\alpha, \beta, b\} = \left\{\widehat{\lambda}, \widehat{h}, 0\right\}$ *if* $\kappa_\lambda \geq \overline{\kappa}_\lambda$*,* $\kappa_h \geq \overline{\kappa}_h$*, and* $\nu \geq \overline{\nu}_0$*. Suppose* $\varepsilon_g'(\cdot) > 0$ *and* $\nu = \overline{\nu}_0$*. With commitment,*

- *If* $\kappa_\lambda > \overline{\kappa}_\lambda$*, then the planner chooses* $\alpha = \widehat{\lambda}$*,* $\beta < \widehat{h}$*, and* $b > 0$ *as* κ_h *is perturbed below* $\overline{\kappa}_h$*.*
- *If* $\kappa_h > \overline{\kappa}_h$*, then the planner chooses* $\alpha < \widehat{\lambda}$*,* $\beta = \widehat{h}$*, and* $b > 0$ *as* κ_λ *is perturbed below* $\overline{\kappa}_\lambda$*.*

Without commitment, the planner chooses $\alpha < \widehat{\lambda}$*,* $\beta < \widehat{h}$*, and* $b > 0$ *in either perturbation.*

There are three takeaways from Proposition 9. First, the planner may use the bailout instrument instead of tightening one regulation to compensate for weakening of the other due to shadow activities. Recall the regulatory complementarity established in Proposition 6 and notice that it is optimally replaced by a bailout for the same perturbations in Proposition 9. Second, the threat of shadow activities leads the planner to use the bailout instrument at some cost parameters ν where he would not otherwise. Recall the absence of a bailout for $\nu = \overline{\nu}_0$ in Proposition 8 and notice the provision of a bailout for $\nu = \overline{\nu}_0$ in Proposition 9. Third, the planner's ability to commit to a bailout size at $t = 0$ affects optimal policy if and only if shadow activities are a threat; the "if" statement follows from Proposition 9 and the "only if" from Proposition 8 where commitment was irrelevant in the absence of shadow activities. If the planner cannot commit when shadow activities are a threat, then a perturbation of either κ_λ or κ_h below their thresholds will lead the planner to choose both $\alpha < \widehat{\lambda}$ and $\beta < \widehat{h}$ alongside $b > 0$. That is, without commitment, the possibility of one shadow activity can motivate the curtailment of both regulations.

The relevance of commitment in Proposition 9 stems from the fact that the bailout size affects the implementation constraints that arise when banks may engage in shadow activities. With commitment, all of the planner's choices are made at $t = 0$ subject to these constraints. The planner internalizes that a larger bailout will decrease the incentives of banks to hold liquidity and apply haircuts and thereby increase their incentives to circumvent regulation, all else the same. Without commitment, the bailout is chosen at $t = 1$ where there are no implementation constraints because regulation has already been set. The planner anticipates that he will provide a larger bailout at $t = 1$ in the absence of commitment, which decreases the marginal benefit of regulation at $t = 0$ and leads him to curtail both regulations even if only one shadow activity is a threat.

Figure 3 provides a graphical illustration of the effect of the bailout instrument on the constrained optimal policy. We use the same parametrization as in Figure 2 and set the parameter for the bailout cost ν equal to $\overline{\nu}_0$ as in Proposition 9. The top row visualizes the effect of varying κ_h while keeping $\kappa_\lambda > \overline{\kappa}_\lambda$; the bottom row visualizes the effect of varying κ_λ while keeping $\kappa_h > \overline{\kappa}_h$. Without the bailout instrument, the results of Proposition 6 apply, namely the compensatory tightening of α for $\kappa_h < \overline{\kappa}_h$ in the top row and the compensatory tightening of β for $\kappa_\lambda < \overline{\kappa}_\lambda$ in the bottom row. With the bailout instrument, the results of Proposition 9 apply, namely the use of bailouts rather than compensatory tightening, with larger bailouts and curtailment of both regulations in the absence of commitment.

Figure 4 plots the constrained optimal policy as a function of both κ_λ and κ_h. We set ν somewhat higher than $\overline{\nu}_0$ to generalize the cross-sections in Figure 3. With $\nu > \overline{\nu}_0$, the planner does not start using the bailout instrument as κ_h is perturbed below $\overline{\kappa}_h$ when κ_λ is high. Thus, the results of Proposition 6 apply even as κ_h falls below $\overline{\kappa}_h$, with the planner increasing α above

$\widehat{\lambda}$ as he decreases β below $\widehat{h}$.[4] Once κ_h falls sufficiently below $\overline{\kappa}_h$, it becomes optimal to use the bailout instrument, at which point the results reflect Proposition 9, namely that α flattens out under commitment (left column) and falls when the planner cannot commit to the size of the bailout (right column). Similar patterns arise for β as κ_λ falls below $\overline{\kappa}_\lambda$ when κ_h is high. Figure 4 also plots the effect of the bailout instrument on welfare. With commitment, the planner will always do at least as well with a bailout instrument as without, and with $\nu > \overline{\nu}_0$, the welfare gains are limited to within the region where at least one implementation constraint binds. In contrast, bailouts without commitment weakly lower welfare compared to the case where the planner does not have a bailout instrument, i.e., Figure 2. With or without commitment though, the welfare effects of reducing the cost of one shadow activity from prohibitively high to zero (while keeping the cost of the other shadow activity prohibitively high) are again the same for both shadow activities, reflecting the assumption of $p\phi'' = -\theta f''$ in the figure.

The next lemma establishes that lower bailout costs can eliminate the threat of moderately costly shadow activities that would otherwise constrain the planner:

Lemma 8 *For $\nu \leq \overline{\nu}_0$, a decrease in ν shrinks the ranges of κ_λ and κ_h for which the planner's implementation constraints bind.*

The intuition is that the planner relies less on regulation and more on bailouts when the bailout cost is low, reducing the marginal benefit to banks of taking costly shadow actions to circumvent regulation. Accordingly, the range of circumvention cost parameters κ_λ and κ_h over which the planner is constrained by the threat of shadow activities falls. As bailouts become more costly, the planner would like to rely more on regulation and hence the threat of shadow activities constrains the optimal policy over more of the parameter space. For any ν though, it remains the case that $\overline{\kappa}_h > \overline{\kappa}_\lambda$ and thus that the threat of shadow activities is more likely to constrain state-contingent regulation than non-contingent regulation for equal circumvention cost parameters.

[4] This increase in α is visible in the second row of Figure 4 with commitment and would be more visible without commitment for larger ν. We notice from the bottom row of Figure 4 that a bailout arises sooner without commitment, at which point the results of Proposition 6 no longer apply.

5 Imperfectly Informed Planner

The analysis so far has assumed that the planner knows the values of the parameters κ_λ and κ_h that govern the costs of the shadow activities to banks. We now relax this assumption. We first consider a planner that ignores (or greatly overestimates) these costs. We then consider a planner that is uncertain about the costs and chooses the constrained optimal policy recognizing this uncertainty. The key departure in both cases is that shadow activities may now occur as part of the regulated equilibrium that the planner implements.

5.1 Welfare Costs of Naive Regulation

A "naive" planner is unaware of the shadow actions available to banks and therefore sets α and β as if κ_λ and κ_h are prohibitively high. As an example, if $\nu \geq \overline{\nu}_0$, a naive planner will set a non-contingent regulation of $\alpha = \widehat{\lambda}$, a state-contingent regulation of $\beta = \widehat{h}$, and intend a bailout of $b = 0$ if the stress state is realized; see Proposition 8. However, these regulations will trigger shadow activities if at least one of κ_λ or κ_h is actually low; see Proposition 5. If the stress state is then realized, the planner will discover that banks' cash shortfall exceeds $\theta\left(1 - \widehat{h}\right) - \widehat{\lambda}$, in which case $b = 0$ may no longer be appropriate. Anticipating that the planner may re-optimize and choose $b > 0$ in the absence of commitment, banks may engage in more shadow activity than implied by Proposition 5 since a bailout in the stress state decreases private incentives to hold more liquid assets or apply higher haircuts; see Lemma 7.

The next lemma explores the marginal welfare loss from triggering a shadow action through naive regulation. We compare specifically the partial derivatives of the social welfare function with respect to ω_λ and ω_h to assess which shadow action is more costly to trigger on the margin. The idea is to isolate the effect on welfare of the same changes in ω_λ and ω_h, not to shock parameters to trigger these activities since the effect on welfare would then also depend on the sensitivity of each shadow action to the parameters being shocked.

Lemma 9 *Suppose $\phi'(0) = 0$. When the stress state is rare, the marginal effect on social welfare of triggering $\omega_h > 0$ is more negative than the marginal effect of triggering $\omega_\lambda > 0$.*

The shadow action $\omega_h > 0$ incurs a direct cost $\theta\kappa_h\omega_h h$ only in the stress state, which is the state of the world where cash is most valuable. This cost exacerbates the cash shortfall that must be covered by project sales to outside investors, decreasing the price q as well as the share of projects retained by banks. In contrast, the direct cost $\kappa_\lambda \omega_\lambda \lambda$ associated with the shadow action $\omega_\lambda > 0$ is incurred upfront at $t = 0$; it decreases project output and hence the share of projects retained by banks in the stress state for a given price but not the price itself. When the stress state is rare, state-contingent regulation is actively used in the absence of shadow activities; recall Lemma 3. Triggering $\omega_h > 0$ is then unambiguously costlier than triggering $\omega_\lambda > 0$ because the direct cost of each shadow action increases with the notion of liquidity being circumvented.

Proposition 10 *(Welfare loss from naive regulation without commitment). When κ_λ and κ_h are low, any planner that cannot commit to a bailout size at $t = 0$ achieves weakly lower welfare than the decentralized equilibrium with no regulation and no bailout instrument. The welfare loss relative to the decentralized equilibrium is highest if α and β are also set naively.*

The first part of Proposition 10 reflects the welfare costs of no commitment. When κ_λ and κ_h are low, regulation is too weak to overcome the negative incentive effects of bailouts on bank liquidity choices. The second part of the proposition highlights that the combination of naive regulation and no commitment can do materially worse than a purely laissez-faire economy.

Figure 5 explores whether the welfare losses from setting α and β naively are symmetric. Panel (a) plots the welfare surface achieved by naive regulation with no bailout instrument under the same parameters as Figure 2. Similar to panel (b) in Figure 2, which plotted welfare under the constrained optimal regulation for each combination of κ_λ and κ_h with no bailout instrument, the welfare surface in panel (a) of Figure 5 exhibits the same overall decline when κ_h falls towards zero as when κ_λ falls towards zero, reflecting the assumption of $p\phi'' = -\theta f''$ in the parametrization. The welfare losses from setting α and β naively are therefore symmetric in the absence of a bailout instrument when the parametrization delivers symmetric usage of the two regulations, i.e., $\widehat{\lambda} - \lambda^* = \theta \left(\widehat{h} - h^* \right)$.

The same does not hold when the planner has a bailout instrument without commitment. Panel (b) in Figure 5 plots the difference between the welfare surface achieved by naive regulation with and without a bailout instrument, assuming no commitment and the same bailout cost parameter ν as Figure 4. The welfare losses from setting α and β naively become asymmetric once the bailout instrument is introduced, even under symmetric usage. This can be seen by comparing the light green and dark green troughs in panel (b) of Figure 5. The entirety of panel (b) is weakly negative, reflecting that a bailout instrument without commitment weakly lowers welfare. The light green trough captures the largest welfare loss when setting the non-contingent regulation α naively if κ_h is large enough that banks will not find it profitable to circumvent the state-contingent regulation. The dark green trough captures the largest welfare loss when setting the state-contingent regulation β naively if κ_λ is large enough that banks will not find it profitable to circumvent the non-contingent regulation. Notice that the dark green trough is deeper than the light green one, indicating asymmetry. Adding the asymmetric differences in panel (b) to the symmetric surface in panel (a) will not deliver a symmetric surface. Instead, the combination of naive regulation with a bailout instrument when the planner cannot commit will lead to bigger welfare losses from naively using state-contingent regulation than naively using non-contingent regulation.

Panel (b) in Figure 5 also exhibits an amplification when banks find it profitable to circumvent both regulations. This can be seen from the blue trough, which captures the largest welfare loss when both regulations are set naively. Notice that the blue trough does not occur at the origin where shadow activities are free; even though the most shadow activity occurs here, no additional resources are consumed. Instead, the blue trough occurs away from the origin but still in the area where both κ_λ and κ_h are low enough that banks engage in both shadow activities. The blue trough is deeper than the sum of the light green and dark green troughs, indicating amplification between regulatory circumvention and bailouts when there are multiple regulations being circumvented.

The combination of naive regulation and lack of commitment on the bailout is key to both the asymmetry and amplification in panel (b). With commitment, the naive planner would announce a bailout of $b = 0$ at $t = 0$ since $\nu \geq \overline{\nu}_0$. Panel (b) would then be flat at zero, i.e., not using the bailout instrument achieves the same welfare as not having the bailout instrument. Consider next the constrained optimal planner without commitment. Panel (c) in Figure 5 plots the difference between the welfare surface achieved by the constrained optimal regulation for each combination of κ_λ and κ_h with and without the same bailout instrument considered in panel (b) when the planner cannot commit. A bailout instrument without commitment again weakly lowers welfare, but the effects in panel (c) are symmetric, unlike those in panel (b).

This discussion highlights an interaction between shadow activities and bailouts that is more pronounced for state-contingent regulation than for non-contingent regulation, even at parameters where the benchmark model delivers symmetric usage. Naive regulation, whether state-contingent or non-contingent, triggers regulatory circumvention via shadow activities, and regulatory circumvention elicits a bailout in the stress state when the planner discovers that the cash shortfall exceeds $\theta\left(1-\widehat{h}\right)-\widehat{\lambda}$ by enough to justify the marginal bailout cost ν. However, the bailout is larger for circumvention of state-contingent regulation because the circumvention cost κ_h directly lowers the price q in the stress state. The circumvention cost κ_λ instead directly lowers project output, which increases the share that banks have to sell in the stress state at a given price.

Panel (d) in Figure 5 isolates the negative effect of naive regulation due to the bailout instrument when the planner cannot commit, plotting panel (b) net of panel (c). In light of the symmetric effects in panel (c), panel (d) inherits the properties of panel (b), namely asymmetry and amplification in the direction of larger welfare losses from the misuse of state-contingent regulation.[5] Recall that these properties emerge despite the symmetric parametrization; the welfare losses from misusing state-contingent regulation in the absence of commitment would be even larger in an environment with $p\phi'' < -\theta f''$, e.g., a very rare but very severe stress state.

5.2 Constrained Optimality Under Uncertainty and No Commitment

We now consider a planner who is aware that banks may engage in shadow activities but uncertain about the cost parameters of these activities. To fix ideas, we consider uncertainty about either κ_λ or κ_h, setting the other high enough that its shadow activity can be non-naively ignored.

Consider first $\kappa_\lambda \to \infty$ so that only the use of state-contingent regulation is potentially constrained. The problem now arises from the fact that the planner does not know the exact value of κ_h. Instead, he only knows that κ_h is between zero and some upper bound $\kappa_h^{max} \in (\overline{\kappa}_h, \infty)$ and forms beliefs defined by a probability density function over $\kappa_h \in [0, \kappa_h^{max}]$. For simplicity, suppose the planner has uniformly distributed beliefs. We compare this to the opposite case of $\kappa_h \to \infty$ with uncertainty about κ_λ so that only the use of non-contingent regulation is potentially constrained. Here, the planner has uniformly distributed beliefs over $\kappa_\lambda \in [0, \kappa_\lambda^{max}]$.

The following proposition establishes some basic properties of the constrained optimal regulation

[5] Notice that panel (d) is positive when only one of κ_λ and κ_h is high. This reflects that a bailout instrument without commitment has a more negative effect on the constrained optimal level of welfare than on the naive level of welfare at these corners. This positive effect would disappear with commitment. With commitment, the surface in panel (c) would be weakly increasing, instead of weakly decreasing, as κ_λ and κ_h fall but would still be symmetric. Panel (b) would be flat at zero, as noted above. Panel (d) would then be the inverse of panel (c) and hence symmetric and never positive.

under uncertainty before we illustrate the effects on welfare and the expected size of the bailout when the planner cannot commit. Setting $\kappa_\lambda^{max} = 1$ delivers a closed-form solution when the use of non-contingent regulation is potentially constrained, so we make that assumption in the proposition and relax it in the numerical results that follow.

Proposition 11 *(Constrained optimal regulation under uncertainty without commitment). Consider no commitment and $\nu = \overline{\nu}_0$ so that the bailout instrument is always weakly used. If $\kappa_\lambda \to \infty$ and $\kappa_h \sim U\left(0, \kappa_h^{\max}\right)$, then the constrained optimal regulation under uncertainty is $\alpha = \widehat{\lambda}$ and $\beta_u\left(\kappa_h^{\max}\right) < \widehat{h}$, where $\beta_u'\left(\kappa_h^{\max}\right) > 0$. If $\kappa_h \to \infty$ and $\kappa_\lambda \sim U\left(0, 1\right)$, then the planner again uses both regulations, but now setting $\alpha_u < \widehat{\lambda}$ and $\beta = \widehat{h}$.*

Compare to the results in Proposition 9, where the planner knew the circumvention costs. There, when the planner could not commit to the size of the bailout, the threat of shadow activity led to both non-contingent and state-contingent regulation being weakened in favor of a bailout. Recall that the larger bailout provided by the planner in the absence of commitment decreased the marginal benefit of regulation (by decreasing the cash shortfall in the stress state) all else the same. Here, all else is not the same; larger bailouts now also incentivize more shadow activity at low circumvention costs, and more shadow activity increases the cash shortfall and hence the marginal benefit of regulation. Accordingly, the planner sets the unaffected regulation (non-contingent if $\kappa_\lambda \to \infty$ and state-contingent if $\kappa_h \to \infty$) at its optimal level from Section 2.3 and only weakens the affected regulation, in addition to providing a bailout. The affected regulation is still used in Proposition 11, despite the fact that it leads to shadow activities with positive probability.

Figure 6 compares the constrained optimal solution under uncertainty about κ_h versus κ_λ. The case where the planner is uncertain about κ_h is depicted in solid black and the solution plotted as a function of κ_h^{max}. The case where the planner is uncertain about κ_λ is depicted in dashed red and plotted as a function of κ_λ^{max}. The figure is drawn for $\nu = \overline{\nu}_0$ and the same parameters as Figure 2. Recall that these parameters delivered symmetric usage of state-contingent and non-contingent regulation in the absence of shadow activities and hence symmetric welfare losses from the implementation constraints introduced by these activities. Figure 6 demonstrates asymmetric usage and asymmetric welfare losses in the presence of uncertainty.

Panels (a) and (b) in Figure 6 illustrate the constrained optimal regulation under uncertainty.

The results are as derived in Proposition 11, with the dashed red lines relaxing the assumption of $\kappa_\lambda^{max} = 1$. Panel (c) then plots the lowest circumvention costs, defined as thresholds $\tilde{\kappa}_\lambda$ and $\tilde{\kappa}_h$, where the relevant shadow activity is not used under the regulations in panels (a) and (b) and the constrained optimal bailout policy, which is illustrated in expectation in panel (d). Notice that $\tilde{\kappa}_h > \tilde{\kappa}_\lambda$ for any $\kappa_\lambda^{max} = \kappa_h^{max} > \overline{\kappa}_h$, similar to $\overline{\kappa}_h > \overline{\kappa}_\lambda$ in Proposition 5 without uncertainty. Also notice that both thresholds are below the thresholds that prevail in the absence of uncertainty, converging only asymptotically towards $\overline{\kappa}_\lambda$ and $\overline{\kappa}_h$ as κ_λ^{max} and κ_h^{max} go to infinity. This reflects that the uncertain planner imposes less regulation, as illustrated in panels (a) and (b).

Panel (e) demonstrates asymmetric welfare losses in the presence of uncertainty. In particular, the constrained optimal planner achieves lower welfare when he is uncertain about κ_h (the cost to banks of circumventing state-contingent regulation) than when he is uncertain about κ_λ (the cost to banks of circumventing non-contingent regulation), unless κ_h^{max} is substantially higher than κ_λ^{max}, i.e., unless the implementation constraint on state-contingent regulation is much less likely to bind than the implementation constraint on non-contingent regulation. This implies that the threat of shadow activities endogenously constrains state-contingent regulation more than non-contingent regulation under uncertainty. This is also visible in panel (f) which illustrates that the constrained optimal planner operating under uncertainty (and without commitment on the bailout instrument) will use state-contingent regulation less than non-contingent regulation relative to their optimal levels, i.e., $\theta\left(\beta_u\left(\kappa_h^{\max}\right) - \widehat{h}\right) < \alpha_u\left(\kappa_\lambda^{\max}\right) - \widehat{\lambda} < 0$ for any $\kappa_\lambda^{\max} = \kappa_h^{\max} > \overline{\kappa}_h$.

As was the case in Section 5.1 with naive regulation, the state-contingent shadow activity ω_h elicits a larger bailout in the stress state because it consumes additional resources precisely when liquidity is most valued, making its emergence more costly in terms of welfare than the emergence of the non-contingent shadow activity ω_λ. This difference was not a concern when the planner knew κ_h and κ_λ because the constrained optimal regulation was designed to not trigger shadow activities; see Proposition 4. The welfare losses from the threat of shadow activities then only stemmed from the implementation constraints they introduced. With uncertainty about the cost parameters, however, the constrained optimal regulation can trigger shadow activities. Triggering circumvention of the state-contingent regulation then leads to a larger expected bailout if the stress state is realized, as shown in panel (d) of Figure 6.

6 Discussion

As noted in the introduction, the academic literature on the circumvention of state-contingent regulation lags behind the implementation of such regulation by policy-makers. A lesson from the more voluminous literature on non-contingent regulation is that private interests will find ways to relax regulatory constraints that bind too tightly. In this section, we discuss some features of the data that our modeling of state-contingent regulation and its circumvention connect to.

We interpret the shadow action ω_h as insurance extended by a bank to its creditors against the state-contingent regulation that imposes a minimum haircut in the stress state. An example of a state-contingent regulation that imposes losses to bolster an intermediary's liquidity is the requirement by the U.S. Securities and Exchange Commission that institutional prime money market funds impose fees on redeeming investors when net redemptions surpass a pre-specified threshold. An example of a state-contingent regulation that imposes losses to bolster a bank's capital is the favorable treatment of contingent convertible bonds by European regulators provided the bonds are structured to automatically absorb losses if a pre-specified negative event occurs.

The current fee structure for U.S. money market funds was only introduced in 2023, so we use contingent convertible bonds ("CoCos") as an empirical setting to explore the broader model ingredients. To this end, it is straightforward to recast our model in terms of capital. For example, rather than choosing an asset mix between cash and investment, the bank chooses a funding mix between deposits and CoCos for each unit of investment. The benefit of deposits is that they are cheaper; the cost is that they contribute more to a cash shortfall in the stress state. And rather than choosing a haircut on deposits, the bank chooses a write-down on the CoCos. The benefit of the write-down is that it forces holders of these bonds to absorb some of the cash shortfall in the stress state; the cost is that it makes the bonds more expensive to issue. The same economic mechanisms will arise: banks will choose excessive deposit funding and insufficient write-downs due to the pecuniary externality, the planner will impose non-contingent regulation on the funding mix and state-contingent regulation on the write-downs, and shadow activities and bailouts will interact in the ways discussed above.

A concrete example of a financial product that can provide insurance properties is a credit line. We first illustrate that our modeling of ω_h is without loss of generality relative to a more explicit modeling of credit lines. We then document an empirical association between the issuance of CoCos by banks and the extension of credit lines, with price movements favoring an interpretation where the credit lines decrease the degree of state-contingency in the bonds. In other words, the credit lines appear to function as insurance against a regulatory haircut (bail-in), undermining the social benefits of these bonds intended by regulators.

6.1 Insurance Against State-Contingent Payoffs

Consider the benchmark model in Section 2 but with an extra decision: the provision of credit lines by banks. Specifically, the representative bank can extend a credit line in the amount of $\ell \in [0, h]$ at $t = 0$ that can be used at $t = 1$ by the recipient and then repaid at gross interest rate r at $t = 2$. The bank earns a fee ξ for each unit of credit line extended and incurs a per-unit capital charge τ when the credit line is used. This charge need not be regulatory; the bank's own risk management models may require provisioning once the credit line becomes a loan on the balance sheet.[6]

By taking out a credit line, a creditor that is subject to a haircut obtains insurance against this haircut. In particular, the θ depositors (henceforth creditors) that experience haircuts if the stress state materializes at $t = 1$ can use their credit lines to recoup liquidity and then repay those lines at $t = 2$ when the haircut is repaid to them. The effective haircut experienced by these creditors is thus $h - \ell$, which is the haircut that enters the bank's cost function $\phi(\cdot)$.[7] The creditors do not use the credit lines if the stress state does not materialize because they do not need the additional liquidity. The expected profit of the bank is then

$$\xi\ell + (1-p)[\lambda + f(1-\lambda) - 1] + p[(1-s) f(1-\lambda) - (1-\theta) - \theta h + r\theta\ell] - p\theta\phi(h-\ell)$$

where s is given by

$$qsf(1-\lambda) = \theta(1 - h + (1+\tau)\ell) - \lambda$$

[6]If raising additional capital in the stress state is not possible, the bank will have to sell an even bigger share of its project to keep its risk-weighted assets (or value-at-risk calculation) flat, i.e., replace the project with the credit line, exacerbating the fire sale externality.

[7]Under Diamond-Dybvig-type preferences, the relevant payoffs to creditors are $1 - h + \ell$ at $t = 1$ or 1 at $t = 2$; those that cash out early do not value consumption at $t = 2$ and hence do not care about having to repay the line. Thus, $\ell > 0$ delivers less variability in their payoffs, which is attractive with risk aversion.

and the price q is

$$q = \frac{\theta\left(1 - h + (1+\tau)\,\ell\right) - \lambda}{g\left(\theta\left(1 - h + (1+\tau)\,\ell\right) - \lambda\right)}$$

Focusing on the state-contingent ingredients of the model, the bank chooses the haircut h and the size of the credit line ℓ to maximize its expected profit subject to the regulatory constraint $h \geq \beta$.

Letting $\mu_h \geq 0$ denote the Lagrange multiplier on this regulatory constraint, the first order condition for h is

$$p\theta\left(1 + \phi'\left(h - \ell\right) - \frac{1}{q}\right) = \mu_h \tag{12}$$

and the derivative of the bank's expected profit with respect to ℓ is

$$\frac{\partial}{\partial \ell} = \xi + p\theta\left(r - \frac{1+\tau}{q} + \phi'\left(h - \ell\right)\right)$$

Suppose $\xi + p\theta\left(r - 1 - \tau\right) \leq 0$. Then the bank will optimally choose $\ell = 0$ when $\mu_h = 0$. That is, there will be no incentive to issue credit lines here without state-contingent regulation. The decentralized equilibrium is then the same as in Definition 1. With the regulation, however, the bank will choose $\ell > 0$ if β is sufficiently high, i.e., if state-contingent regulation is sufficiently strict, in which case $\mu_h > 0$. Notice from Eq. (12) that $\frac{\partial \mu_h}{\partial \ell} = -p\theta\phi''\left(h - \ell\right) < 0$, so extending more credit lines relaxes the constraint $h \geq \beta$. In other words, the bank takes the action ℓ to relax its state-contingent regulatory constraint despite incurring a cost $\left(1+\tau\right)\ell$ in the stress state to do so. The provision of credit lines therefore represents a specific example of the more general shadow activity ω_h studied earlier.

6.2 Empirical Application: Contingent Convertible Bonds

Under many European implementations of Basel III, contingent convertible bonds with a mechanical trigger ("CoCos") qualify as additional Tier 1 capital for regulatory purposes. CoCos are a fixed-income instrument that absorb losses either by converting into equity or taking a write-down when a pre-specified event occurs. The payoffs to CoCo buyers are thus state-contingent. The shadow activities that banks may use to circumvent these state-contingent requirements have not been studied in existing work on CoCos (see Avdjiev et al. (2020) and Fiordelisi et al. (2020) for prominent examples) so we explore this question empirically here.

We collect data from Bloomberg and Thomson Reuters Eikon on CoCos issued by European banks. We also collect financial statement information from Bloomberg and S&P Capital IQ to construct a proxy for shadow activities that can provide insurance against a CoCo trigger. The proxy loosely follows Boyd and Gertler (1995) by using the non-interest income of banks to make inferences about their off-balance-sheet activities. We specifically collect income statement data to construct the ratio of fee and commission income to total income; a higher share of fee and commission income is consistent with the sale of more insurance-like products, e.g., credit lines. We also construct the ratio of trading income to total income and use it as a placebo; a higher share of trading income also indicates more off-balance-sheet activity but these activities are unlikely to provide the type of insurance we are interested in. The resulting sample is a quarterly panel of 203 European banks from 2009Q1 to 2023Q2.

To explore the potential use of state-contingent shadow activities by banks, we run regressions of the form

$$\begin{aligned} Shadow_{i,t} &= \gamma_1 CoCo_{i,t} + \gamma_2 CoCo_{i,t} \times CapRatio_{i,t-1} \\ &\quad + \gamma_3 CapRatio_{i,t-1} + \gamma_4 Shadow_{i,t-1} + \zeta_i + \epsilon_{i,t} \end{aligned}$$

where $Shadow_{i,t}$ is the proxy of shadow activity for bank i in quarter t, $CoCo_{i,t}$ is CoCo issuance by bank i in quarter t measured as either a dummy variable for whether at least one CoCo was issued (extensive margin) or a count of the number of CoCos issued or the dollar value of issuance relative to bank assets (intensive margin), $CapRatio_{i,t-1}$ is the bank's capital ratio in the prior quarter, and ζ_i are fixed effects that control for unobserved heterogeneity in bank business models.

The coefficients of interest are γ_1 and γ_2. If $\gamma_1 > 0$, then shadow activities increase alongside CoCo issuance. If $\gamma_2 < 0$, then this association is driven by banks with low capital ratios, i.e., banks that are most likely to be issuing CoCos to satisfy regulatory requirements. Table 1 presents the results. The coefficient γ_1 is indeed positive and statistically significant when $CoCo_{i,t}$ is constructed as either a dummy variable or a count, capturing the extensive margin and one notion of the intensive margin. The coefficient γ_2 is negative and statistically significant for the same specifications.[8] Using the trading income placebo as the dependent variable instead produces statistically

[8]In unreported results available on request, we find $\gamma_1 > 0$ and $\gamma_2 < 0$ with statistical significance at standard

insignificant results in all specifications. Thus, more CoCo issuance by banks more constrained by regulation tends to be associated with more shadow activity, specifically activity that could be undermining the state-contingency in CoCos by providing insurance.

Are the shadow activities captured by our proxy in fact providing insurance against variability in CoCo payoffs, or are banks just issuing CoCos to raise capital against these activities? Disentangling the two hypotheses amounts to disentangling demand and supply. If the shadow activities provide insurance, then they make CoCos more attractive to buyers, which increases the demand for the CoCos being supplied. Otherwise, only the supply of the CoCos increases. We can therefore disentangle the hypotheses by looking at CoCo pricing. To this end, we focus on bank-quarter pairs that have at least one CoCo issuance and run regressions of the form

$$\begin{aligned} WgtAvgCoupon_{i,t} \quad = \quad & \delta_1 \% \Delta Shadow_{i,t} + \delta_2 \% \Delta Shadow_{i,t} \times CapRatio_{i,t-1} + \delta_3 CapRatio_{i,t-1} \\ & + \delta_4 CoCoAmount_{i,t} + \delta_5 BankSize_{i,t-1} + \delta_6 SovYield_{i,t} + \delta_7 UST_t + \zeta_i + \epsilon_{i,t} \end{aligned}$$

where $WgtAvgCoupon_{i,t}$ is the weighted average coupon rate for new CoCos issued by bank i in quarter t, $\%\Delta Shadow_{i,t}$ is the percentage change in the proxy of shadow activity for bank i from the prior quarter, $CoCoAmount_{i,t}$ is the dollar value of CoCos issued by bank i in quarter t, $BankSize_{i,t-1}$ is the total assets of the bank in the prior quarter, $SovYield_{i,t}$ is the 10-year sovereign bond yield in bank i's country of headquarter during quarter t, and UST_t is the 10-year U.S. Treasury yield.[9] We use the coupon rate as a proxy for CoCo pricing because it is the most consistently populated pricing field in the data. An increase in the demand for CoCos would allow the bank to issue the CoCos at lower coupons.

The coefficients of interest are δ_1 and δ_2. If $\delta_1 < 0$, then coupons decrease as shadow activities increase. If $\delta_2 > 0$, then this association is again driven by banks that are most likely to be issuing CoCos to satisfy regulatory requirements. Table 2 presents the results. We indeed find $\delta_1 < 0$ and $\delta_2 > 0$, with both statistically significant. Using the trading income placebo instead of the relevant proxy again produces statistically insignificant results. Thus, more shadow activity by

levels when $CoCo_{i,t}$ is constructed as a dollar value measure if the proxy for shadow activities is also a dollar value measure, namely credit and liquidity facilities relative to bank assets as reported in banks' Pillar 3 disclosures under Basel III. We leave further exploration for future work.

[9] Sovereign debt and U.S. Treasury yield data are quarterly averages from Haver Analytics.

CoCo-issuing banks more constrained by regulation is associated with lower CoCo coupons—that is, less costly issuance. Taken together, Tables 1 and 2 provide strong suggestive evidence that the threat of shadow activities extends to state-contingent regulation.

7 Conclusion

This paper has developed a theoretical framework to study the optimal mix of corrective regulation when banks make socially inefficient choices but may find it profitable to engage in shadow activities that circumvent the aim of regulation. We distinguished between regulations that introduce state-contingency into the returns of short-term creditors (e.g., dynamic liquidity fees or bail-ins of debt instruments) and regulations that do not (e.g., minimum liquidity or capital requirements). We demonstrated an endogenous asymmetry in favor of non-contingent regulation when the planner has imperfect information about the cost parameters of the shadow activities available to banks and limited commitment to a bailout policy, even for cost parameters where both regulations would otherwise be equally important corrective tools.

The need for corrective regulation in our model stems from a classic pecuniary externality. If an aggregate stress state occurs, banks experience significant early withdrawals and raise cash from outside investors by selling assets at an endogenously-determined market price. Banks take this price as given when choosing both their ex ante liquidity ratios and the haircut they impose on investors in the stress state, not internalizing that fewer sales help other banks. Accordingly, the planner introduces a floor on the liquidity ratio (non-contingent regulation) and a floor on the haircut that must be applied (state-contingent regulation). Without the threat of shadow activities, we show that it is optimal for the planner to rely more on state-contingent regulation than non-contingent regulation when the stress state is severe but unlikely, underscoring that the two instruments are not perfect substitutes.

The threat of shadow activities introduces constraints on the design of regulation. Shadow activities are privately beneficial to banks in the presence of binding regulation but socially wasteful, so the planner never finds it optimal to trigger them. He therefore chooses each regulation subject to the constraint that the marginal benefit of shadow activity is lower than the marginal cost. We show that state-contingent regulation is more likely to be constrained by the threat of shadow activities. Specifically, the range of circumvention cost parameters over which the planner cannot implement the unconstrained optimum is larger than for non-contingent regulation. This reflects that the marginal cost of the state-contingent shadow activity is only incurred by banks in the stress state and at a price that neglects the externality. However, since no shadow activity is triggered by the planner, the total welfare loss from introducing one shadow activity at a time is the same for both regulations if they would be used equally without the threat of shadow activities.

When the planner has imperfect information about the circumvention costs, shadow activities may occur as part of the regulated equilibrium that the planner implements. The emergence of shadow activities in equilibrium increases the size of the bailout when the planner lacks commitment, and this interaction is more pronounced for state-contingent regulation because the associated circumvention cost directly lowers the sale price of assets in the stress state. The planner thus achieves lower welfare when he is uncertain about the cost to banks of circumventing state-contingent regulation than when he is uncertain about the cost to banks of circumventing non-contingent regulation. A planner who ignores these costs altogether also generates a larger welfare loss from naively using state-contingent regulation than naively using non-contingent regulation, as well as an amplification of welfare losses when both regulations are being circumvented because the bailout triggered by the circumvention of one regulation increases the incentive of banks to circumvent the other.

Financial regulators in the U.S. and Europe have already implemented state-contingent regulation in the form of either dynamic liquidity fees or bail-ins of debt instruments. Whether these regulations will have the intended effect depends on whether banks will engage in shadow activities to circumvent them, and we have shown that a planner with imperfect information about the shadow technologies available to banks will elicit such activities with positive probability as part of the constrained efficient policy. The data support the presence of these activities; we find evidence that the threat of shadow activities extends to state-contingent regulation, complementing a growing literature on the circumvention of non-contingent regulation. Further empirical work on this question is an important avenue for research in light of our theoretical results.

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Figure 1: Timeline

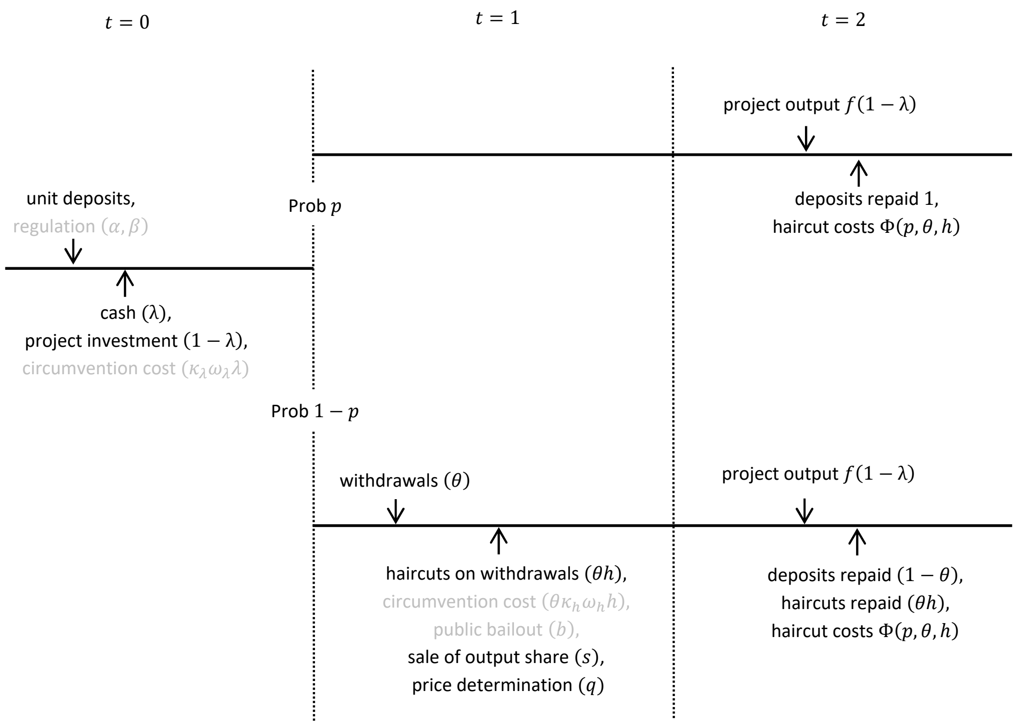

Figure 2: Constrained optimal solution as a function of κ_λ and κ_h (no bailout instrument, $\varepsilon_g'(\cdot) > 0$, $p\phi'' = -\theta f''$). Cost thresholds $\{\overline{\kappa}_\lambda, \overline{\kappa}_h\}$ are as defined in Proposition 5. $\{\lambda^*, h^*, \Pi^*\}$ and $\{\widehat{\lambda}, \widehat{h}, \widehat{\Pi}\}$ refer to liquidity ratio, haircut, and welfare in the decentralized equilibrium and in the planner's unconstrained solution, respectively.

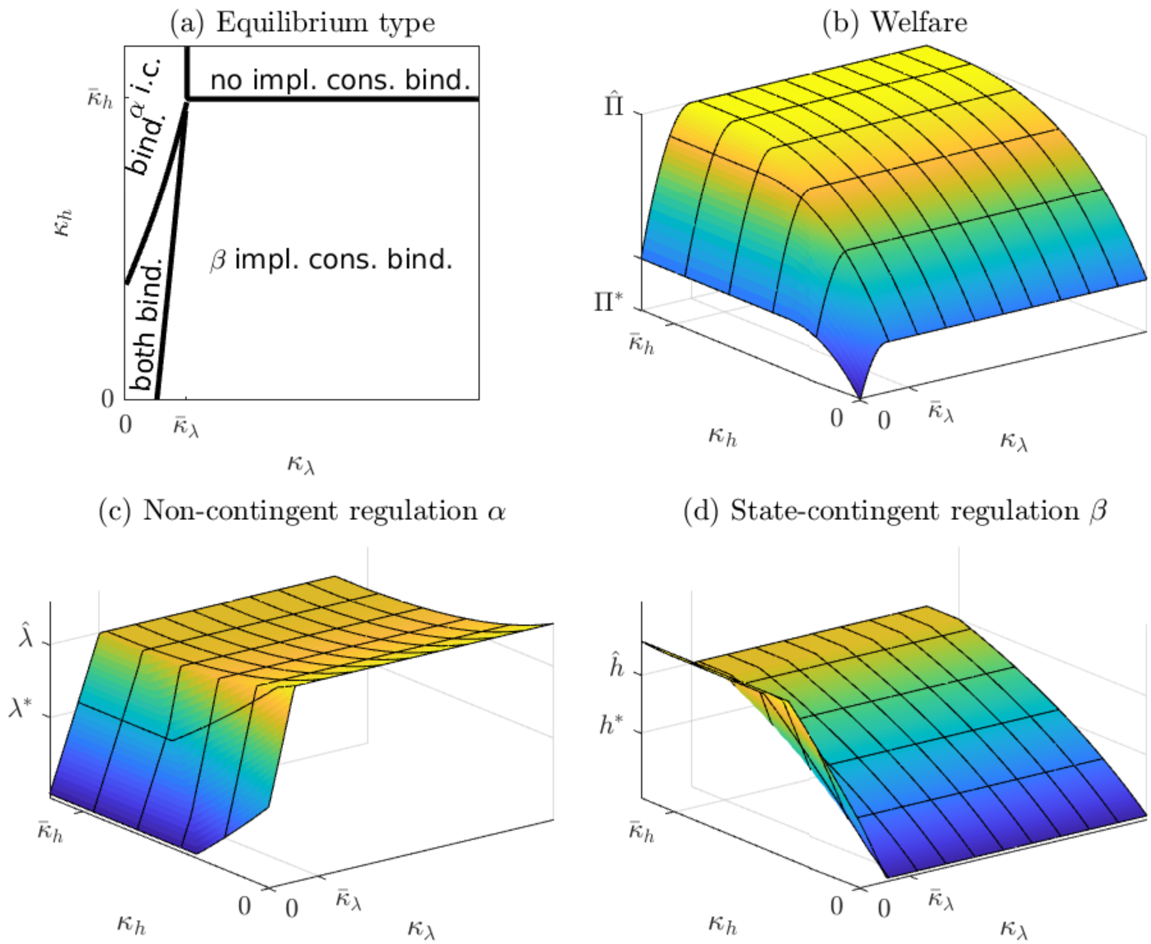

Figure 3: Constrained optimal solution as a function of either κ_h or κ_λ for various bailout conditions ($\varepsilon_g'(\cdot) > 0$, $p\phi'' = -\theta f''$; $\nu = \overline{\nu}_0$ when the bailout instrument is available). The top row is drawn for $\kappa_\lambda > \overline{\kappa}_\lambda$; the bottom row is drawn for $\kappa_h > \overline{\kappa}_h$.

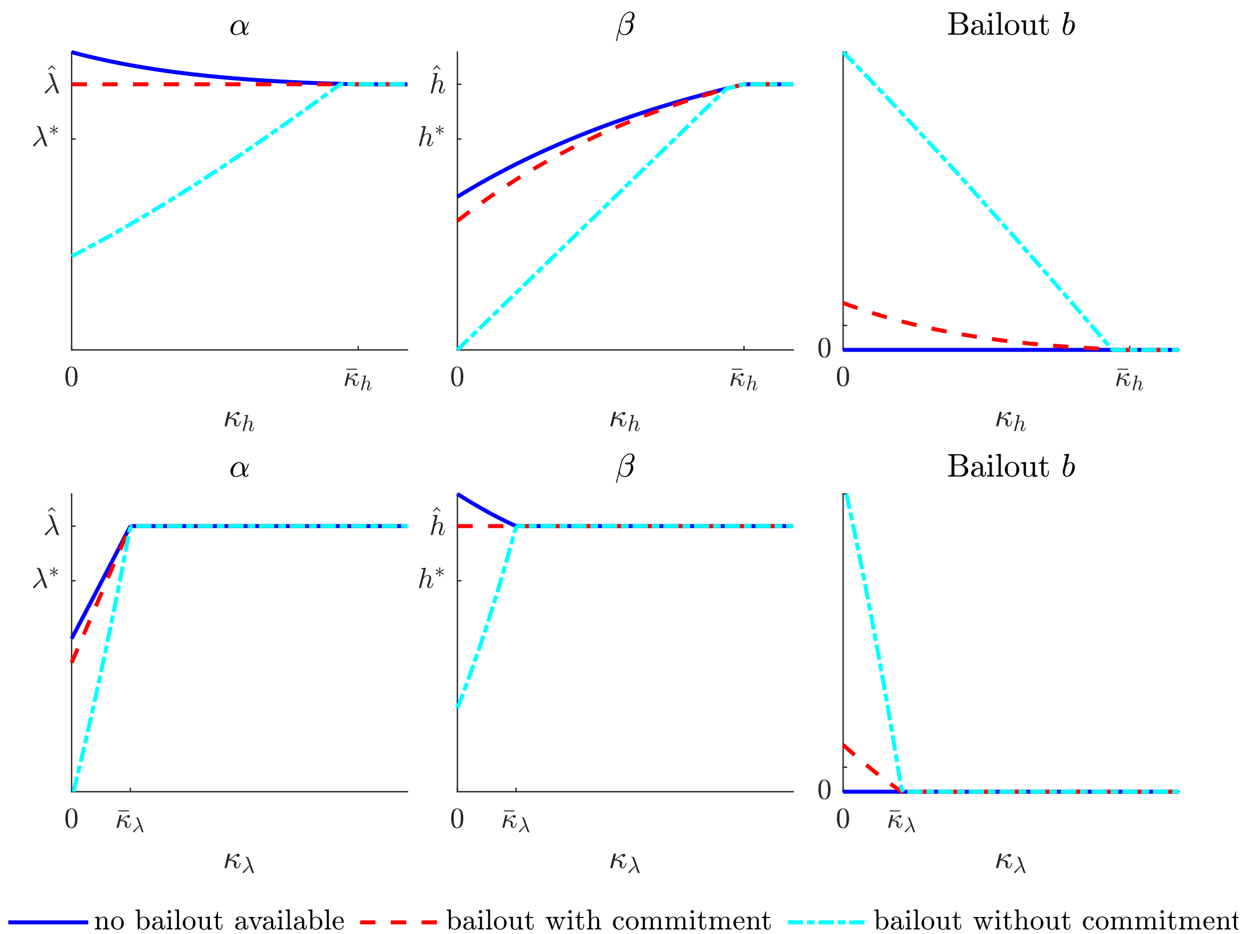

Figure 4: Constrained optimal solution as a function of κ_λ, κ_h, and ability to commit to bailout size ($\varepsilon_g'(\cdot) > 0$, $p\phi'' = -\theta f''$, $\nu > \overline{\nu}_0$).

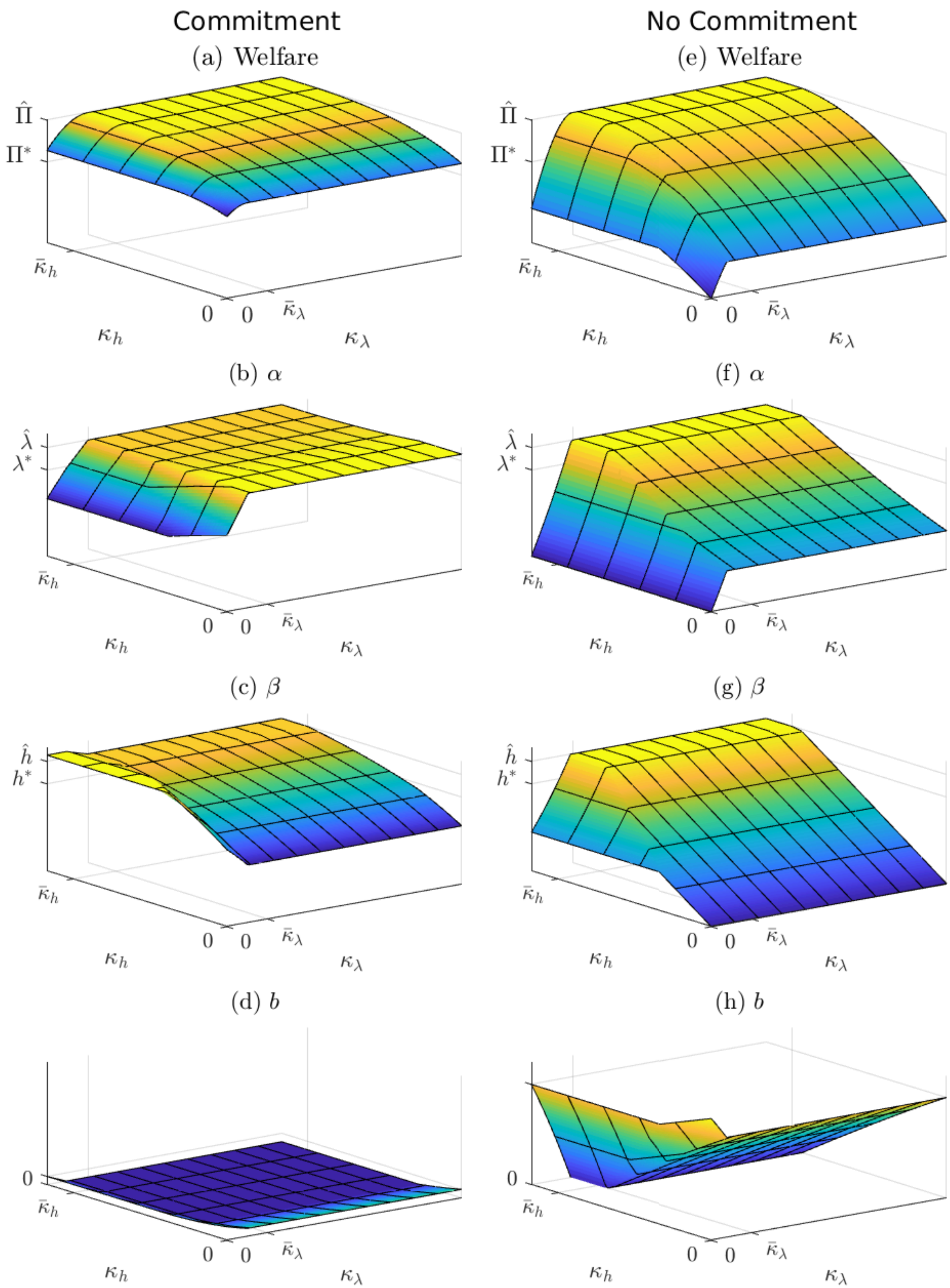

Figure 5: Welfare under constrained optimal versus naive regulation ($\varepsilon_g'(\cdot) > 0$, $p\phi'' = -\theta f''$; no commitment and $\nu > \overline{\nu}_0$ when the bailout instrument is available). Panel (a) plots the welfare surface assuming naive regulation with no bailout instrument. Panel (b) plots the difference between the welfare surface assuming naive regulation with the bailout instrument and panel (a). Panel (c) plots the difference between the welfare surface assuming constrained optimal regulation with the bailout instrument (Fig. 4(e)) and the welfare surface assuming constrained optimal regulation with no bailout instrument (Fig. 2(b)). Panel (d) plots the difference between panels (b) and (c).

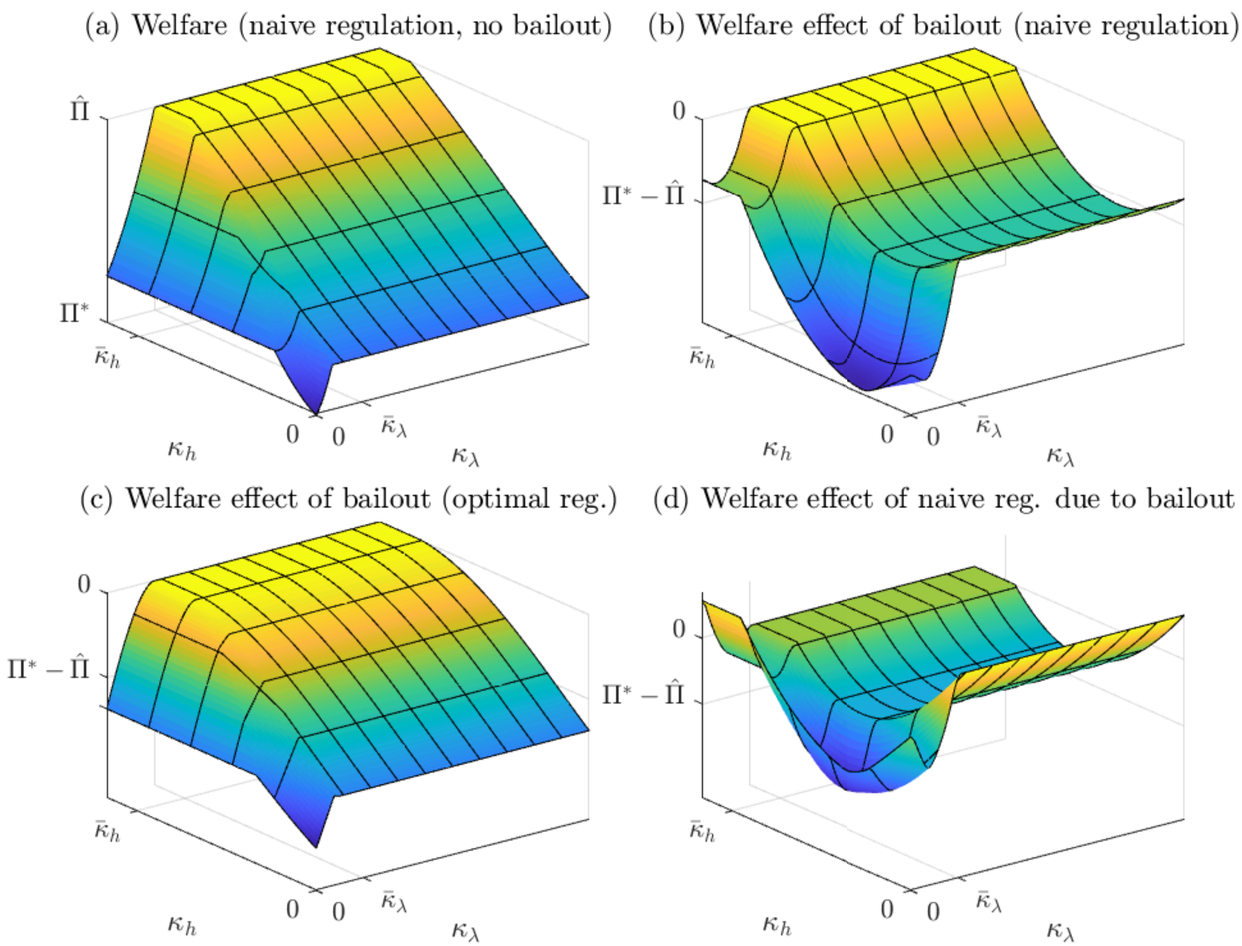

Figure 6: Constrained optimal solution with uncertainty ($\varepsilon_g'(\cdot) > 0$, $p\phi'' = -\theta f''$, no commitment, $\nu = \overline{\nu}_0$). Solid black lines are drawn as functions of κ_h^{max} and denote the constrained optimal solution when the planner is uncertain about κ_h with $\kappa_\lambda \to \infty$. Dashed red lines are drawn as functions of κ_λ^{max} and denote the constrained optimal solution when the planner is uncertain about κ_λ with $\kappa_h \to \infty$. In both cases, the horizontal axis is drawn up to $10\overline{\kappa}_h$. The notation $\tilde{\kappa}_h$ refers to the lowest κ_h for which banks do not take the state-contingent shadow action at the relevant constrained optimal solution; the analogous object for the non-contingent shadow action is $\tilde{\kappa}_\lambda$.

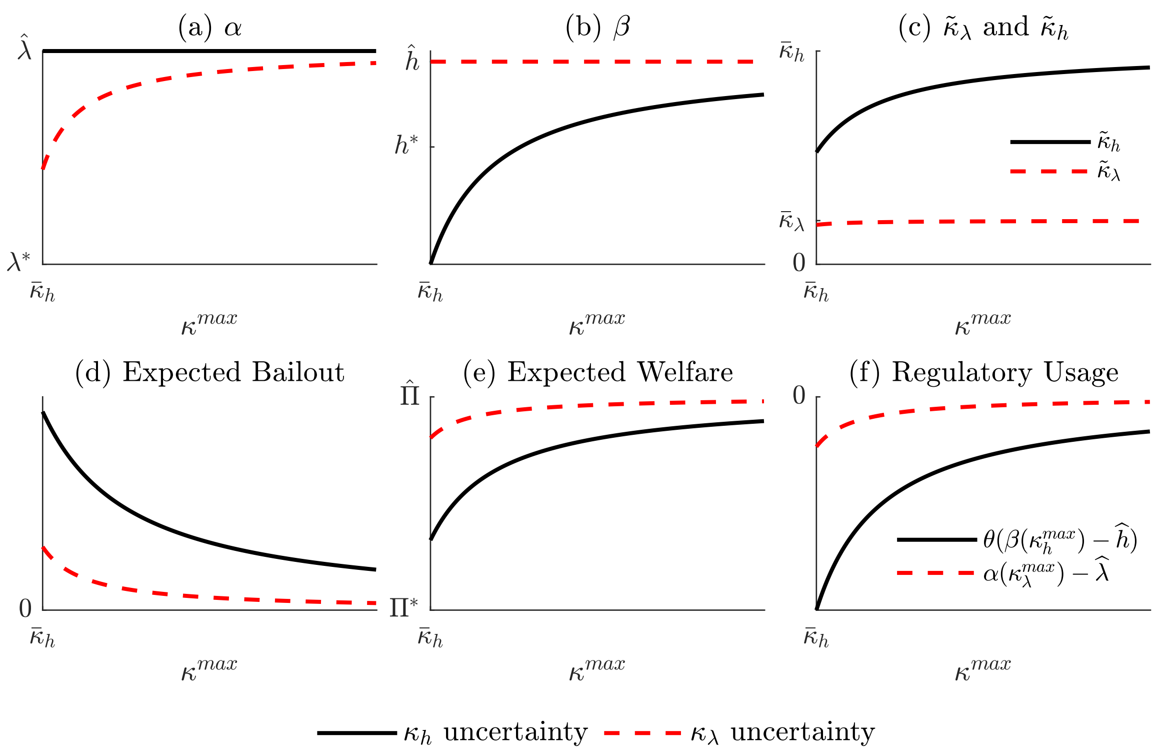

Table 1: Shadow Activity Regression

	$\frac{\text{Fee+Commission Income}}{\text{Total Income}}$			$\frac{\text{Trading Income}}{\text{Total Income}}$		
	(1)	(2)	(3)	(4)	(5)	(6)
Bank Issued CoCo	0.030**			−0.007		
	(0.013)			(0.013)		
Capital Ratio (Lag-1) x Bank Issued CoCo	−0.337**			0.039		
	(0.151)			(0.130)		
Number CoCos Issued		0.018**			−0.007	
		(0.008)			(0.008)	
Capital Ratio (Lag-1) x Number CoCos Issued		−0.218*			0.025	
		(0.111)			(0.092)	
Value CoCos Over Assets			−0.442			1.258
			(1.627)			(1.497)
Capital Ratio (Lag-1) x Value CoCos Over Assets			0.834			−14.52
			(18.22)			(14.12)
Capital Ratio (Lag-1)	0.202	0.202	0.201	0.008	0.008	0.009
	(0.143)	(0.143)	(0.142)	(0.146)	(0.146)	(0.146)
Dependent Variable (Lag-1)	−0.009	−0.009	−0.009	−0.051***	−0.051***	−0.051***
	(0.024)	(0.024)	(0.024)	(0.006)	(0.006)	(0.006)
Bank Fixed Effects	Y	Y	Y	Y	Y	Y
Number of Banks	203	203	203	203	203	203
Observations	5,611	5,611	5,611	5,611	5,611	5,611
R^2	0.244	0.244	0.244	0.041	0.041	0.041

Notes: The table presents panel regression results of the relationship between CoCo issuance and a proxy for shadow activity at the quarterly frequency within a panel of European banks from 2009Q1 to 2023Q2. Columns (1) to (3) use the proxy as the dependent variable; columns (4) to (6) use a placebo. CoCo issuance is measured as an indicator variable (columns (1) and (4)), a count of issued instruments (columns (2) and (5)), or the value of issued instruments as a fraction of the bank's total assets (columns (3) and (6)). CoCos include all contingent convertible AT1 instruments issued by the depository institution. Standard errors are clustered by bank and reported in parentheses. $^{*}p < 0.1;^{**} p < 0.05;^{***} p < 0.01$

Table 2: Coupon Rate Regression

	Weighted Average Coupon Rate on New CoCo Issues			
	Proxy: $\frac{\text{Fee+Commission Income}}{\text{Total Income}}$		Placebo: $\frac{\text{Trading Income}}{\text{Total Income}}$	
	(1)	(2)	(3)	(4)
%Δ Shadow ∈ {Proxy, Placebo}	-6.69^{***}	-7.99^{***}	0.05	-0.05
	(2.38)	(2.13)	(0.05)	(0.11)
Capital Ratio (Lag-1) x %Δ Shadow	58.04^{**}	96.20^{***}	-0.33	0.80
	(24.92)	(22.73)	(0.54)	(1.23)
Capital Ratio (Lag-1)	30.46^{***}	20.85^{***}	30.38^{***}	22.60^{***}
	(4.51)	(5.10)	(4.49)	(6.18)
CoCo Amount ($M)	-0.00	-0.00	-0.00	-0.00
	(0.00)	(0.00)	(0.00)	(0.00)
Bank Assets ($M, Lag-1)	0.00^{***}	0.00^{***}	0.00^{***}	0.00^{***}
	(0.00)	(0.00)	(0.00)	(0.00)
10Y Sovereign Debt Yield	0.32^{***}	0.13	0.34^{***}	0.18
	(0.10)	(0.16)	(0.10)	(0.16)
10Y U.S. Treasury Yield	0.30^{**}	0.50^{**}	0.40^{***}	0.56^{***}
	(0.14)	(0.20)	(0.13)	(0.19)
Bank Fixed Effects	N	Y	N	Y
Number of Banks	115	71	115	71
Observations	315	271	315	271
R^2	0.49	0.70	0.47	0.68

Note: The table presents regression results of the relationship between coupon rates of newly issued CoCo instruments and a proxy for shadow activity at the quarterly frequency among European banks from 2009Q1 to 2023Q2. Coupon rates are expressed as a value-weighted average of all CoCo instruments issued by a bank in a given quarter. Columns (1) and (2) calculate $\%\Delta Shadow$ using the indicated proxy; columns (3) and (4) use the indicated placebo. Standard errors are clustered by bank and reported in parentheses. ${}^{*}p < 0.1; {}^{**} p < 0.05; {}^{***} p < 0.01$

Appendix A – Proofs

Proof of Lemma 1

Consider the bank's problem subject to the constraint $\theta(1-h)-\lambda \geq 0$. Letting $\psi \geq 0$ denote the Lagrange multiplier on this constraint, the partial derivatives of the Lagrangian are

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \Pi}{\partial \lambda} - \psi = (1-p) - f'(1-\lambda) + \tfrac{p}{q} - \psi$$

$$\frac{\partial \mathcal{L}}{\partial h} = \frac{\partial \Pi}{\partial h} - \psi\theta = p\theta\left(\tfrac{1}{q} - 1 - \phi'(h)\right) - \psi\theta$$

If $\lambda = h = 0$ is a solution, then $\psi = 0$ and $q = \frac{\theta}{g(\theta)}$ and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (1-p) - f'(1) + p\tfrac{g(\theta)}{\theta} \leq 0$$

$$\frac{\partial \mathcal{L}}{\partial h} = p\theta\left(\tfrac{g(\theta)}{\theta} - 1 - \phi'(0)\right) \leq 0$$

so either $f'(1) - 1 < p\left(\frac{g(\theta)}{\theta} - 1\right)$ or $\frac{g(\theta)}{\theta} > 1 + \phi'(0)$ rules out $\lambda = h = 0$. ■

Proof of Lemma 2

If $\lambda > 0$ and $h = 0$ is a solution, then

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \longrightarrow \psi_{h=0} = (1-p) - f'(1-\lambda) + \tfrac{p}{q_{h=0}}$$

$$\frac{\partial \mathcal{L}}{\partial h} = \theta\left[f'(1-\lambda) - 1 - p\phi'(0)\right] \leq 0 \tag{A.1}$$

If $\lambda = 0$ and $h > 0$ is a solution, then

$$\frac{\partial \mathcal{L}}{\partial h} = 0 \longrightarrow \psi_{\lambda=0} = p\left(\tfrac{1}{q_{\lambda=0}} - 1 - \phi'(h)\right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 + p\phi'(h) - f'(1) \leq 0 \tag{A.2}$$

Suppose $f'(1) = 1 + p\phi'(0)$. Then condition (A.1) is contradicted by $\lambda > 0$ and $f''(\cdot) < 0$ while condition (A.2) is contradicted by $h > 0$ and $\phi''(\cdot) > 0$. Thus, $\lambda > 0$ and $h > 0$.

If $f'(1) > 1 + p\phi'(0)$, then condition (A.1) still cannot hold, i.e., $\lambda > 0$ and $h = 0$ is not a solution, but condition (A.2) may hold, i.e., $\lambda = 0$ and $h > 0$ may be a solution.

If $f'(1) < 1 + p\phi'(0)$, then condition (A.2) still cannot hold, i.e., $\lambda = 0$ and $h > 0$ is not a solution, but condition (A.1) may hold, i.e., $\lambda > 0$ and $h = 0$ may be a solution. ■

Proof of Proposition 1

At $\lambda = \theta(1-h)$, the price in Eq. (3) is

$$q_0 \equiv \lim_{x \to 0} \frac{x}{g(x)} = \frac{1}{g'(0)}$$

by l'Hopital's rule, so

$$\left.\frac{\partial \Pi}{\partial h}\right|_{q=q_0} \overset{sign}{=} g'(0) - 1 - \phi'(h) < g'(0) - 1 - \phi'(0)$$

where the inequality follows from $h > 0$ and $\phi''(\cdot) > 0$.

Therefore, $g'(0) \leq 1 + \phi'(0)$ implies $\left.\frac{\partial \Pi}{\partial h}\right|_{q=q_0} < 0$, which means any bank at $\lambda = \theta(1-h)$ wants to lower its choice of h. This establishes that the decentralized equilibrium has $\lambda < \theta(1-h)$.

If also $g'(0) \geq 1$, then $g'(x) > 1$ for all $x > 0$ from $g''(\cdot) > 0$. Accordingly, $q < q_0 \leq 1$. ■

Proof of Proposition 2

Recall $\frac{\partial q}{\partial \lambda} = \frac{1}{\theta}\frac{\partial q}{\partial h}$ from Section 2.1, so Eqs. (7) and (8) imply

$$f'(1-\lambda) = 1 + p\phi'(h) \tag{A.3}$$

which is also implied by Eqs. (5) and (6). That is, both the decentralized equilibrium and the planner's solution satisfy Eq. (A.3), which equalizes the marginal costs of using λ and h. From $f''(\cdot) < 0$ and $\phi''(\cdot) > 0$, it follows that Eq. (A.3) defines a positive relationship between λ and h.

The decentralized equilibrium is the intersection between Eqs. (6) and (A.3), where Eq. (6), with q as per Eq. (3), defines a negative relationship between λ and h.

If the planner's solution satisfies $\theta(1-h) > \lambda$, then the planner's solution is the intersection between Eqs. (8) and (A.3), where Eq. (8) implies a higher λ than Eq. (6) for any h. Therefore, on a two-dimensional graph with h on the horizontal axis and λ on the vertical axis, the decentralized equilibrium (h^*, λ^*) is an intersection between an upward-sloping curve and a downward-sloping curve, whereas the planner's solution $\left(\widehat{h}, \widehat{\lambda}\right)$ is an intersection between the same upward-sloping curve and another curve that lies to the right of the downward-sloping curve. It follows immediately that $\lambda^* < \widehat{\lambda}$ and $h^* < \widehat{h}$.

If the planner's solution satisfies $\theta(1-h) = \lambda$, then the planner's solution is the intersection between Eq. (A.3) and $\theta(1-h) = \lambda$, where the latter defines a negative relationship between λ and h. Since the decentralized equilibrium has $\theta(1-h^*) > \lambda^*$, at least one of $\lambda^* < \widehat{\lambda}$ and $h^* < \widehat{h}$ must be true, but since (h^*, λ^*) and $\left(\widehat{h}, \widehat{\lambda}\right)$ lie on the same upward-sloping curve, one cannot be true without the other. ■

Proof of Proposition 3

Let $\mu_\lambda \geq 0$ and $\mu_h \geq 0$ denote the Lagrange multipliers on the regulatory constraints $\lambda \geq \alpha$ and $h \geq \beta$ respectively. Then the first order conditions of the bank's problem are

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \Pi}{\partial \lambda} + \mu_\lambda = \underbrace{\frac{p}{q} + \mu_\lambda}_{MB_\lambda^{private,reg}} - \underbrace{\left[f'(1-\lambda) - (1-p)\right]}_{MC_\lambda}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h} = \frac{\partial \Pi}{\partial h} + \mu_h = \theta \left(\underbrace{\frac{p}{q} + \frac{\mu_h}{\theta}}_{MB_h^{private,reg}} - \underbrace{p\left(1 + \phi'(h)\right)}_{MC_h} \right)$$

alongside Eq. (3) for the price q.

To implement the planner's solution, the shadow prices μ_λ and μ_h must satisfy

$$\mu_\lambda = \frac{\mu_h}{\theta} = \frac{\theta(1-h) - \lambda}{q} \frac{\partial q}{\partial \lambda} \frac{p}{q} \tag{A.4}$$

when evaluated at $\left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$. Notice that Eq. (A.4) implies $\mu_\lambda > 0$ and $\mu_h > 0$ at $\left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$, so the planner simply needs to set $\alpha = \widehat{\lambda}$ and $\beta = \widehat{h}$. ■

Proof of Lemma 3

Use q as in Eq. (3) and $\frac{1}{\theta}\frac{\partial q}{\partial h}$ as in Eq. (4) to rewrite Eq. (8) as

$$g'(\theta(1-h) - \lambda) = 1 + \phi'(h) \tag{A.5}$$

Notice that $\left(\widehat{\lambda}, \widehat{h}\right)$ solves Eqs. (A.3) and (A.5). Totally differentiate Eqs. (A.3) and (A.5) with respect to p to get

$$f''\left(1 - \widehat{\lambda}\right) \frac{d\widehat{\lambda}}{dp} + \phi'\left(\widehat{h}\right) + p\phi''\left(\widehat{h}\right) \frac{d\widehat{h}}{dp} = 0$$

and

$$\frac{d\widehat{\lambda}}{dp} = -\left(\theta + \frac{\phi''\left(\widehat{h}\right)}{g''\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right)}\right) \frac{d\widehat{h}}{dp}$$

respectively. It then follows from $\phi''(\cdot) > 0$ and $g''(\cdot) > 0$ that $\frac{d\widehat{\lambda}}{dp}$ and $\frac{d\widehat{h}}{dp}$ have opposite signs. Combining the differentiated equations gives

$$\left[\left(\theta + \frac{\phi''\left(\widehat{h}\right)}{g''\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right)}\right) f''\left(1-\widehat{\lambda}\right) - p\phi''\left(\widehat{h}\right)\right] \frac{d\widehat{h}}{dp} = \phi'\left(\widehat{h}\right)$$

Recalling $f''(\cdot) < 0$ and $\phi'(\cdot) > 0$ leads to $\frac{d\widehat{h}}{dp} < 0$ and therefore $\frac{d\widehat{\lambda}}{dp} > 0$.

Next, totally differentiate Eqs. (A.3) and (A.5) with respect to θ to get

$$-f''\left(1-\widehat{\lambda}\right)\frac{d\widehat{\lambda}}{d\theta} = p\phi''\left(\widehat{h}\right)\frac{d\widehat{h}}{d\theta}$$

and

$$1-\widehat{h}-\frac{d\widehat{\lambda}}{d\theta} = \left(\theta + \frac{\phi''\left(\widehat{h}\right)}{g''\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}\right)\frac{d\widehat{h}}{d\theta}$$

respectively. It then follows that $\frac{d\widehat{\lambda}}{d\theta}$ and $\frac{d\widehat{h}}{d\theta}$ have the same sign. Combining the differentiated equations gives

$$\left(\theta + \frac{\phi''\left(\widehat{h}\right)}{g''\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)} - \frac{p\phi''\left(\widehat{h}\right)}{f''\left(1-\widehat{\lambda}\right)}\right)\frac{d\widehat{h}}{d\theta} = 1-\widehat{h}$$

Therefore, $\frac{d\widehat{h}}{d\theta} > 0$ and $\frac{d\widehat{\lambda}}{d\theta} > 0$.

To get the total derivative of $\widehat{q}$ with respect to the parameter $a \in \{p, \theta\}$, totally differentiate Eq. (A.5) as

$$g''\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)\frac{d\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}{da} = \phi''\left(\widehat{h}\right)\frac{d\widehat{h}}{da}$$

which implies that $\frac{d\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}{da}$ has the same sign as $\frac{d\widehat{h}}{da}$, then totally differentiate Eq. (3) as

$$\frac{d\widehat{q}}{da} = \frac{1}{g\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}\underbrace{\left(1-\frac{\theta\left(1-\widehat{h}\right)-\widehat{\lambda}}{g\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}g'\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)\right)}_{\text{negative by the elasticity of } g(\cdot)}\frac{d\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}{da}$$

Therefore, $\frac{d\widehat{q}}{dp} > 0$ and $\frac{d\widehat{q}}{d\theta} < 0$. ■

Proof of Lemma 4

Recall that both $\{\lambda^*, h^*\}$ and $\left\{\widehat{\lambda}, \widehat{h}\right\}$ satisfy Eq. (A.3). First order Taylor approximations of $f'(1-\lambda)$ and $\phi'(h)$ around $\left(1-\widehat{\lambda}\right)$ and $\widehat{h}$ respectively yield

$$f'(1-\lambda) \approx f'\left(1-\widehat{\lambda}\right) + f''\left(1-\widehat{\lambda}\right)\left((1-\lambda)-\left(1-\widehat{\lambda}\right)\right)$$

$$\phi'(h) \approx \phi'\left(\widehat{h}\right) + \phi''\left(\widehat{h}\right)\left(h-\widehat{h}\right)$$

and therefore

$$f'(1-\lambda^*) \approx f'\left(1-\widehat{\lambda}\right) + f''\left(1-\widehat{\lambda}\right)\left(\widehat{\lambda}-\lambda^*\right)$$

$$\phi'(h^*) \approx \phi'\left(\widehat{h}\right) + \phi''\left(\widehat{h}\right)\left(h^* - \widehat{h}\right)$$

Substituting these approximations into Eq. (A.3) evaluated at $\{\lambda^*, h^*\}$ then yields

$$f'\left(1-\widehat{\lambda}\right) + f''\left(1-\widehat{\lambda}\right)\left(\widehat{\lambda} - \lambda^*\right) \approx 1 + p\phi'\left(\widehat{h}\right) + p\phi''\left(\widehat{h}\right)\left(h^* - \widehat{h}\right)$$

which simplifies to

$$f''\left(1-\widehat{\lambda}\right)\left(\widehat{\lambda} - \lambda^*\right) \approx p\phi''\left(\widehat{h}\right)\left(h^* - \widehat{h}\right)$$

after invoking Eq. (A.3) evaluated at $\left\{\widehat{\lambda}, \widehat{h}\right\}$. Thus,

$$\left(\widehat{h} - h^*\right) \approx -\frac{f''\left(1-\widehat{\lambda}\right)}{p\phi''\left(\widehat{h}\right)}\left(\widehat{\lambda} - \lambda^*\right)$$

The approximation is exact if $f'''(\cdot) = \phi'''(\cdot) = 0$, in which case $f''(\cdot)$ and $\phi''(\cdot)$ are constants as in the statement of the lemma. ■

Proof of Proposition 4

We establish by contradiction that the planner does not want to trigger shadow activities.

When both shadow activities are potentially used by banks, the bank's objective function is given by $\widetilde{\Pi}(\cdot)$ as in Section 3.1, i.e.,

$$\begin{aligned}\widetilde{\Pi}(\lambda, h, \omega_\lambda, \omega_h; q) \equiv\ & (1-p)\left[\lambda + f(1-\lambda-\kappa_\lambda\omega_\lambda\lambda) - 1\right] \\ & + p\left[f(1-\lambda-\kappa_\lambda\omega_\lambda\lambda) - \frac{\theta(1-h+\kappa_h\omega_h h) - \lambda}{q} - (1-\theta) - \theta h\right] - p\theta\phi(h)\end{aligned}$$

and the Lagrangian of the bank problem is

$$\mathcal{L} = \widetilde{\Pi}(\lambda, h, \omega_\lambda, \omega_h; q) + \mu_\lambda\left[(1+\omega_\lambda)\lambda - \alpha\right] + \mu_h\left[(1+\omega_h)h - \beta\right] + \eta_{\omega_\lambda}\omega_\lambda + \eta_{\omega_h}\omega_h$$

where $\mu_\lambda, \mu_h \geq 0$ are Lagrange multipliers on the regulatory constraints and $\eta_{\omega_\lambda}, \eta_{\omega_h} \geq 0$ are Lagrange multipliers on the non-negativity constraints for the shadow activities. If the planner's regulation triggers shadow activities, then $\eta_{\omega_\lambda} = 0$ and/or $\eta_{\omega_h} = 0$.

The bank's problem yields

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \frac{p}{q} + \mu_\lambda(1+\omega_\lambda) - (1+\kappa_\lambda\omega_\lambda)f'(1-\lambda-\kappa_\lambda\omega_\lambda\lambda) + (1-p)$$

$$\frac{\partial\mathcal{L}}{\partial h} = p\theta\left(\frac{1-\kappa_h\omega_h}{q} - 1 - \phi'(h)\right) + \mu_h(1+\omega_h)$$

$$\frac{\partial\mathcal{L}}{\partial\omega_\lambda} = \mu_\lambda\lambda + \eta_{\omega_\lambda} - \kappa_\lambda\lambda f'(1-\lambda-\kappa_\lambda\omega_\lambda\lambda)$$

$$\frac{\partial \mathcal{L}}{\partial \omega_h} = \mu_h h + \eta_{\omega_h} - \frac{p\theta\kappa_h h}{q}$$

Setting $\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial h} = \frac{\partial \mathcal{L}}{\partial \omega_\lambda} = \frac{\partial \mathcal{L}}{\partial \omega_h} = 0$ and using complementary slackness on the non-negativity constraints ($\eta_{\omega_\lambda}\omega_\lambda = 0$ and $\eta_{\omega_h}\omega_h = 0$), the bank's first order conditions simplify to

$$\frac{p}{q} + \mu_\lambda = f'\left(1 - \lambda - \kappa_\lambda \omega_\lambda \lambda\right) - (1-p) \tag{A.6}$$

$$\frac{p}{q} + \frac{\mu_h}{\theta} = p\left(1 + \phi'(h)\right) \tag{A.7}$$

$$\eta_{\omega_\lambda} = \left[\kappa_\lambda f'\left(1 - \lambda - \kappa_\lambda \omega_\lambda \lambda\right) - \mu_\lambda\right]\lambda \tag{A.8}$$

$$\eta_{\omega_h} = \theta\left(\frac{p\kappa_h}{q} - \frac{\mu_h}{\theta}\right)h \tag{A.9}$$

For parameters where both shadow activities are used, i.e., $\omega_\lambda > 0$ and $\omega_h > 0$, complementary slackness implies $\eta_{\omega_\lambda} = 0$ and $\eta_{\omega_h} = 0$ and hence

$$\mu_\lambda = \kappa_\lambda f'\left(1 - \lambda - \kappa_\lambda \omega_\lambda \lambda\right) > 0$$

$$\mu_h = \frac{p\theta\kappa_h}{q} > 0$$

from Eqs. (A.8) and (A.9). Thus, conditional on α and β, the bank's choices of λ, h, ω_λ, and ω_h solve

$$\frac{p}{q} = (1-\kappa_\lambda) f'\left(1 - \lambda - \kappa_\lambda(\alpha - \lambda)\right) - (1-p) \tag{A.10}$$

$$\frac{p}{q} = \frac{p\left(1 + \phi'(h)\right)}{1 + \kappa_h} \tag{A.11}$$

$$(1 + \omega_\lambda)\lambda = \alpha \tag{A.12}$$

$$(1 + \omega_h)h = \beta \tag{A.13}$$

where q is given by Eq. (10). Notice that $\kappa_\lambda = \kappa_h = 0$ returns the decentralized equilibrium $\{\lambda^*, h^*, q^*\}$ with the entire effect of regulation absorbed by the shadow activities ω_λ and ω_h.

Using Eqs. (A.12) and (A.13) to substitute ω_λ and ω_h out of $\widetilde{\Pi}(\cdot)$ defines

$$\begin{aligned}\widehat{\Pi}(\lambda, h, q; \alpha, \beta) \equiv\ & (1-p)\left[\lambda + f\left(1 - \lambda - \kappa_\lambda(\alpha - \lambda)\right) - 1\right] \\ & + p\left[f\left(1 - \lambda - \kappa_\lambda(\alpha - \lambda)\right) - \frac{\theta\left(1 - h + \kappa_h(\beta - h)\right) - \lambda}{q} - (1-\theta) - \theta h\right] - p\theta\phi(h)\end{aligned}$$

The planner therefore chooses α and β to maximize $\widehat{\Pi}(\lambda, h, q; \alpha, \beta)$ subject to Eqs. (A.10), (A.11), and

$$q = \frac{\theta\left(1 - h + \kappa_h(\beta - h)\right) - \lambda}{g\left(\theta\left(1 - h + \kappa_h(\beta - h)\right) - \lambda\right)} \tag{A.14}$$

For α:

$$\frac{d\widehat{\Pi}}{d\alpha}=\frac{\partial\widehat{\Pi}}{\partial\alpha}+\frac{\partial\widehat{\Pi}}{\partial\lambda}\frac{d\lambda}{d\alpha}+\frac{\partial\widehat{\Pi}}{\partial h}\frac{dh}{d\alpha}+\frac{\partial\widehat{\Pi}}{\partial q}\frac{dq}{d\alpha}$$

where total differentiation of Eqs. (A.10), (A.11), and (A.14) determines $\frac{d\lambda}{d\alpha}$, $\frac{dh}{d\alpha}$, and $\frac{dq}{d\alpha}$. The envelope condition implies that $\frac{\partial\widehat{\Pi}}{\partial\lambda}=\frac{\partial\widehat{\Pi}}{\partial h}=0$ at the solution to these equations, hence we only need to sign

$$\frac{d\widehat{\Pi}}{d\alpha}=\frac{\partial\widehat{\Pi}}{\partial\alpha}+\frac{\partial\widehat{\Pi}}{\partial q}\frac{dq}{d\alpha}$$

The relevant derivatives are

$$\frac{\partial\widehat{\Pi}}{\partial\alpha}=-\kappa_\lambda f'\left(1-\lambda-\kappa_\lambda\left(\alpha-\lambda\right)\right)<0$$

$$\frac{\partial\widehat{\Pi}}{\partial q}=p\frac{\theta\left(1-h+\kappa_h\left(\beta-h\right)\right)-\lambda}{q^2}>0$$

$$\frac{dq}{d\alpha}=-\frac{\frac{\kappa_\lambda}{1-\kappa_\lambda}}{\frac{g(\theta(1-h+\kappa_h(\beta-h))-\lambda)}{g'(\theta(1-h+\kappa_h(\beta-h))-\lambda)\frac{\theta(1-h+\kappa_h(\beta-h))-\lambda}{g(\theta(1-h+\kappa_h(\beta-h))-\lambda)}-1}+\frac{\theta(1+\kappa_h)^2}{\phi''(h)}\frac{1}{q^2}-\frac{1}{(1-\kappa_\lambda)^2f''(1-\lambda-\kappa_\lambda(\alpha-\lambda))}\frac{p}{q^2}}<0$$

Therefore, $\frac{d\widehat{\Pi}}{d\alpha}<0$. This means that α is welfare-reducing and hence not used, contradicting that it triggers shadow activities.

The proof proceeds similarly for β. Specifically, the envelope condition implies that we only need to sign

$$\frac{d\widehat{\Pi}}{d\beta}=\frac{\partial\widehat{\Pi}}{\partial\beta}+\frac{\partial\widehat{\Pi}}{\partial q}\frac{dq}{d\beta}$$

where total differentiation of Eqs. (A.10), (A.11), and (A.14) determines $\frac{dq}{d\beta}$. The relevant derivatives are

$$\frac{\partial\widehat{\Pi}}{\partial\beta}=-\frac{p\theta\kappa_h}{q}<0$$

$$\frac{dq}{d\beta}=-\frac{\theta\kappa_h}{\frac{g(\theta(1-h+\kappa_h(\beta-h))-\lambda)}{g'(\theta(1-h+\kappa_h(\beta-h))-\lambda)\frac{\theta(1-h+\kappa_h(\beta-h))-\lambda}{g(\theta(1-h+\kappa_h(\beta-h))-\lambda)}-1}+\frac{\theta(1+\kappa_h)^2}{\phi''(h)}\frac{1}{q^2}-\frac{1}{(1-\kappa_\lambda)^2f''(1-\lambda-\kappa_\lambda(\alpha-\lambda))}\frac{p}{q^2}}<0$$

which, together with $\frac{\partial\widehat{\Pi}}{\partial q}>0$, imply $\frac{d\widehat{\Pi}}{d\beta}<0$. That is, β is welfare-reducing and hence not used, contradicting that it triggers shadow activities.

The proof for parameters where only one shadow activity is used, i.e., either $\omega_\lambda>0$ or $\omega_h>0$, proceeds in the same way, as outlined next.

If only the non-contingent shadow activity is used, i.e., $\omega_\lambda>0$ and $\omega_h=0$, then Eqs. (A.10) and (A.12) stay the same while the bank's first order condition for h is

$$\mu_h\left(h-\beta\right)=0,\mu_h=p\theta\left(1+\phi'\left(h\right)-\frac{1}{q}\right)\geq 0,h\geq\beta \tag{A.15}$$

with complementary slackness. The price q is given by Eq. (3). The relevant $\widehat{\Pi}(\cdot)$ still has the properties $\frac{\partial\widehat{\Pi}}{\partial\alpha}<0$ and $\frac{\partial\widehat{\Pi}}{\partial q}>0$ so it will be sufficient to show $\frac{dq}{d\alpha}<0$ from total differentiation of Eqs. (3), (A.10), and the first order condition for h. There are two cases to consider: (i) β low enough that $h>\beta$ with $\frac{1}{q}=1+\phi'(h)$ and (ii) β high enough that $h=\beta$. Going through the algebra confirms $\frac{dq}{d\alpha}<0$ for each case. Therefore, $\frac{d\widehat{\Pi}}{d\alpha}<0$, so α is welfare-reducing and hence not used, contradicting that it triggers the non-contingent shadow activity.

If only the state-contingent shadow activity is used, i.e., $\omega_\lambda=0$ and $\omega_h>0$, then Eqs. (A.11) and (A.13) stay the same while the bank's first order condition for λ is

$$\mu_\lambda(\lambda-\alpha)=0, \mu_\lambda=f'(1-\lambda)-(1-p)-\frac{p}{q}\geq 0, \lambda\geq\alpha \tag{A.16}$$

with complementary slackness. The price q is given by Eq. (A.14). The relevant $\widehat{\Pi}(\cdot)$ still has the properties $\frac{\partial\widehat{\Pi}}{\partial\beta}<0$ and $\frac{\partial\widehat{\Pi}}{\partial q}>0$ so it will be sufficient to show $\frac{dq}{d\beta}<0$ from total differentiation of Eqs. (A.11), (A.14), and the first order condition for λ. There are again two cases to consider: (i) α low enough that $\lambda>\alpha$ with $\frac{p}{q}=f'(1-\lambda)-(1-p)$ and (ii) α high enough that $\lambda=\alpha$. Going through the algebra confirms $\frac{dq}{d\beta}<0$ for each case. Therefore, $\frac{d\widehat{\Pi}}{d\beta}<0$, so β is welfare-reducing and hence not used, contradicting that it triggers the state-contingent shadow activity. ■

Proof of Proposition 5

Recall the bank's first order conditions in Eqs. (A.6) to (A.9) from the proof of Proposition 4. Use (A.6) to substitute μ_λ out of (A.8) and get

$$\eta_{\omega_\lambda}=\left[\frac{p}{q}-(1-\kappa_\lambda)f'(1-\lambda-\kappa_\lambda\omega_\lambda\lambda)+(1-p)\right]\lambda$$

then use (A.7) to substitute μ_h out of (A.9) and get

$$\eta_{\omega_h}=\theta\left(\frac{p(1+\kappa_h)}{q}-p\left(1+\phi'(h)\right)\right)h$$

Proposition 4 established that the planner never finds it optimal to trigger shadow activities, which means his choices of α and β implement $\eta_{\omega_\lambda}\geq 0$ and $\eta_{\omega_h}\geq 0$ with $\omega_\lambda=\omega_h=0$. Thus,

$$\frac{p}{q(\lambda,h)}\geq(1-\kappa_\lambda)f'(1-\lambda)-(1-p) \tag{A.17}$$

$$\frac{1+\kappa_h}{q(\lambda,h)}\geq 1+\phi'(h) \tag{A.18}$$

with q given by Eq. (3) and written as $q(\lambda,h)$ to make explicit the dependencies. Evaluated at $\lambda=\alpha$ and $h=\beta$, these inequalities constitute implementation constraints on the planner's problem.

Note that the implementation constraints can be rearranged to get

$$\kappa_\lambda \geq 1 - \frac{1}{f'(1-\lambda)} \left(\frac{p}{q(\lambda, h)} + (1-p) \right)$$

$$\kappa_h \geq q(\lambda, h) \left(1 + \phi'(h)\right) - 1$$

which, when evaluated at $\widehat{\lambda}$ and $\widehat{h}$, yields

$$\kappa_\lambda \geq 1 - \frac{1}{f'\left(1-\widehat{\lambda}\right)} \left(\frac{p}{\widehat{q}} + (1-p) \right) \equiv \overline{\kappa}_\lambda$$

$$\kappa_h \geq \widehat{q} \left(1 + \phi'\left(\widehat{h}\right)\right) - 1 \equiv \overline{\kappa}_h$$

Therefore, if $\kappa_\lambda \geq \overline{\kappa}_\lambda$ and $\kappa_h \geq \overline{\kappa}_h$, then $\alpha = \widehat{\lambda}$ and $\beta = \widehat{h}$ satisfy the implementation constraints, meaning that the planner can achieve $\left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$ without triggering shadow activities.

Finally, $\overline{\kappa}_h > \overline{\kappa}_\lambda$ if and only if

$$\widehat{q}\left(1 + \phi'\left(\widehat{h}\right)\right) - 1 > 1 - \frac{1}{f'\left(1-\widehat{\lambda}\right)} \left(\frac{p}{\widehat{q}} + (1-p) \right)$$

Using Eqs. (A.3) and (A.5), this inequality becomes

$$\widehat{q} g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) - 1 > 1 - \frac{1}{p g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) + 1 - p} \left(\frac{p}{\widehat{q}} + (1-p) \right)$$

and thus

$$\frac{\theta\left(1-\widehat{h}\right) - \widehat{\lambda}}{g\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right)} g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) - 1 > \frac{p\left(g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) - \frac{g\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right)}{\theta\left(1-\widehat{h}\right)-\widehat{\lambda}}\right)}{p g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) + 1 - p}$$

after using Eq. (3) to substitute out $\widehat{q}$ and simplifying. The last inequality further simplifies to

$$\left(g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) - \frac{g\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right)}{\theta\left(1-\widehat{h}\right) - \widehat{\lambda}} + \frac{1-p}{p}\right)\left(g'\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right) - \frac{g\left(\theta\left(1-\widehat{h}\right) - \widehat{\lambda}\right)}{\theta\left(1-\widehat{h}\right) - \widehat{\lambda}}\right) > 0$$

which is true, proving $\overline{\kappa}_h > \overline{\kappa}_\lambda$. ■

Proof of Proposition 6

The planner seeks to maximize $\widetilde{\Pi}(\lambda, h, 0, 0; q)$, taking into account q as per Eq. (3), subject to the implementation constraints (A.17) and (A.18). Without loss of generality, we can consider the case where the regulatory constraints hold with equality, i.e., $\lambda = \alpha$ and $h = \beta$. The nature of the externality is that the representative bank always undervalues liquidity relative to the planner who internalizes the effect on the sale price of projects, so for a given λ, the planner will want the bank to choose a higher h, and for a given h, he will want it to choose a higher λ. In other words, the planner will never want the bank to choose lower λ or h than would prevail without intervention.

To make the algebra more compact, define $x \equiv \theta(1-\beta) - \alpha$ to be the cash shortfall that must be covered by project sales. Then Eq. (3) is simply

$$\frac{1}{q} = \frac{g(x)}{x}$$

and the implementation constraints (A.17) and (A.18) can be expressed as

$$p\frac{g(x)}{x} \geq (1-\kappa_\lambda) f'(1-\alpha) - (1-p)$$

and

$$(1+\kappa_h)\frac{g(x)}{x} \geq 1 + \phi'\left(1 - \frac{\alpha + x}{\theta}\right)$$

respectively. Also rewrite the planner's objective function

$$\begin{aligned}
& \widetilde{\Pi}(\alpha, \beta, 0, 0; q) \\
= \;& (1-p)[\alpha + f(1-\alpha) - 1] + p\left[f(1-\alpha) - \frac{\theta(1-\beta)-\alpha}{q} - (1-\theta) - \theta\beta\right] - p\theta\phi(\beta) \\
= \;& (1-p)[\alpha + f(1-\alpha) - 1] + p[f(1-\alpha) - g(\theta(1-\beta)-\alpha) - (1-\theta) - \theta\beta] - p\theta\phi(\beta) \\
= \;& (1-p)[\alpha + f(1-\alpha) - 1] + p\left[f(1-\alpha) - g(x) - (1-\theta) - \theta\left(1 - \frac{\alpha+x}{\theta}\right)\right] - p\theta\phi\left(1 - \frac{\alpha+x}{\theta}\right) \\
= \;& \alpha + f(1-\alpha) - pg(x) + px - p\theta\phi\left(1 - \frac{\alpha+x}{\theta}\right) - 1
\end{aligned}$$

where the first equality follows from the expression for $\widetilde{\Pi}(\cdot)$ in the proof of Proposition 4, the second uses Eq. (3) to substitute out q, the third uses $x \equiv \theta(1-\beta) - \alpha$ to substitute out for β, and the last line simplifies.

Consider first the case where only (A.18) binds; this corresponds to κ_λ sufficiently high that (A.17) is slack. Then the planner's problem is

$$\max_x \left\{\alpha(x) + f(1-\alpha(x)) - pg(x) + px - p\theta\phi\left(1 - \frac{\alpha(x)+x}{\theta}\right)\right\} \tag{A.19}$$

where the binding implementation constraint

$$(1+\kappa_h)\frac{g(x)}{x} = 1 + \phi'\left(1 - \frac{\alpha+x}{\theta}\right) \tag{A.20}$$

implicitly defines $\alpha(x)$, with

$$\alpha'(x) = -1 - \frac{\theta(1+\kappa_h)\left(g'(x) - \frac{g(x)}{x}\right)}{x\phi''\left(1 - \frac{\alpha+x}{\theta}\right)}$$

The first order condition for the problem in (A.19) is

$$g'(x) = 1 + \phi'\left(1 - \frac{\alpha+x}{\theta}\right) + \frac{1}{p}\left(1 + p\phi'\left(1 - \frac{\alpha+x}{\theta}\right) - f'(1-\alpha)\right)\alpha'(x)$$

which reduces to

$$g'(x) = 1 + \phi'\left(1 - \frac{\alpha+x}{\theta}\right) + \frac{1}{p}\left(f'(1-\alpha) - 1 - p\phi'\left(1 - \frac{\alpha+x}{\theta}\right)\right)\left(1 + \frac{\theta(1+\kappa_h)\left(g'(x) - \frac{g(x)}{x}\right)}{x\phi''\left(1 - \frac{\alpha+x}{\theta}\right)}\right) \tag{A.21}$$

after substituting in the expression for $\alpha'(x)$. The planner's solution is a pair $\{\alpha, x\}$ solving Eqs. (A.20) and (A.21). Define $\widehat{x} \equiv \theta\left(1 - \widehat{h}\right) - \widehat{\lambda}$ and note from Eqs. (A.3) and (A.5) that $\left\{\widehat{\lambda}, \widehat{x}\right\}$ solves

$$f'\left(1 - \widehat{\lambda}\right) = 1 + p\phi'\left(1 - \frac{\widehat{\lambda} + \widehat{x}}{\theta}\right)$$

$$g'(\widehat{x}) = 1 + \phi'\left(1 - \frac{\widehat{\alpha} + \widehat{x}}{\theta}\right)$$

Therefore, $\left\{\widehat{\lambda}, \widehat{x}\right\}$ satisfies Eqs. (A.20) and (A.21) if $\kappa_h = \overline{\kappa}_h$. To see what happens to $\{\alpha, x\}$ as κ_h is perturbed below $\overline{\kappa}_h$, differentiate Eqs. (A.20) and (A.21) to get $\frac{d\alpha}{d\kappa_h}$ and $\frac{dx}{d\kappa_h}$ evaluated at $\kappa_h = \overline{\kappa}_h$ where we know the planner chooses $\left\{\widehat{\lambda}, \widehat{x}\right\}$. If $\kappa_\lambda > \overline{\kappa}_\lambda$, then (A.17) is slack with strict inequality when $\kappa_h = \overline{\kappa}_h$ and thus remains so with a slight perturbation. This gives

$$\frac{g(\widehat{x})}{\widehat{x}}d\kappa_h + \frac{1+\overline{\kappa}_h}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)dx = -\frac{\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx)$$

$$g''(\widehat{x})\,dx = -\frac{\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx) + \frac{1}{p}\left(-f''\left(1 - \widehat{\lambda}\right)d\alpha + \frac{p\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx)\right)\underbrace{\left(1 + \frac{\theta(1+\overline{\kappa}_h)\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)}{\widehat{x}\phi''\left(\widehat{h}\right)}\right)}_{\equiv 1+K}$$

$\Longrightarrow$

$$d\alpha = -\frac{\theta g(\widehat{x})}{\widehat{x}\phi''\left(\widehat{h}\right)}d\kappa_h - (1+K)\,dx$$

$$g''(\widehat{x})\,dx = \frac{\phi''\left(\widehat{h}\right)K}{\theta}(d\alpha + dx) + \frac{-f''\left(1-\widehat{\lambda}\right)(1+K)}{p}d\alpha$$

$\Longrightarrow$

$$g''(\widehat{x})\,dx = -\frac{\phi''\left(\widehat{h}\right)K}{\theta}\left(\frac{\theta g(\widehat{x})}{\widehat{x}\phi''\left(\widehat{h}\right)}d\kappa_h + K dx\right) - \frac{-f''\left(1-\widehat{\lambda}\right)(1+K)}{p}\left(\frac{\theta g(\widehat{x})}{\widehat{x}\phi''\left(\widehat{h}\right)}d\kappa_h + (1+K)\,dx\right)$$

$\Longrightarrow$

$$\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{\frac{g(\widehat{x})}{\widehat{x}}\left(K + \frac{-f''\left(1-\widehat{\lambda}\right)\theta(1+K)}{p\phi''\left(\widehat{h}\right)}\right)}{g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)K^2}{\theta} + \frac{-f''\left(1-\widehat{\lambda}\right)(1+K)^2}{p}} < 0$$

$\Longrightarrow$

$$\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{\theta g(\widehat{x})}{\widehat{x}\phi''\left(\widehat{h}\right)} - (1+K)\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{\frac{\theta g(\widehat{x})}{\widehat{x}\phi''\left(\widehat{h}\right)}\left(g''(\widehat{x}) - \frac{1+\overline{\kappa}_h}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)\right)}{g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)K^2}{\theta} + \frac{-f''\left(1-\widehat{\lambda}\right)(1+K)^2}{p}}$$

So $g''(\cdot) >> 0$ implies $\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} < 0$ and hence

$$\left.\frac{d\beta}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{1}{\theta}\left(\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} + \left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h}\right) > 0$$

Therefore, a perturbation of κ_h below $\overline{\kappa}_h$ leads to $\alpha > \widehat{\lambda}$ and $\beta < \widehat{h}$.

Next consider the case where only (A.17) binds; this corresponds to κ_h sufficiently high that (A.18) is slack. The planner's problem is still given by (A.19) but now the binding implementation constraint that implicitly defines $\alpha(x)$ is

$$p\frac{g(x)}{x} = (1-\kappa_\lambda) f'(1-\alpha) - (1-p) \tag{A.22}$$

with

$$\alpha'(x) = \frac{\frac{p}{x}\left(g'(x) - \frac{g(x)}{x}\right)}{-(1-\kappa_\lambda) f''(1-\alpha)}$$

Accordingly, the planner's first order condition for x is

$$g'(x) = 1 + \phi'\left(1 - \frac{\alpha+x}{\theta}\right) + \left(1 + p\phi'\left(1 - \frac{\alpha+x}{\theta}\right) - f'(1-\alpha)\right)\frac{\frac{1}{x}\left(g'(x) - \frac{g(x)}{x}\right)}{-(1-\kappa_\lambda) f''(1-\alpha)} \tag{A.23}$$

The planner's solution is a pair $\{\alpha, x\}$ solving Eqs. (A.22) and (A.23). Similar to the previous case, $\left\{\widehat{\lambda}, \widehat{x}\right\}$ satisfies Eqs. (A.22) and (A.23) if $\kappa_\lambda = \overline{\kappa}_\lambda$. To see what happens to $\{\alpha, x\}$ as κ_λ is perturbed below $\overline{\kappa}_\lambda$, differentiate Eqs. (A.20) and (A.21) to get $\frac{d\alpha}{d\kappa_\lambda}$ and $\frac{dx}{d\kappa_\lambda}$ evaluated at $\kappa_\lambda = \overline{\kappa}_\lambda$ where we know the planner chooses $\left\{\widehat{\lambda}, \widehat{x}\right\}$. If $\kappa_h > \overline{\kappa}_h$, then (A.18) is slack with strict inequality when $\kappa_\lambda = \overline{\kappa}_\lambda$ and thus remains so with a slight perturbation. This gives

$$\frac{p}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)dx = -f'\left(1-\widehat{\lambda}\right)d\kappa_\lambda - (1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)d\alpha$$

$$g''(\widehat{x})\,dx = -\frac{\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx) - \left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}(d\alpha+dx) - f''\left(1-\widehat{\lambda}\right)d\alpha\right)\underbrace{\frac{\frac{1}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)}{-(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)}}_{\equiv\widetilde{K}}$$

$\Longrightarrow$

$$d\alpha = \frac{f'\left(1-\widehat{\lambda}\right)}{-(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)}d\kappa_\lambda + p\widetilde{K}dx$$

$$g''(\widehat{x})\,dx = -\frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)}{\theta}(d\alpha+dx) + f''\left(1-\widehat{\lambda}\right)\widetilde{K}d\alpha$$

$\Longrightarrow$

$$g''(\widehat{x})\,dx = -\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(\frac{f'\left(1-\widehat{\lambda}\right)\left(1+p\widetilde{K}\right)}{-(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)}d\kappa_\lambda + \left(1+p\widetilde{K}\right)^2 dx\right) - \frac{f'\left(1-\widehat{\lambda}\right)\widetilde{K}}{1-\overline{\kappa}_\lambda}d\kappa_\lambda + f''\left(1-\widehat{\lambda}\right)p\widetilde{K}^2 dx$$

$\Longrightarrow$

$$\left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} = -\frac{\frac{f'\left(1-\widehat{\lambda}\right)}{1-\overline{\kappa}_\lambda}\left(\widetilde{K} + \frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)}{-\theta f''\left(1-\widehat{\lambda}\right)}\right)}{g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)^2}{\theta} - f''\left(1-\widehat{\lambda}\right)p\widetilde{K}^2} < 0$$

$\Longrightarrow$

$$\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} = \frac{f'\left(1-\widehat{\lambda}\right)}{-(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)} + p\widetilde{K}\left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} = \frac{\frac{f'\left(1-\widehat{\lambda}\right)}{-(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)}\left(g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)}{\theta}\right)}{g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)^2}{\theta} - f''\left(1-\widehat{\lambda}\right)p\widetilde{K}^2} > 0$$

$\Longrightarrow$

$$\left.\frac{d\beta}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} = -\frac{1}{\theta}\left(\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} + \left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda}\right) = -\frac{\frac{f'\left(1-\widehat{\lambda}\right)}{-\theta(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)}\left(g''(\widehat{x}) - \frac{g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}}{(1-\overline{\kappa}_\lambda)\widehat{x}}\right)}{g''(\widehat{x}) + \frac{\phi''\left(\widehat{h}\right)\left(1+p\widetilde{K}\right)^2}{\theta} - f''\left(1-\widehat{\lambda}\right)p\widetilde{K}^2}$$

So $g''(\cdot) >> 0$ implies $\left.\frac{d\beta}{d\kappa_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda} < 0$ and hence a perturbation of κ_λ below $\overline{\kappa}_\lambda$ leads to $\alpha < \widehat{\lambda}$ and $\beta > \widehat{h}$.

The last step is to show that $\varepsilon_g'(x) > 0$ implies $g''(\cdot) >> 0$. Note that the exact level of $g''(\cdot) >> 0$ needed is

$$g''(\widehat{x}) > \frac{\max\left\{1+\overline{\kappa}_h, \frac{1}{1-\overline{\kappa}_\lambda}\right\}}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)$$

Using the expressions for $\overline{\kappa}_\lambda$ and $\overline{\kappa}_h$ from the proof of Proposition 5,

$$\max\left\{1+\overline{\kappa}_h, \frac{1}{1-\overline{\kappa}_\lambda}\right\} = \max\left\{\widehat{q}\left(1+\phi'\left(\widehat{h}\right)\right), \frac{f'\left(1-\widehat{\lambda}\right)}{\frac{p}{\widehat{q}}+(1-p)}\right\}$$

Using Eqs. (3), (A.3), and (A.5) to substitute out $\left\{\widehat{\lambda}, \widehat{h}, \widehat{q}\right\}$,

$$\max\left\{1+\overline{\kappa}_h, \frac{1}{1-\overline{\kappa}_\lambda}\right\} = \max\left\{\frac{\widehat{x}g'\left(\widehat{x}\right)}{g\left(\widehat{x}\right)}, \frac{pg'\left(\widehat{x}\right)+1-p}{p\frac{g(\widehat{x})}{\widehat{x}}+1-p}\right\} = \frac{\widehat{x}g'\left(\widehat{x}\right)}{g\left(\widehat{x}\right)}$$

Therefore, the level of $g''(\cdot) >> 0$ needed is

$$g''\left(\widehat{x}\right) > \frac{g'\left(\widehat{x}\right)}{g\left(\widehat{x}\right)}\left(g'\left(\widehat{x}\right)-\frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)$$

which is implied by $\varepsilon_g'(x) \equiv \frac{x}{g(x)}\left(g''(x)-\frac{g'(x)}{g(x)}\left(g'(x)-\frac{g(x)}{x}\right)\right) > 0$. ■

Proof of Proposition 7

The planner chooses α and x (recall $x \equiv \theta(1-\beta)-\alpha$) to solve the problem associated with the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= \alpha + f(1-\alpha) - pg(x) + px - p\theta\phi\left(1-\frac{\alpha+x}{\theta}\right) \\ &\quad +\psi_\alpha\left[p\frac{g(x)}{x}-(1-\kappa_\lambda)f'(1-\alpha)+(1-p)\right]+\psi_\beta\left[(1+\kappa_h)\frac{g(x)}{x}-1-\phi'\left(1-\frac{\alpha+x}{\theta}\right)\right]\end{aligned}$$

where $\psi_\beta, \psi_\alpha \geq 0$ are Lagrange multipliers on the implementation constraints (A.20) and (A.22).

The planner's first order conditions are

$$f'(1-\alpha) = 1 + p\phi'\left(1-\frac{\alpha+x}{\theta}\right) + \psi_\alpha(1-\kappa_\lambda)f''(1-\alpha) + \psi_\beta\phi''\left(1-\frac{\alpha+x}{\theta}\right)\frac{1}{\theta}$$

$$g'(x) = 1 + \phi'\left(1-\frac{\alpha+x}{\theta}\right) + \left(\psi_\alpha + \psi_\beta\frac{1+\kappa_h}{p}\right)\frac{1}{x}\left(g'(x)-\frac{g(x)}{x}\right) + \psi_\beta\phi''\left(1-\frac{\alpha+x}{\theta}\right)\frac{1}{p\theta}$$

If $\psi_\alpha > 0$ and $\psi_\beta > 0$, i.e., if both implementation constraints are binding, then the planner's solution is simply the pair $\{\alpha, x\}$ solving Eqs. (A.20) and (A.22). If $\kappa_\lambda = \kappa_h = 0$, then those equations collapse to Eqs. (6) and (A.3), which returns the decentralized equilibrium, i.e., $\alpha_0 = \lambda^*$ and $x_0 = \theta(1-h^*) - \lambda^*$, where we recall $\lambda^* < \widehat{\lambda}$ and $h^* < \widehat{h}$.

The next step is to confirm that both implementation constraints are binding at $\kappa_\lambda = \kappa_h = 0$. Evaluating the planner's first order conditions at α_0 and x_0 yields

$$0 = \psi_\alpha(1-\kappa_\lambda)f''(1-\lambda^*) + \psi_\beta\phi''(h^*)\frac{1}{\theta}$$

$$g'(x_0) = \frac{g(x_0)}{x_0} + \left(\psi_\alpha + \psi_\beta \frac{1+\kappa_h}{p}\right)\frac{1}{x}\left(g'(x_0) - \frac{g(x_0)}{x_0}\right) + \psi_\beta \phi''(h^*)\frac{1}{p\theta}$$

which pins down ψ_α and ψ_β as

$$\psi_\alpha = \frac{\phi''(h^*)\frac{1}{\theta}}{-(1-\kappa_\lambda) f''(1-\lambda^*)}\psi_\beta > 0$$

$$\psi_\beta = \frac{g'(x_0) - \frac{g(x_0)}{x_0}}{\left(\frac{\phi''(h^*)\frac{1}{\theta}}{-(1-\kappa_\lambda)f''(1-\lambda^*)} + \frac{1+\kappa_h}{p}\right)\frac{1}{x}\left(g'(x_0) - \frac{g(x_0)}{x_0}\right) + \phi''(h^*)\frac{1}{p\theta}} > 0$$

thereby confirming that both implementation constraints are indeed binding at $\kappa_\lambda = \kappa_h = 0$.

By continuity, it follows that (i) $\psi_\alpha > 0$ and $\psi_\beta > 0$ and (ii) $\alpha < \widehat{\lambda}$ and $\beta < \widehat{h}$ for sufficiently low but positive κ_λ and κ_h. ■

Proof of Lemma 5

Define the welfare function

$$W(\alpha,\beta) \equiv \alpha + f(1-\alpha) - pg(\theta(1-\beta)-\alpha) + p(\theta(1-\beta)-\alpha) - p\theta\phi(\beta)$$

A second order Taylor expansion around the unconstrained solution $\left(\widehat{\lambda},\widehat{h}\right)$ delivers

$$\begin{aligned} W(\alpha,\beta) \approx\;& W\left(\widehat{\lambda},\widehat{h}\right) + W'_\alpha\left(\widehat{\lambda},\widehat{h}\right)\left(\alpha-\widehat{\lambda}\right) + W'_\beta\left(\widehat{\lambda},\widehat{h}\right)\left(\beta-\widehat{h}\right) \\ &+\frac{1}{2}\left[W''_{\alpha\alpha}\left(\widehat{\lambda},\widehat{h}\right)\left(\alpha-\widehat{\lambda}\right)^2 + 2W''_{\alpha\beta}\left(\widehat{\lambda},\widehat{h}\right)\left(\alpha-\widehat{\lambda}\right)\left(\beta-\widehat{h}\right) + W''_{\beta\beta}\left(\widehat{\lambda},\widehat{h}\right)\left(\beta-\widehat{h}\right)^2\right] \end{aligned}$$

where we recall that the planner's first order conditions from the unconstrained problem deliver $W'_\alpha\left(\widehat{\lambda},\widehat{h}\right) = W'_\beta\left(\widehat{\lambda},\widehat{h}\right) = 0$. Therefore,

$$W(\alpha,\beta) \approx W\left(\widehat{\lambda},\widehat{h}\right) + \frac{1}{2}\left[W''_{\alpha\alpha}\left(\widehat{\lambda},\widehat{h}\right)\left(\alpha-\widehat{\lambda}\right)^2 + 2W''_{\alpha\beta}\left(\widehat{\lambda},\widehat{h}\right)\left(\alpha-\widehat{\lambda}\right)\left(\beta-\widehat{h}\right) + W''_{\beta\beta}\left(\widehat{\lambda},\widehat{h}\right)\left(\beta-\widehat{h}\right)^2\right]$$

where

$$W''_{\alpha\alpha}(\alpha,\beta) = f''(1-\alpha) - pg''(\theta(1-\beta)-\alpha)$$

$$W''_{\alpha\beta}(\alpha,\beta) = -p\theta g''(\theta(1-\beta)-\alpha)$$

$$W''_{\beta\beta}(\alpha,\beta) = -p\theta^2 g''(\theta(1-\beta)-\alpha) - p\theta\phi''(\beta)$$

With $f'''(\cdot) = \phi'''(\cdot) = g'''(\cdot) = 0$, the second derivatives of the welfare function are constants and hence the second order Taylor expansion is exact, i.e.,

$$W(\alpha,\beta) = W\left(\widehat{\lambda},\widehat{h}\right) - \frac{1}{2}\left[\left(\alpha-\widehat{\lambda}\right)^2(pg''-f'') + 2\left(\alpha-\widehat{\lambda}\right)\left(\beta-\widehat{h}\right)p\theta g'' + \left(\beta-\widehat{h}\right)^2\left(p\theta^2 g'' + p\theta\phi''\right)\right]$$

where $f'' < 0$, $\phi'' > 0$, and $g'' > 0$ are scalars.

Denote by $(\alpha_\lambda, \beta_\lambda)$ the constrained optimal regulation when $\kappa_\lambda = 0$ and $\kappa_h \to \infty$, and denote by (α_h, β_h) the constrained optimal regulation when $\kappa_\lambda \to \infty$ and $\kappa_h = 0$. We then need to show $W(\alpha_\lambda, \beta_\lambda) = W(\alpha_h, \beta_h)$. With $-f'' = \frac{p\phi''}{\theta}$, this amounts to showing

$$
\begin{aligned}
& \left(\alpha_\lambda - \widehat{\lambda}\right)^2 \left(pg'' + \frac{p\phi''}{\theta}\right) + 2\left(\alpha_\lambda - \widehat{\lambda}\right)\left(\beta_\lambda - \widehat{h}\right) p\theta g'' + \left(\beta_\lambda - \widehat{h}\right)^2 \left(p\theta^2 g'' + p\theta\phi''\right) \\
= \; & \left(\alpha_h - \widehat{\lambda}\right)^2 \left(pg'' + \frac{p\phi''}{\theta}\right) + 2\left(\alpha_h - \widehat{\lambda}\right)\left(\beta_h - \widehat{h}\right) p\theta g'' + \left(\beta_h - \widehat{h}\right)^2 \left(p\theta^2 g'' + p\theta\phi''\right)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& \left(\alpha_h + \theta\beta_h - \theta\beta_\lambda - \alpha_\lambda\right)\left(\alpha_\lambda + \theta\beta_\lambda + \alpha_h + \theta\beta_h - 2\widehat{\lambda} - 2\theta\widehat{h}\right)\frac{\theta g''}{\phi''} \\
= \; & \left(\alpha_\lambda - \alpha_h\right)\left(\alpha_\lambda + \alpha_h - 2\widehat{\lambda}\right) + \theta^2\left(\beta_\lambda - \beta_h\right)\left(\beta_\lambda + \beta_h - 2\widehat{h}\right)
\end{aligned}
$$

$\Longleftrightarrow$

$$
\left(x_h - x_\lambda\right)\left(2\widehat{x} - x_\lambda - x_h\right)\frac{\theta g''}{\phi''} + \left(\alpha_\lambda - \alpha_h\right)\left(\alpha_\lambda + \alpha_h - 2\widehat{\lambda}\right) + \theta^2\left(\beta_\lambda - \beta_h\right)\left(\beta_\lambda + \beta_h - 2\widehat{h}\right) = 0
$$

$\Longleftrightarrow$

$$
\left(x_h - x_\lambda\right)\left(\left(2\widehat{x} - x_\lambda - x_h\right)\frac{\theta g''}{\phi''} + \alpha_\lambda + \alpha_h - 2\widehat{\lambda}\right) + \theta\left(\beta_\lambda - \beta_h\right)\left(\theta\beta_\lambda + \theta\beta_h - 2\theta\widehat{h} - \alpha_\lambda - \alpha_h + 2\widehat{\lambda}\right) = 0
$$

$\Longleftrightarrow$

$$
\left(x_h - x_\lambda\right)\left(\left(2\widehat{x} - x_\lambda - x_h\right)\frac{\theta g''}{\phi''} + \alpha_\lambda + \alpha_h - 2\widehat{\lambda} - \theta\left(\beta_\lambda - \beta_h\right)\right) + 2\theta\left(\beta_\lambda - \beta_h\right)\left(\widehat{\lambda} - \alpha_h - \theta\left(\widehat{h} - \beta_\lambda\right)\right) = 0
$$

where $x_h \equiv \theta(1 - \beta_h) - \alpha_h$, $x_\lambda \equiv \theta(1 - \beta_\lambda) - \alpha_\lambda$, and $\widehat{x} \equiv \theta\left(1 - \widehat{h}\right) - \widehat{\lambda}$. Therefore, to show $W(\alpha_\lambda, \beta_\lambda) = W(\alpha_h, \beta_h)$, it will suffice to show $x_h = x_\lambda$ and $\widehat{\lambda} - \alpha_h = \theta\left(\widehat{h} - \beta_\lambda\right)$.

We recall from the proof of Proposition 7 that the planner's first order conditions (expressed here in terms of α and β rather than α and x) are

$$
f'(1 - \alpha) = 1 + p\phi'(\beta) + \psi_\alpha(1 - \kappa_\lambda) f'' + \psi_\beta \frac{\phi''}{\theta} \tag{A.24}
$$

$$
g'(\theta(1 - \beta) - \alpha) = 1 + \phi'(\beta) + \left(\psi_\alpha + \psi_\beta \frac{1 + \kappa_h}{p}\right)\frac{g'(\theta(1 - \beta) - \alpha) - \frac{g(\theta(1-\beta)-\alpha)}{\theta(1-\beta)-\alpha}}{\theta(1 - \beta) - \alpha} + \psi_\beta \frac{\phi''}{p\theta} \tag{A.25}
$$

If $\kappa_\lambda = 0$ and $\kappa_h \to \infty$, then $\psi_\alpha > 0$ and $\psi_\beta = 0$. Use Eq. (A.24) to substitute ψ_α out of Eq. (A.25) and get one equation in terms of only α_λ and β_λ. The other equation comes from the complementary slackness condition on $\psi_\alpha > 0$, i.e., the binding implementation constraint (A.22) with $\kappa_\lambda = 0$. Thus, $(\alpha_\lambda, \beta_\lambda)$ solves

$$g'(\theta(1-\beta_\lambda)-\alpha_\lambda) = 1+\phi'(\beta_\lambda)+\frac{f'(1-\alpha_\lambda)-1-p\phi'(\beta_\lambda)}{f''}\left(\frac{g'(\theta(1-\beta_\lambda)-\alpha_\lambda)-\frac{g(\theta(1-\beta_\lambda)-\alpha_\lambda)}{\theta(1-\beta_\lambda)-\alpha_\lambda}}{\theta(1-\beta_\lambda)-\alpha_\lambda}\right)$$

$$p\frac{g(\theta(1-\beta_\lambda)-\alpha_\lambda)}{\theta(1-\beta_\lambda)-\alpha_\lambda} = f'(1-\alpha_\lambda)-(1-p)$$

These equations can be written more compactly. Define

$$G(x) \equiv \frac{g'(x)-\frac{g(x)}{x}}{1+\frac{1}{x}\left(g'(x)-\frac{g(x)}{x}\right)\frac{\theta}{\phi''}}$$

where $G'(x) > 0$ follows from $\varepsilon_g'(x) > 0$. Then $-f'' = \frac{p\phi''}{\theta}$ implies that $(\alpha_\lambda, \beta_\lambda)$ solves

$$\frac{g(x_\lambda)}{x_\lambda} + G(x_\lambda) = 1+\phi'(\beta_\lambda) \tag{A.26}$$

$$\frac{g(x_\lambda)}{x_\lambda} = \frac{f'(1-\alpha_\lambda)-(1-p)}{p} \tag{A.27}$$

If $\kappa_\lambda \to \infty$ and $\kappa_h = 0$, then $\psi_\alpha = 0$ and $\psi_\beta > 0$. Use Eq. (A.24) to substitute ψ_β out of Eq. (A.25) and get one equation in terms of only α_h and β_h. The other equation comes from the complementary slackness condition on $\psi_\beta > 0$, i.e., the binding implementation constraint (A.20) with $\kappa_h = 0$. Thus, (α_h, β_h) solves

$$g'(\theta(1-\beta_h)-\alpha_h) = 1+\phi'(\beta_h)+\frac{f'(1-\alpha_h)-1-p\phi'(\beta_h)}{\frac{p\phi''}{\theta}}\left(\frac{g'(\theta(1-\beta_h)-\alpha_h)-\frac{g(\theta(1-\beta_h)-\alpha_h)}{\theta(1-\beta_h)-\alpha_h}}{\theta(1-\beta_h)-\alpha_h}+\frac{\phi''}{\theta}\right)$$

$$\frac{g(\theta(1-\beta_h)-\alpha_h)}{\theta(1-\beta_h)-\alpha_h} = 1+\phi'(\beta_h)$$

or equivalently

$$\frac{g(x_h)}{x_h} = 1+\phi'(\beta_h) \tag{A.28}$$

$$\frac{g(x_h)}{x_h} + G(x_h) = \frac{f'(1-\alpha_h)-(1-p)}{p} \tag{A.29}$$

Recall that $\left(\widehat{\lambda}, \widehat{h}\right)$ solves Eqs. (A.3) and (A.5), which can be written as

$$g'(\widehat{x}) = 1+\phi'\left(\widehat{h}\right) \tag{A.30}$$

$$g'(\widehat{x}) = \frac{f'\left(1-\widehat{\lambda}\right)-(1-p)}{p} \tag{A.31}$$

Subtract Eq. (A.30) from Eqs. (A.28) and (A.26) then apply $\phi'''(\cdot) = 0$ to get

$$g'(\widehat{x}) - \frac{g(x_h)}{x_h} = \left(\widehat{h} - \beta_h\right)\phi'' \tag{A.32}$$

$$g'(\widehat{x}) - \frac{g(x_\lambda)}{x_\lambda} - G(x_\lambda) = \left(\widehat{h} - \beta_\lambda\right)\phi'' \tag{A.33}$$

Next, subtract Eq. (A.31) from Eqs. (A.29) and (A.27) then apply $f'''(\cdot) = 0$ and $-f'' = \frac{p\phi''}{\theta}$ to get

$$g'(\widehat{x}) - \frac{g(x_h)}{x_h} - G(x_h) = \left(\widehat{\lambda} - \alpha_h\right)\frac{\phi''}{\theta} \tag{A.34}$$

$$g'(\widehat{x}) - \frac{g(x_\lambda)}{x_\lambda} = \left(\widehat{\lambda} - \alpha_\lambda\right)\frac{\phi''}{\theta} \tag{A.35}$$

Subtracting Eq. (A.32) from Eq. (A.35) gives

$$\frac{g(x_h)}{x_h} - \frac{g(x_\lambda)}{x_\lambda} = \left(\widehat{\lambda} - \alpha_\lambda - \theta\left(\widehat{h} - \beta_h\right)\right)\frac{\phi''}{\theta} \tag{A.36}$$

while subtracting Eq. (A.33) from Eq. (A.34) gives

$$\frac{g(x_\lambda)}{x_\lambda} + G(x_\lambda) - \left(\frac{g(x_h)}{x_h} + G(x_h)\right) = \left(\widehat{\lambda} - \alpha_h - \theta\left(\widehat{h} - \beta_\lambda\right)\right)\frac{\phi''}{\theta} \tag{A.37}$$

We now show that $x_h = x_\lambda$. The proof proceeds by contradiction. Suppose $x_h > x_\lambda$. Then $\widehat{\lambda} - \alpha_\lambda > \theta\left(\widehat{h} - \beta_h\right)$ from Eq. (A.36) and $\widehat{\lambda} - \alpha_h < \theta\left(\widehat{h} - \beta_\lambda\right)$ from Eq. (A.37). Accordingly, $\left(\widehat{\lambda} - \alpha_\lambda\right) - \left(\widehat{\lambda} - \alpha_h\right) > \theta\left(\widehat{h} - \beta_h\right) - \theta\left(\widehat{h} - \beta_\lambda\right)$ or equivalently $\alpha_h - \alpha_\lambda > \theta(\beta_\lambda - \beta_h)$. But then $\theta(1 - \beta_\lambda) - \alpha_\lambda > \theta(1 - \beta_h) - \alpha_h$, which means $x_\lambda > x_h$. This is a contradiction. Suppose instead $x_\lambda > x_h$. Then $\widehat{\lambda} - \alpha_\lambda < \theta\left(\widehat{h} - \beta_h\right)$ from Eq. (A.36) and $\widehat{\lambda} - \alpha_h > \theta\left(\widehat{h} - \beta_\lambda\right)$ from Eq. (A.37). Accordingly, $\left(\widehat{\lambda} - \alpha_h\right) - \left(\widehat{\lambda} - \alpha_\lambda\right) > \theta\left(\widehat{h} - \beta_\lambda\right) - \theta\left(\widehat{h} - \beta_h\right)$ or equivalently $\alpha_\lambda - \alpha_h > \theta(\beta_h - \beta_\lambda)$. But then $\theta(1 - \beta_h) - \alpha_h > \theta(1 - \beta_\lambda) - \alpha_\lambda$, which means $x_h > x_\lambda$. This is also a contradiction. Therefore, $x_h = x_\lambda$ and hence $\widehat{\lambda} - \alpha_h = \theta\left(\widehat{h} - \beta_\lambda\right)$ from Eq. (A.37). ■

Proof of Lemma 6

Recall from the proof of Proposition 6 that we can rewrite $\overline{\kappa}_h$ as

$$1 + \overline{\kappa}_h = \frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}$$

where the right-hand side is equivalent to the elasticity $\varepsilon_g(\widehat{x})$. Therefore,

$$d\overline{\kappa}_h = \varepsilon_g'(\widehat{x})\,d\widehat{x}$$

which implies

$$\frac{d\overline{\kappa}_h}{dp} \overset{sign}{=} \frac{d\widehat{x}}{dp} \text{ and } \frac{d\overline{\kappa}_h}{d\theta} \overset{sign}{=} \frac{d\widehat{x}}{d\theta}$$

Next, recall that $\left\{\widehat{\lambda}, \widehat{h}, \widehat{x}\right\}$ solves the system

$$f'\left(1-\widehat{\lambda}\right) = 1 + p\phi'\left(\widehat{h}\right)$$

$$g'\left(\widehat{x}\right) = 1 + \phi'\left(\widehat{h}\right)$$

$$\widehat{x} = \theta\left(1-\widehat{h}\right) - \widehat{\lambda}$$

Differentiating yields

$$-f''\left(1-\widehat{\lambda}\right) d\widehat{\lambda} = \phi'\left(\widehat{h}\right) dp + p\phi''\left(\widehat{h}\right) d\widehat{h}$$

$$g''\left(\widehat{x}\right) d\widehat{x} = \phi''\left(\widehat{h}\right) d\widehat{h}$$

$$d\widehat{x} = \left(1-\widehat{h}\right) d\theta - \theta d\widehat{h} - d\widehat{\lambda}$$

which combines to give

$$\left(1 + \left(\frac{\theta}{\phi''\left(\widehat{h}\right)} + \frac{p}{-f''\left(1-\widehat{\lambda}\right)}\right) g''\left(\widehat{x}\right)\right) d\widehat{x} = \left(1-\widehat{h}\right) d\theta - \frac{\phi'\left(\widehat{h}\right)}{-f''\left(1-\widehat{\lambda}\right)} dp$$

Therefore, $\frac{d\widehat{x}}{dp} < 0$ and $\frac{d\widehat{x}}{d\theta} > 0$.

Turning next to $\overline{\kappa}_\lambda$, recall again from the proof of Proposition 6 that

$$\frac{1}{1-\overline{\kappa}_\lambda} = \frac{pg'\left(\widehat{x}\right) + 1 - p}{p\frac{g(\widehat{x})}{\widehat{x}} + 1 - p}$$

which rearranges to

$$\overline{\kappa}_\lambda = \frac{\varepsilon_g\left(\widehat{x}\right) - 1}{\varepsilon_g\left(\widehat{x}\right)\left(1 + \frac{1-p}{pg'(\widehat{x})}\right)}$$

Therefore,

$$\frac{d\overline{\kappa}_\lambda}{d\theta} = \frac{\varepsilon_g'\left(\widehat{x}\right)\left(1 + \frac{1-p}{p}\frac{\widehat{x}}{g(\widehat{x})}\right) + \left(\varepsilon_g\left(\widehat{x}\right) - 1\right)^2 \frac{1-p}{p}\frac{1}{g(\widehat{x})}}{\left(\varepsilon_g\left(\widehat{x}\right) + \frac{1-p}{p}\frac{\widehat{x}}{g(\widehat{x})}\right)^2} \frac{d\widehat{x}}{d\theta}$$

which implies $\frac{d\overline{\kappa}_\lambda}{d\theta} \overset{sign}{=} \frac{d\widehat{x}}{d\theta}$ and hence $\frac{d\overline{\kappa}_\lambda}{d\theta} > 0$. Moreover,

$$\frac{\overline{\kappa}_h}{\overline{\kappa}_\lambda} = \varepsilon_g\left(\widehat{x}\right)\left(1 + \frac{1-p}{pg'\left(\widehat{x}\right)}\right)$$

and thus

$$\frac{d\left(\frac{\overline{\kappa}_h}{\overline{\kappa}_\lambda}\right)}{dp} = \varepsilon_g'\left(\widehat{x}\right)\left(1+\frac{1-p}{pg'\left(\widehat{x}\right)}\right)\frac{d\widehat{x}}{dp} - \frac{\varepsilon_g\left(\widehat{x}\right)}{pg'\left(\widehat{x}\right)}\left(1+\frac{1-p}{pg'\left(\widehat{x}\right)}\frac{d\left(pg'\left(\widehat{x}\right)\right)}{dp}\right)$$

The first term on the right-hand side is negative by $\frac{d\widehat{x}}{dp} < 0$. A sufficient condition for the second term to also be negative is $\frac{d(pg'(\widehat{x}))}{dp} > 0$. From (A.3) and (A.5), we can write

$$pg'\left(\widehat{x}\right) = f'\left(1-\widehat{\lambda}\right) - (1-p)$$

and hence

$$\frac{d\left(pg'\left(\widehat{x}\right)\right)}{dp} = -f''\left(1-\widehat{\lambda}\right)\frac{d\widehat{\lambda}}{dp} + 1 > 0$$

where the inequality follows from $\frac{d\widehat{\lambda}}{dp} > 0$ as established in Lemma 3. Therefore, $\frac{d\left(\frac{\overline{\kappa}_h}{\overline{\kappa}_\lambda}\right)}{dp} < 0$. ■

Proof of Lemma 7

Bank first order conditions are still given by Eqs. (5) and (6) but with

$$q = \frac{\theta\left(1-h\right)-\lambda-b}{g\left(\theta\left(1-h\right)-\lambda-b\right)}$$

Therefore, the decentralized equilibrium is now characterized by Eq. (A.3) and

$$\frac{g\left(\theta\left(1-h\right)-\lambda-b\right)}{\theta\left(1-h\right)-\lambda-b} = 1+\phi'\left(h\right)$$

Differentiating these equations gives

$$\frac{d\lambda}{db} = \frac{p\phi''\left(h\right)}{-f''\left(1-\lambda\right)}\frac{dh}{db}$$

$$-\frac{g'\left(\theta\left(1-h\right)-\lambda-b\right)-\frac{g(\theta(1-h)-\lambda-b)}{\theta(1-h)-\lambda-b}}{\theta\left(1-h\right)-\lambda-b}\left(\theta\frac{dh}{db}+\frac{d\lambda}{db}+1\right) = \phi''\left(h\right)\frac{dh}{db}$$

which rearranges to give $\frac{dh}{db} < 0$ and hence $\frac{d\lambda}{db} < 0$. ■

Proof of Proposition 8

Assume the planner can commit to a level of b at $t=0$, at the same time that he is choosing the regulations α and β (or equivalently α and x). The planner's objective function, $\widetilde{\Pi}\left(\cdot\right) - p\nu b$, can be expressed as

$$\alpha + f\left(1-\alpha\right) - pg\left(x-b\right) + px - p\theta\phi\left(1-\frac{\alpha+x}{\theta}\right) - p\nu b - 1$$

following similar steps to the proof of Proposition 6. Absent implementation constraints, the Lagrangian is

$$\mathcal{L} = \alpha + f(1-\alpha) - pg(x-b) + px - p\theta\phi\left(1 - \frac{\alpha + x}{\theta}\right) - p\nu b + \eta_b b$$

where $\eta_b \geq 0$ is the Lagrange multiplier on the non-negativity constraint $b \geq 0$.

The first order conditions with respect to α, x, and b are respectively

$$f'(1-\alpha) = 1 + p\phi'(\beta)$$

$$g'(\theta(1-\beta) - \alpha - b) = 1 + \phi'(\beta)$$

$$\eta_b = p\left(\nu - g'(\theta(1-\beta) - \alpha - b)\right)$$

where we have substituted out $x \equiv \theta(1-\beta) - \alpha$. If $b = 0$, then (i) $\{\alpha, \beta\} = \left\{\widehat{\lambda}, \widehat{h}\right\}$ and (ii) $\eta_b \geq 0$, so confirming $b = 0$ requires

$$\nu \geq g'\left(\theta\left(1 - \widehat{h}\right) - \widehat{\lambda}\right) \equiv \overline{\nu}_0$$

If instead $\nu < \overline{\nu}_0$, then it must be the case that $b > 0$ with $\eta_b = 0$, i.e.,

$$g'(\theta(1-\beta) - \alpha - b) = \nu$$

and therefore

$$1 + \phi'(\beta) = \nu < \overline{\nu}_0 = 1 + \phi'\left(\widehat{h}\right)$$

Thus, $\nu < \overline{\nu}_0$ implies $\beta < \widehat{h}$ which further implies $\alpha < \widehat{\lambda}$ from $f'(1-\alpha) = 1 + p\phi'(\beta)$.

Now assume the planner cannot commit to a level of b at $t = 0$. Then b is chosen at $t = 1$ to maximize $-g(x-b) - \nu b + \eta_b b$ if the stress state is realized while α and x are still chosen at $t = 0$ anticipating the choice of b. The resulting first order conditions are exactly as above, so the optimal policy is the same. ■

Proof of Proposition 9

The planner's implementation constraints are

$$\frac{p}{q} \geq (1-\kappa_\lambda) f'(1-\lambda) - (1-p) \longrightarrow p\frac{g(x-b)}{x-b} \geq (1-\kappa_\lambda) f'(1-\alpha) - (1-p) \tag{A.38}$$

$$\frac{1+\kappa_h}{q} \geq 1 + \phi'(h) \longrightarrow (1+\kappa_h)\frac{g(x-b)}{x-b} \geq 1 + \phi'\left(1 - \frac{\alpha + x}{\theta}\right) \tag{A.39}$$

If both implementation constraints are slack, then the planner solves the same problem as in the proof of Proposition 8. Accordingly, $\{\alpha, \beta, b\} = \left\{\widehat{\lambda}, \widehat{h}, 0\right\}$ if $\nu \geq \overline{\nu}_0$ and it follows that both implementation constraints are slack if $\kappa_\lambda \geq \overline{\kappa}_\lambda$ and $\kappa_h \geq \overline{\kappa}_h$.

Consider next (A.38) slack and (A.39) binding, assuming commitment. Then the planner's problem is

$$\max_{x,b}\left\{\alpha\left(x,b\right)+f\left(1-\alpha\left(x,b\right)\right)-pg\left(x-b\right)+px-p\theta\phi\left(1-\frac{\alpha\left(x,b\right)+x}{\theta}\right)-p\nu b+\eta_b b\right\}$$

where the binding implementation constraint implicitly defines $\alpha\left(x,b\right)$. The first order conditions for x and b are respectively

$$g'\left(x-b\right)=1+\phi'\left(1-\frac{\alpha+x}{\theta}\right)+\frac{1}{p}\left(f'\left(1-\alpha\right)-1-p\phi'\left(1-\frac{\alpha+x}{\theta}\right)\right)\left(1+K_b\left(\alpha,x,b;\kappa_h\right)\right) \tag{A.40}$$

$$\eta_b=p\left(\nu-g'\left(x-b\right)\right)-\left(1+p\phi'\left(1-\frac{\alpha+x}{\theta}\right)-f'\left(1-\alpha\right)\right)K_b\left(\alpha,x,b;\kappa_h\right) \tag{A.41}$$

where

$$K_b\left(\alpha,x,b;\kappa_h\right)\equiv\frac{\theta\left(1+\kappa_h\right)\left(g'\left(x-b\right)-\frac{g(x-b)}{x-b}\right)}{\left(x-b\right)\phi''\left(1-\frac{\alpha+x}{\theta}\right)}$$

The planner's solution is a triple $\{\alpha,x,b\}$ solving Eqs. (A.40), (A.41), and (A.39) with equality. Notice that $\left\{\widehat{\lambda},\widehat{x},0\right\}$ satisfies these equations if $\kappa_h=\overline{\kappa}_h$ and $\nu=\overline{\nu}_0$. Differentiate to get $\frac{d\alpha}{d\kappa_h}$, $\frac{dx}{d\kappa_h}$, and $\frac{db}{d\kappa_h}$ then evaluate at $\kappa_h=\overline{\kappa}_h$ to get

$$g''\left(\widehat{x}\right)\left(dx-db\right)=-\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)+\frac{1}{p}\left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)-f''\left(1-\widehat{\lambda}\right)d\alpha\right)\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)$$

$$d\eta_b=-pg''\left(\widehat{x}\right)\left(dx-db\right)+\left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)-f''\left(1-\widehat{\lambda}\right)d\alpha\right)K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)$$

$$\frac{g\left(\widehat{x}\right)}{\widehat{x}}d\kappa_h+\frac{1+\overline{\kappa}_h}{\widehat{x}}\left(g'\left(\widehat{x}\right)-\frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)\left(dx-db\right)=-\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)$$

$\Longrightarrow$

$$g''\left(\widehat{x}\right)\left(dx-db\right)=\frac{\phi''\left(\widehat{h}\right)}{\theta}K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\left(d\alpha+dx\right)-\frac{f''\left(1-\widehat{\lambda}\right)}{p}\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)d\alpha$$

$$d\eta_b=f''\left(1-\widehat{\lambda}\right)d\alpha$$

$$\frac{\theta}{\phi''\left(\widehat{h}\right)}\frac{g\left(\widehat{x}\right)}{\widehat{x}}d\kappa_h+\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)dx-K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)db=-d\alpha$$

If $d\eta_b=0$, then

$$\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h}=0$$

$$\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{g''\left(\widehat{x}\right)\frac{\theta}{\phi''\left(\widehat{h}\right)}\frac{g(\widehat{x})}{\widehat{x}}}{g''\left(\widehat{x}\right)+\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)^2} < 0$$

$$\left.\frac{db}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = \frac{\frac{g(\widehat{x})}{\widehat{x}}\left(K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)-\frac{\theta g''(\widehat{x})}{\phi''\left(\widehat{h}\right)}\right)}{g''\left(\widehat{x}\right)+\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)^2} = -\frac{\frac{\theta}{\phi''\left(\widehat{h}\right)}\left(\frac{g(\widehat{x})}{\widehat{x}}\right)^2\varepsilon_g'\left(\widehat{x}\right)}{g''\left(\widehat{x}\right)+\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)^2} < 0$$

Therefore, a perturbation of κ_h below $\overline{\kappa}_h$ leads to $b > 0$, which requires $\eta_b = 0$, confirming $d\eta_b = 0$ when $\nu = \overline{\nu}_0$. Moreover, $\alpha = \widehat{\lambda}$ and $x > \widehat{x}$, which together imply $\beta < \widehat{h}$. The case of $d\eta_b \neq 0$ can be ruled out as follows. If $\left.\frac{d\eta_b}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} \neq 0$, then $\left.\frac{db}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = 0$ and hence

$$\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{\frac{\theta}{\phi''\left(\widehat{h}\right)}\left(\frac{g(\widehat{x})}{\widehat{x}}\right)^2\varepsilon_g'\left(\widehat{x}\right)}{g''\left(\widehat{x}\right)+\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)^2+\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)^2} < 0$$

$$\left.\frac{d\eta_b}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = f''\left(1-\widehat{\lambda}\right)\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} > 0$$

But then a perturbation of κ_h below $\overline{\kappa}_h$ leads to $\eta_b < 0$ which is impossible since $\eta_b \geq 0$ is a Lagrange multiplier.

Now consider (A.39) slack and (A.38) binding, again assuming commitment. The planner's problem is the same but with $\alpha\left(x,b\right)$ implicitly defined by (A.38) binding. The first order conditions for x and b are respectively

$$g'\left(x-b\right) = 1+\phi'\left(1-\frac{\alpha+x}{\theta}\right)+\frac{1}{p}\left(1+p\phi'\left(1-\frac{\alpha+x}{\theta}\right)-f'\left(1-\alpha\right)\right)\widetilde{K}_b\left(\alpha,x,b;\kappa_\lambda\right) \quad \text{(A.42)}$$

$$\eta_b = p\left(\nu-g'\left(x-b\right)\right)+\left(1+p\phi'\left(1-\frac{\alpha+x}{\theta}\right)-f'\left(1-\alpha\right)\right)\widetilde{K}_b\left(\alpha,x,b;\kappa_\lambda\right) \quad \text{(A.43)}$$

where

$$\widetilde{K}_b\left(\alpha,x,b;\kappa_\lambda\right) \equiv \frac{p\left(g'\left(x-b\right)-\frac{g(x-b)}{x-b}\right)}{-\left(1-\kappa_\lambda\right)f''\left(1-\alpha\right)\left(x-b\right)}$$

The planner's solution is a triple $\{\alpha,x,b\}$ solving Eqs. (A.42), (A.43), and (A.38) with equality. Notice that $\left\{\widehat{\lambda},\widehat{x},0\right\}$ satisfies these equations if $\kappa_\lambda = \overline{\kappa}_\lambda$ and $\nu = \overline{\nu}_0$. Differentiate to get $\frac{d\alpha}{d\kappa_\lambda}$, $\frac{dx}{d\kappa_\lambda}$, and $\frac{db}{d\kappa_\lambda}$ then evaluate at $\kappa_\lambda = \overline{\kappa}_\lambda$ to get

$$g''\left(\widehat{x}\right)\left(dx-db\right) = -\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)-\frac{1}{p}\left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)-f''\left(1-\widehat{\lambda}\right)d\alpha\right)\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_\lambda\right)$$

$$d\eta_b = -pg''\left(\widehat{x}\right)\left(dx-db\right)-\left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}\left(d\alpha+dx\right)-f''\left(1-\widehat{\lambda}\right)d\alpha\right)\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_\lambda\right)$$

$$\frac{p}{\widehat{x}}\left(g'\left(\widehat{x}\right)-\frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)(dx-db)=-f'\left(1-\widehat{\lambda}\right)d\kappa_{\lambda}-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)d\alpha$$

$\Longrightarrow$

$$g''\left(\widehat{x}\right)(dx-db)=-\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(1+\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)(d\alpha+dx)+\frac{f''\left(1-\widehat{\lambda}\right)}{p}\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)d\alpha$$

$$d\eta_b=\frac{p\phi''\left(\widehat{h}\right)}{\theta}(d\alpha+dx)$$

$$\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)(dx-db)+\frac{f'\left(1-\widehat{\lambda}\right)}{-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)}d\kappa_{\lambda}=d\alpha$$

If $d\eta_b=0$, then

$$\left.\frac{d\alpha}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}=-\left.\frac{dx}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}>0$$

$$\left.\frac{dx}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}=-\frac{\frac{f'\left(1-\widehat{\lambda}\right)g''\left(\widehat{x}\right)}{-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)}}{g''\left(\widehat{x}\right)+\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\left(\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)^2}<0$$

$$\begin{aligned}\left.\frac{db}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}} &= -\frac{\frac{f'\left(1-\widehat{\lambda}\right)}{-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)}\left(g''\left(\widehat{x}\right)-\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)}{g''\left(\widehat{x}\right)+\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\left(\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)^2}\\ &= -\frac{\frac{f'\left(1-\widehat{\lambda}\right)}{-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)}\left(\frac{g\left(\widehat{x}\right)}{\widehat{x}}\varepsilon_g'\left(\widehat{x}\right)+\frac{(1-p)\left(g'\left(\widehat{x}\right)-\frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)^2}{g\left(\widehat{x}\right)\left(p\frac{g\left(\widehat{x}\right)}{\widehat{x}}+1-p\right)}\right)}{g''\left(\widehat{x}\right)+\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\left(\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)^2}<0\end{aligned}$$

Therefore, a perturbation of κ_{λ} below $\overline{\kappa}_{\lambda}$ leads to $b>0$, which requires $\eta_b=0$, confirming $d\eta_b=0$ when $\nu=\overline{\nu}_0$. Moreover, $\alpha<\widehat{\lambda}$ and $\alpha+x=\widehat{\lambda}+\widehat{x}$, which implies $\beta=\widehat{h}$. The case of $d\eta_b\neq 0$ can be ruled out as follows. If $\left.\frac{d\eta_b}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}\neq 0$, then $\left.\frac{db}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}=0$ and hence

$$\left.\frac{d\eta_b}{d\kappa_{\lambda}}\right|_{\kappa_{\lambda}=\overline{\kappa}_{\lambda}}=\frac{\frac{f'\left(1-\widehat{\lambda}\right)}{-\left(1-\overline{\kappa}_{\lambda}\right)f''\left(1-\widehat{\lambda}\right)}\frac{p\phi''\left(\widehat{h}\right)}{\theta}\left(\frac{g\left(\widehat{x}\right)}{\widehat{x}}\varepsilon_g'\left(\widehat{x}\right)+\frac{(1-p)\left(g'\left(\widehat{x}\right)-\frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)^2}{g\left(\widehat{x}\right)\left(p\frac{g\left(\widehat{x}\right)}{\widehat{x}}+1-p\right)}\right)}{g''\left(\widehat{x}\right)+\frac{\phi''\left(\widehat{h}\right)}{\theta}\left(1+\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)^2+\frac{-f''\left(1-\widehat{\lambda}\right)}{p}\left(\widetilde{K}_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_{\lambda}\right)\right)^2}>0$$

But then a perturbation of κ_{λ} below $\overline{\kappa}_{\lambda}$ leads to $\eta_b<0$ which is impossible since $\eta_b\geq 0$ is a Lagrange multiplier.

Now assume the planner cannot commit. Without commitment, the choice of b is governed by

$$\frac{\eta_b}{p} = \nu - g'(x-b) \tag{A.44}$$

while the choices of α and x are still governed by the same equations as with commitment.

In the first case where (A.38) is slack and (A.39) is binding, the planner's solution is a triple $\{\alpha, x, b\}$ solving Eqs. (A.40), (A.44), and (A.39) with equality. Once again, $\left\{\widehat{\lambda}, \widehat{x}, 0\right\}$ satisfies these equations if $\kappa_h = \overline{\kappa}_h$ and $\nu = \overline{\nu}_0$. Differentiate to get $\frac{d\alpha}{d\kappa_h}$, $\frac{dx}{d\kappa_h}$, and $\frac{db}{d\kappa_h}$ then evaluate at $\kappa_h = \overline{\kappa}_h$ to get

$$g''(\widehat{x})(dx-db) = \frac{\phi''\left(\widehat{h}\right)}{\theta} K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)(d\alpha + dx) - \frac{f''\left(1-\widehat{\lambda}\right)}{p}\left(1 + K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)\right) d\alpha$$

$$\frac{g(\widehat{x})}{\widehat{x}} d\kappa_h + \frac{1+\overline{\kappa}_h}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right)(dx-db) = -\frac{\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx)$$

$$d\eta_b = -pg''(\widehat{x})(dx-db)$$

If $d\eta_b \neq 0$, then $db = 0$ and hence

$$\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{\frac{g(\widehat{x})}{\widehat{x}}}{\frac{1+\overline{\kappa}_h}{\widehat{x}}\left(g'(\widehat{x}) - \frac{g(\widehat{x})}{\widehat{x}}\right) + \frac{g''(\widehat{x}) - \frac{f''(1-\widehat{\lambda})}{p}\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)}{K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right) - \frac{f''(1-\widehat{\lambda})}{p}\left(1+K_b\left(\widehat{\lambda},\widehat{x},0;\overline{\kappa}_h\right)\right)\frac{\theta}{\phi''(\widehat{h})}}} < 0$$

$$\left.\frac{d\eta_b}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -pg''(\widehat{x})\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} > 0$$

But then a perturbation of κ_h below $\overline{\kappa}_h$ leads to $\eta_b < 0$ which is impossible since $\eta_b \geq 0$ is a Lagrange multiplier. Therefore, $d\eta_b = 0$, which implies

$$\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{p}{f''\left(1-\widehat{\lambda}\right)}\frac{g(\widehat{x})}{\widehat{x}}\frac{K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)}{1+K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)} > 0$$

$$\left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\left(1 - \frac{f''\left(1-\widehat{\lambda}\right)}{p}\frac{\theta}{\phi''\left(\widehat{h}\right)}\frac{1+K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)}{K_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_h\right)}\right)\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} < 0$$

$$\left.\frac{db}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = \left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} < 0$$

$$\left.\frac{d\beta}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} = -\frac{1}{\theta}\left(\left.\frac{d\alpha}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h} + \left.\frac{dx}{d\kappa_h}\right|_{\kappa_h=\overline{\kappa}_h}\right) = \frac{1}{\phi''\left(\widehat{h}\right)}\frac{g(\widehat{x})}{\widehat{x}} > 0$$

A perturbation of κ_h below $\overline{\kappa}_h$ still leads to $b > 0$, which requires $\eta_b = 0$, confirming $d\eta_b = 0$ when

$\nu = \overline{\nu}_0$. However, now $\alpha < \widehat{\lambda}$ and $\beta < \widehat{h}$.

In the second case where (A.39) is slack and (A.38) is binding, the planner's solution is a triple $\{\alpha, x, b\}$ solving Eqs. (A.42), (A.44), and (A.38) with equality. Once again, $\left\{\widehat{\lambda}, \widehat{x}, 0\right\}$ satisfies these equations if $\kappa_\lambda = \overline{\kappa}_\lambda$ and $\nu = \overline{\nu}_0$. Differentiate to get $\frac{d\alpha}{d\kappa_\lambda}$, $\frac{dx}{d\kappa_\lambda}$, and $\frac{db}{d\kappa_\lambda}$ then evaluate at $\kappa_\lambda = \overline{\kappa}_\lambda$ to get

$$g''\left(\widehat{x}\right)(dx - db) = -\frac{\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx) - \frac{1}{p}\left(\frac{p\phi''\left(\widehat{h}\right)}{\theta}(d\alpha + dx) - f''\left(1 - \widehat{\lambda}\right)d\alpha\right)\widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)$$

$$\frac{p}{\widehat{x}}\left(g'\left(\widehat{x}\right) - \frac{g\left(\widehat{x}\right)}{\widehat{x}}\right)(dx - db) = -f'\left(1 - \widehat{\lambda}\right)d\kappa_\lambda - \left(1 - \overline{\kappa}_\lambda\right)f''\left(1 - \widehat{\lambda}\right)d\alpha$$

$$d\eta_b = -pg''\left(\widehat{x}\right)(dx - db)$$

If $d\eta_b \neq 0$, then $db = 0$ and hence

$$\left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = -\frac{f'\left(1 - \widehat{\lambda}\right)}{\frac{p}{\widehat{x}}\left(g'\left(\widehat{x}\right) - \frac{g(\widehat{x})}{\widehat{x}}\right) - \frac{(1-\overline{\kappa}_\lambda)f''\left(1-\widehat{\lambda}\right)\left(g''(\widehat{x}) + \frac{\phi''(\widehat{h})}{\theta}\left(1 + \widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)\right)\right)}{\frac{\phi''(\widehat{h})}{\theta}\left(1 + \widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)\right) - \frac{f''(1-\widehat{\lambda})}{p}\widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)}} < 0$$

$$\left.\frac{d\eta_b}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = -pg''\left(\widehat{x}\right)\left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} > 0$$

But then a perturbation of κ_λ below $\overline{\kappa}_\lambda$ leads to $\eta_b < 0$ which is impossible since $\eta_b \geq 0$ is a Lagrange multiplier. Therefore, $d\eta_b = 0$, which implies

$$\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = \frac{f'\left(1 - \widehat{\lambda}\right)}{-\left(1 - \overline{\kappa}_\lambda\right)f''\left(1 - \widehat{\lambda}\right)} > 0$$

$$\left.\frac{db}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = \left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = -\left(1 - \frac{f''\left(1 - \widehat{\lambda}\right)}{p}\frac{\theta}{\phi''\left(\widehat{h}\right)}\frac{\widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)}{1 + \widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)}\right)\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} < 0$$

$$\left.\frac{d\beta}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = -\frac{1}{\theta}\left(\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} + \left.\frac{dx}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda}\right) = -\frac{f''\left(1 - \widehat{\lambda}\right)}{p}\frac{1}{\phi''\left(\widehat{h}\right)}\frac{\widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)}{1 + \widetilde{K}_b\left(\widehat{\lambda}, \widehat{x}, 0; \overline{\kappa}_\lambda\right)}\left.\frac{d\alpha}{d\kappa_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} > 0$$

A perturbation of κ_λ below $\overline{\kappa}_\lambda$ still leads to $b > 0$, which requires $\eta_b = 0$, confirming $d\eta_b = 0$ when $\nu = \overline{\nu}_0$. However, now $\alpha < \widehat{\lambda}$ and $\beta < \widehat{h}$. ■

Proof of Lemma 8

The planner implements $\eta_{\omega_\lambda} \geq 0$ and $\eta_{\omega_h} \geq 0$ in Eqs. (A.6) to (A.9) with $\omega_\lambda = \omega_h = 0$. The only change to the expressions for $\overline{\kappa}_\lambda$ and $\overline{\kappa}_h$ in the proof of Proposition 5 is that q is now given by

$$q = \frac{\theta(1-h) - \lambda - b}{g(\theta(1-h) - \lambda - b)}$$

Therefore,

$$\widetilde{\overline{\kappa}}_\lambda \equiv 1 - \frac{p\frac{g(\widetilde{x})}{\widetilde{x}} + 1 - p}{pg'(\widetilde{x}) + 1 - p}$$

and

$$\widetilde{\overline{\kappa}}_h \equiv \varepsilon_g(\widetilde{x}) - 1$$

where $\widetilde{x} \equiv \theta(1-\beta) - \alpha - b$ with $\{\alpha, \beta, b\}$ as characterized in the proof of Proposition 8. If $\nu \leq \overline{\nu}_0$, then $\widetilde{x}$ solves $g'(\widetilde{x}) = \nu$, which implies $\frac{d\widetilde{x}}{d\nu} > 0$. Combined with

$$\frac{d\widetilde{\overline{\kappa}}_\lambda}{d\widetilde{x}} = \frac{\frac{p}{\widetilde{x}}\left(\varepsilon_g'(\widetilde{x})\, g(\widetilde{x})\left(p\frac{g(\widetilde{x})}{\widetilde{x}} + 1 - p\right) + (1-p)(\varepsilon_g(\widetilde{x}) - 1)\left(g'(\widetilde{x}) - \frac{g(\widetilde{x})}{\widetilde{x}}\right)\right)}{(pg'(\widetilde{x}) + 1 - p)^2} > 0$$

and

$$\frac{d\widetilde{\overline{\kappa}}_h}{d\widetilde{x}} = \varepsilon_g'(\widetilde{x}) > 0$$

this means that $\frac{d\widetilde{\overline{\kappa}}_\lambda}{d\nu} > 0$ and $\frac{d\widetilde{\overline{\kappa}}_h}{d\nu} > 0$. ■

Proof of Lemma 9

Away from the optimal regulation, shadow activities may be triggered. In the presence of bailouts, the bank's objective function is a slightly modified version of $\widetilde{\Pi}(\cdot)$ in Section 3.1, namely

$$\begin{aligned}\widetilde{\Pi}_b(\cdot) &= (1-p)\left[\lambda + f(1 - \lambda - \kappa_\lambda\omega_\lambda\lambda) - 1\right] \\ &\quad + p\left[f(1 - \lambda - \kappa_\lambda\omega_\lambda\lambda) - \frac{\theta(1 - h + \kappa_h\omega_h h) - \lambda - b}{q} - (1-\theta) - \theta h\right] - p\theta\phi(h)\end{aligned}$$

which takes into account the effect of the bailout b on the cash shortfall that must be covered via project sales. The bank's first order conditions are as in the proof of Proposition 4, namely Eqs. (A.6) to (A.9), but with the price q given by

$$q = \frac{\theta(1 - h + \kappa_h\omega_h h) - \lambda - b}{g(\theta(1 - h + \kappa_h\omega_h h) - \lambda - b)}$$

instead of Eq. (10).

Social welfare is $\widetilde{\Pi}_b(\cdot) - p\nu b$, the endogenous component of which can be expressed as

$$\mathcal{W} \equiv (1-p)\lambda + f(1 - \lambda - \kappa_\lambda\omega_\lambda\lambda) - pg(\theta(1 - h + \kappa_h\omega_h h) - \lambda - b) - p\theta(h + \phi(h)) - p\nu b$$

after subbing in for q, simplifying, and dropping constant terms.

Recall from the proof of Proposition 8 that a planner operating in the absence of shadow activities chooses $\{\alpha, \beta, b\}$ solving

$$f'(1-\alpha) = 1 + p\phi'(\beta) \tag{A.45}$$

$$g'(\theta(1-\beta) - \alpha - b) = 1 + \phi'(\beta) \tag{A.46}$$

$$1 + \phi'(\beta) = \nu \tag{A.47}$$

if $\nu < \overline{\nu}_0$. If instead $\nu \geq \overline{\nu}_0$, then the planner chooses $\{\alpha, \beta, b\}$ solving Eqs. (A.45) and (A.46) with $b = 0$, which returns $\{\alpha, \beta, b\} = \left\{\widehat{\lambda}, \widehat{h}, 0\right\}$.

The first case to consider is $\omega_\lambda > 0$ and $\omega_h = 0$. Eqs. (A.10) and (A.12) apply along with the complementary slackness condition in (A.15). If $h > \beta$, then (A.15) implies $1 + \phi'(h) = \frac{1}{q}$, which when combined with (A.10), gives

$$(1-\kappa_\lambda) f'(1-\lambda-\kappa_\lambda(\alpha-\lambda)) = 1 + p\phi'(h) > 1 + p\phi'(\beta) = f'(1-\alpha)$$

but this requires $1-\lambda-\kappa_\lambda(\alpha-\lambda) < 1-\alpha$, or equivalently $(1-\kappa_\lambda)(\alpha-\lambda) < 0$, which is false. Therefore, $h = \beta$. Welfare is then

$$\mathcal{W}_1 = (1-p)\frac{\alpha}{1+\omega_\lambda} + f\left(1 - \frac{1+\kappa_\lambda\omega_\lambda}{1+\omega_\lambda}\alpha\right) - pg\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right) - p\theta(\beta+\phi(\beta)) - p\nu b$$

where ω_λ solves

$$p\frac{g\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right)}{\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b} = (1-\kappa_\lambda) f'\left(1 - \frac{1+\kappa_\lambda\omega_\lambda}{1+\omega_\lambda}\alpha\right) - (1-p)$$

The partial derivative is

$$\begin{aligned}
\frac{\partial \mathcal{W}_1}{\partial \omega_\lambda} &= \left[(1-\kappa_\lambda) f'\left(1 - \frac{1+\kappa_\lambda\omega_\lambda}{1+\omega_\lambda}\alpha\right) - (1-p) - pg'\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right)\right]\frac{\alpha}{(1+\omega_\lambda)^2} \\
&= -p\left(g'\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right) - \frac{g\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right)}{\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b}\right)\frac{\alpha}{(1+\omega_\lambda)^2}
\end{aligned}$$

where the second line follows from the solution for ω_λ. Evaluating at $\kappa_\lambda = \overline{\kappa}_\lambda$ where $\omega_\lambda = 0$,

$$\left.\frac{\partial \mathcal{W}_1}{\partial \omega_\lambda}\right|_{\kappa_\lambda = \overline{\kappa}_\lambda} = -p\left(g'(\theta(1-\beta) - \alpha - b) - \frac{g(\theta(1-\beta) - \alpha - b)}{\theta(1-\beta) - \alpha - b}\right)\alpha < 0$$

The second case to consider is $\omega_\lambda = 0$ and $\omega_h > 0$. Eqs. (A.11) and (A.13) apply along with the complementary slackness condition in (A.16). If $\lambda > \alpha$, then (A.16) implies $f'(1-\lambda) - (1-p) = \frac{p}{q}$, which when combined with (A.11), gives

$$1+p\phi'(h)=f'(1-\lambda)+\frac{\kappa_h}{1+\kappa_h}p\left(1+\phi'(h)\right)>f'(1-\lambda)>f'(1-\alpha)=1+p\phi'(\beta)$$

but this requires $h>\beta$, which is false. Therefore, $\lambda=\alpha$. Welfare is then

$$\mathcal{W}_2=(1-p)\,\alpha+f(1-\alpha)-pg\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)-p\theta\left(\frac{\beta}{1+\omega_h}+\phi\left(\frac{\beta}{1+\omega_h}\right)\right)-p\nu b$$

where ω_h solves

$$(1+\kappa_h)\frac{g\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)}{\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b}=1+\phi'\left(\frac{\beta}{1+\omega_h}\right)$$

The partial derivative is

$$\begin{aligned}\frac{\partial\mathcal{W}_2}{\partial\omega_h}&=-p\theta\left[(1+\kappa_h)\,g'\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)-\left(1+\phi'\left(\frac{\beta}{1+\omega_h}\right)\right)\right]\frac{\beta}{(1+\omega_h)^2}\\&=-p\theta\,(1+\kappa_h)\left(g'\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)-\frac{g\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)}{\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b}\right)\frac{\beta}{(1+\omega_h)^2}\end{aligned}$$

where the second line follows from the solution for ω_h. Evaluating at $\kappa_h=\overline{\kappa}_h$ where $\omega_h=0$,

$$\left.\frac{\partial\mathcal{W}_2}{\partial\omega_h}\right|_{\kappa_h=\overline{\kappa}_h}=-p\theta\,(1+\overline{\kappa}_h)\left(g'(\theta(1-\beta)-\alpha-b)-\frac{g(\theta(1-\beta)-\alpha-b)}{\theta(1-\beta)-\alpha-b}\right)\beta<0$$

Therefore,

$$\left.\frac{\partial\mathcal{W}_2}{\partial\omega_h}\right|_{\kappa_h=\overline{\kappa}_h}=(1+\overline{\kappa}_h)\frac{\theta\beta}{\alpha}\left.\frac{\partial\mathcal{W}_1}{\partial\omega_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda}$$

Since both partial derivatives are negative, $\left.\frac{\partial\mathcal{W}_2}{\partial\omega_h}\right|_{\kappa_h=\overline{\kappa}_h}<\left.\frac{\partial\mathcal{W}_1}{\partial\omega_\lambda}\right|_{\kappa_\lambda=\overline{\kappa}_\lambda}$ if and only if $(1+\overline{\kappa}_h)\frac{\theta\beta}{\alpha}>1$. Recall from Lemma 3 that $\frac{d}{dp}\left(\frac{\widehat{h}}{\widehat{\lambda}}\right)<0$, which applies for $\nu\geq\overline{\nu}_0$. If instead $\nu<\overline{\nu}_0$, then differentiate Eqs. (A.45), (A.46), and (A.47) to also find $\frac{d}{dp}\left(\frac{\beta}{\alpha}\right)<0$. As $p\to 0$, Eq. (A.45) implies $\alpha\to 0$. If $\nu\geq\overline{\nu}_0$ so that $b=0$, then $\beta>0$ is implied from Eq. (A.46) by $g'(\theta)>1$ and the assumption of $\phi'(0)=0$. If $\nu<\overline{\nu}_0$ so that $b>0$, then $\beta>0$ is implied from Eq. (A.47) by $\nu>1$ and the assumption of $\phi'(0)=0$. Accordingly, $(1+\overline{\kappa}_h)\frac{\theta\beta}{\alpha}>1$ is satisfied for low p. ■

Proof of Proposition 10

The Lagrangian of a planner who takes into account the threat of both shadow activities (informed planner) is

$$\begin{aligned}\mathcal{L} &= \alpha + f(1-\alpha) - pg(x-b) + px - p\theta\phi\left(1-\frac{\alpha+x}{\theta}\right) - p\nu b + \eta_b b \\ &\quad +\psi_\alpha\left[p\frac{g(x-b)}{x-b} - (1-\kappa_\lambda) f'(1-\alpha) + (1-p)\right] + \psi_\beta\left[(1+\kappa_h)\frac{g(x-b)}{x-b} - 1 - \phi'\left(1-\frac{\alpha+x}{\theta}\right)\right]\end{aligned}$$

If the planner can commit to a bailout level at $t=0$, then the first order condition for b is given by

$$\frac{\eta_b}{p} = \nu - g'(x-b) + \left(\psi_\alpha + \psi_\beta\frac{1+\kappa_h}{p}\right)\frac{g'(x-b) - \frac{g(x-b)}{x-b}}{x-b} \tag{A.48}$$

Otherwise, it is given by Eq. (A.44).

Consider $\kappa_\lambda = \kappa_h = 0$, in which case regulation is ineffective, i.e., $\lambda = \lambda^*$ and $h = h^*$ for any α and β. Without commitment, Eq. (A.44) implies that the planner chooses $b > 0$ as ν is perturbed below $g'(x^*)$. With commitment, Eq. (A.48) implies that the planner chooses $b = 0$ as ν is perturbed below $g'(x^*)$. The case with commitment therefore recovers the decentralized equilibrium with no regulation and no bailout in Section 2.2. Since the planner cannot do worse with commitment than without, it follows that the decentralized equilibrium achieves higher welfare than the case without commitment.

To prove the second part of the proposition, it will suffice to show that an informed planner achieves strictly higher welfare than a naive planner as κ_λ or κ_h increases from zero. That an informed planner cannot achieve strictly lower welfare than a naive planner follows immediately from the fact that any policy chosen by the naive planner is available to the informed planner, so we only need to rule out that these two planners achieve the same welfare as κ_λ or κ_h is perturbed above zero. This is straightforward, but the formal proof follows.

Assume no commitment and $\nu \le g'(x^*)$. Welfare achieved by a naive planner who triggers both shadow activities is

$$\begin{aligned}\mathcal{W}_n &\equiv (1-p)\frac{\alpha}{1+\omega_\lambda} + f\left(1-\frac{(1+\kappa_\lambda\omega_\lambda)\alpha}{1+\omega_\lambda}\right) - pg\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right) - \frac{\alpha}{1+\omega_\lambda} - b\right) \\ &\quad -p\theta\left(\frac{\beta}{1+\omega_h} + \phi\left(\frac{\beta}{1+\omega_h}\right)\right) - p\nu b\end{aligned}$$

where

$$\begin{aligned}\frac{d\mathcal{W}_n}{d\kappa_\lambda} &= \begin{pmatrix} (1-\kappa_\lambda) f'\left(1-\frac{(1+\kappa_\lambda\omega_\lambda)\alpha}{1+\omega_\lambda}\right) - (1-p) \\ -pg'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right) - \frac{\alpha}{1+\omega_\lambda} - b\right)\end{pmatrix}\frac{\alpha}{(1+\omega_\lambda)^2}\frac{d\omega_\lambda}{d\kappa_\lambda} \\ &\quad +p\theta\begin{pmatrix} 1+\phi'\left(\frac{\beta}{1+\omega_h}\right) \\ -(1+\kappa_h) g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right) - \frac{\alpha}{1+\omega_\lambda} - b\right)\end{pmatrix}\frac{\beta}{(1+\omega_h)^2}\frac{d\omega_h}{d\kappa_\lambda} \\ &\quad +p\left(g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right) - \frac{\alpha}{1+\omega_\lambda} - b\right) - \nu\right)\frac{db}{d\kappa_\lambda} - f'\left(1-\frac{(1+\kappa_\lambda\omega_\lambda)\alpha}{1+\omega_\lambda}\right)\frac{\alpha\omega_\lambda}{1+\omega_\lambda}\end{aligned}$$

and

$$\begin{aligned}\frac{d\mathcal{W}_n}{d\kappa_h} &= \begin{pmatrix} (1-\kappa_\lambda) f'\left(1-\frac{(1+\kappa_\lambda\omega_\lambda)\alpha}{1+\omega_\lambda}\right)-(1-p) \\ -pg'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\end{pmatrix}\frac{\alpha}{(1+\omega_\lambda)^2}\frac{d\omega_\lambda}{d\kappa_h} \\ &\quad +p\theta\begin{pmatrix} 1+\phi'\left(\frac{\beta}{1+\omega_h}\right) \\ -(1+\kappa_h) g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\end{pmatrix}\frac{\beta}{(1+\omega_h)^2}\frac{d\omega_h}{d\kappa_h} \\ &\quad +p\left(g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)-\nu\right)\frac{db}{d\kappa_h} \\ &\quad -p\theta g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\frac{\beta\omega_h}{1+\omega_h}\end{aligned}$$

Use the bank first order conditions in Eqs. (A.10) to (A.13) and the relevant expression for q in the proof of Lemma 9 to reduce these derivatives to

$$\begin{aligned}\frac{d\mathcal{W}_n}{d\kappa_\lambda} &= p\begin{pmatrix} \frac{g\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)}{\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b} \\ -g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\end{pmatrix}\left(\frac{\alpha}{(1+\omega_\lambda)^2}\frac{d\omega_\lambda}{d\kappa_\lambda}+\frac{\theta(1+\kappa_h)\beta}{(1+\omega_h)^2}\frac{d\omega_h}{d\kappa_\lambda}\right) \\ &\quad +p\left(g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)-\nu\right)\frac{db}{d\kappa_\lambda} \\ &\quad -f'\left(1-\frac{(1+\kappa_\lambda\omega_\lambda)\alpha}{1+\omega_\lambda}\right)\frac{\alpha\omega_\lambda}{1+\omega_\lambda}\end{aligned}$$

and

$$\begin{aligned}\frac{d\mathcal{W}_n}{d\kappa_h} &= p\begin{pmatrix} \frac{g\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)}{\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b} \\ -g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\end{pmatrix}\left(\frac{\alpha}{(1+\omega_\lambda)^2}\frac{d\omega_\lambda}{d\kappa_h}+\frac{\theta(1+\kappa_h)\beta}{(1+\omega_h)^2}\frac{d\omega_h}{d\kappa_h}\right) \\ &\quad +p\left(g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)-\nu\right)\frac{db}{d\kappa_h} \\ &\quad -p\theta g'\left(\theta\left(1-\frac{(1-\kappa_h\omega_h)\beta}{1+\omega_h}\right)-\frac{\alpha}{1+\omega_\lambda}-b\right)\frac{\beta\omega_h}{1+\omega_h}\end{aligned}$$

For $\kappa_\lambda=\kappa_h=0$,

$$\begin{aligned}\left.\frac{d\mathcal{W}_n}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0} &= -p\left(g'(x^*-b)-\frac{g(x^*-b)}{x^*-b}\right)\left(\frac{(\lambda^*)^2}{\widehat{\lambda}}\left.\frac{d\omega_\lambda}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}+\theta\frac{(h^*)^2}{\widehat{h}}\left.\frac{d\omega_h}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}\right) \\ &\quad -f'(1-\lambda^*)\left(\widehat{\lambda}-\lambda^*\right)\end{aligned}$$

and

$$\left.\frac{d\mathcal{W}_n}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0} = -p\left(g'\left(x^*-b\right)-\frac{g\left(x^*-b\right)}{x^*-b}\right)\left(\frac{\left(\lambda^*\right)^2}{\widehat{\lambda}}\left.\frac{d\omega_\lambda}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}+\theta\frac{\left(h^*\right)^2}{\widehat{h}}\left.\frac{d\omega_h}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}\right)$$
$$-p\theta g'\left(x^*-b\right)\left(\widehat{h}-h^*\right)$$

where we have used the fact that the naive planner sets $\{\alpha,\beta\}=\left\{\widehat{\lambda},\widehat{h}\right\}$, banks choose $\{\lambda,h\}=\{\lambda^*,h^*\}$, and b is then pinned down by $g'\left(x^*-b\right)=\nu$. As κ_λ or κ_h is perturbed above zero, b is pinned down by

$$g'\left(\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b\right)=\nu$$

so

$$\frac{d\left(\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b\right)}{d\kappa_\lambda}=\frac{d\left(\theta\left(1-h+\kappa_h\omega_h h\right)-\lambda-b\right)}{d\kappa_h}=0$$

and therefore

$$\frac{dq}{d\kappa_\lambda}=\frac{dq}{d\kappa_h}=0$$

Differentiating Eqs. (A.10) to (A.13) and evaluating at $\kappa_\lambda=\kappa_h=0$ then yields

$$f'\left(1-\lambda^*\right)=-f''\left(1-\lambda^*\right)\left(\widehat{\lambda}-\lambda^*-\frac{\left(\lambda^*\right)^2}{\widehat{\lambda}}\left.\frac{d\omega_\lambda}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}\right)$$

$$\left.\frac{d\omega_h}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}=0$$

and

$$\left.\frac{d\omega_\lambda}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}=0$$

$$\frac{g\left(x^*-b\right)}{x^*-b}=-\phi''\left(h^*\right)\frac{\left(h^*\right)^2}{\widehat{h}}\left.\frac{d\omega_h}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}$$

Accordingly,

$$\left.\frac{d\mathcal{W}_n}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}=-p\left(g'\left(x^*-b\right)-\frac{g\left(x^*-b\right)}{x^*-b}\right)\left(\widehat{\lambda}-\lambda^*+\frac{f'\left(1-\lambda^*\right)}{f''\left(1-\lambda^*\right)}\right)-f'\left(1-\lambda^*\right)\left(\widehat{\lambda}-\lambda^*\right)$$

and

$$\left.\frac{d\mathcal{W}_n}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}=p\theta\left(g'\left(x^*-b\right)-\frac{g\left(x^*-b\right)}{x^*-b}\right)\frac{g\left(x^*-b\right)}{x^*-b}\frac{1}{\phi''\left(h^*\right)}-p\theta g'\left(x^*-b\right)\left(\widehat{h}-h^*\right)$$

Now consider the welfare achieved by an informed planner around $\kappa_\lambda=\kappa_h=0$,

$$\mathcal{W}_i\equiv\alpha+f\left(1-\alpha\right)-pg\left(x-b\right)+px-p\theta\phi\left(1-\frac{\alpha+x}{\theta}\right)-p\nu b$$

where α and x solve the implementation constraints (A.38) and (A.39) with equality and b is pinned down by $g'(x-b)=\nu$. Notice that changes in x must be exactly offset by changes in b so that $g'(x-b)=\nu$ continues to hold. We can then see from (A.38) binding that α cannot change with κ_h and from (A.39) binding that $\alpha+x$ cannot change with κ_λ, hence

$$\frac{d\mathcal{W}_i}{d\kappa_\lambda}=\left(1-p-f'(1-\alpha)+p\nu\right)\frac{f'(1-\alpha)}{-(1-\kappa_\lambda)f''(1-\alpha)}$$

and

$$\frac{d\mathcal{W}_i}{d\kappa_h}=-p\theta\left(1+\phi'\left(1-\frac{\alpha+x}{\theta}\right)-\nu\right)\frac{g(x-b)}{x-b}\frac{1}{\phi''\left(1-\frac{\alpha+x}{\theta}\right)}$$

Evaluating at $\kappa_\lambda=\kappa_h=0$ and using $g'(x^*-b)=\nu$ to substitute out for ν gives

$$\left.\frac{d\mathcal{W}_i}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}=\left(1-p-f'(1-\lambda^*)+pg'(x^*-b)\right)\frac{f'(1-\lambda^*)}{-f''(1-\lambda^*)}$$

and

$$\left.\frac{d\mathcal{W}_i}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}=-p\theta\left(1+\phi'(h^*)-g'(x^*-b)\right)\frac{g(x^*-b)}{x^*-b}\frac{1}{\phi''(h^*)}$$

Finally, use Eqs. (5) and (6) to conclude

$$\left.\frac{d\mathcal{W}_i}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}=p\left(g'(x^*-b)-\frac{g(x^*-b)}{x^*-b}\right)\frac{f'(1-\lambda^*)}{-f''(1-\lambda^*)}>\left.\frac{d\mathcal{W}_n}{d\kappa_\lambda}\right|_{\kappa_\lambda=\kappa_h=0}$$

and

$$\left.\frac{d\mathcal{W}_i}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}=p\theta\left(g'(x^*-b)-\frac{g(x^*-b)}{x^*-b}\right)\frac{g(x^*-b)}{x^*-b}\frac{1}{\phi''(h^*)}>\left.\frac{d\mathcal{W}_n}{d\kappa_h}\right|_{\kappa_\lambda=\kappa_h=0}$$

which completes the proof. ■

Proof of Proposition 11

We derive expressions for any $\nu\in[g'(0),\overline{\nu}_0]$ before evaluating the solution at $\nu=\overline{\nu}_0$.

Start with $\kappa_\lambda\to\infty$ and $\kappa_h\sim U(0,\kappa_h^{\max})$. For a given κ_h, recall from $\mathcal{W}_2$ in the proof of Lemma 9 that welfare when $\omega_\lambda=0$ and $\omega_h\geq 0$ is

$$(1-p)\alpha+f(1-\alpha)-pg\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)-p\theta\left(\frac{\beta}{1+\omega_h}+\phi\left(\frac{\beta}{1+\omega_h}\right)\right)-p\nu b$$

Define the threshold

$$\overline{\kappa}_h(\alpha,\beta,b)\equiv\frac{\theta(1-\beta)-\alpha-b}{g(\theta(1-\beta)-\alpha-b)}\left(1+\phi'(\beta)\right)-1 \tag{A.49}$$

If $\kappa_h < \overline{\kappa}_h(\alpha, \beta, b)$, then $\omega_h(\kappa_h) > 0$ solves

$$(1+\kappa_h)\frac{g\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right)}{\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b} = 1+\phi'\left(\frac{\beta}{1+\omega_h}\right) \tag{A.50}$$

If instead $\kappa_h \geq \overline{\kappa}_h(\alpha, \beta, b)$, then $\omega_h = 0$.

Without commitment, the planner chooses b at $t = 1$ after discovering the value of κ_h. If

$$g'\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha\right) \geq \nu \tag{A.51}$$

then $b(\kappa_h) \geq 0$ solves

$$g'\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b\right) = \nu \tag{A.52}$$

That condition (A.51) is true for all $\kappa_h \in [0, \kappa_h^{\max}]$ follows from $\nu \leq \overline{\nu}_0$, i.e.,

$$g'\left(\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha\right) \geq g'(\theta(1-\beta)-\alpha) \geq g'\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right) \equiv \overline{\nu}_0$$

That Eq. (A.52) holds with strict equality for all $\kappa_h \in [0, \kappa_h^{\max}]$ follows from $\nu \geq g'(0)$.

The planner chooses α and β at $t = 0$ to maximize expected welfare, which is given by

$$\frac{1}{\kappa_h^{\max}}\int_0^{\kappa_h^{\max}}\left(\begin{array}{c}(1-p)\alpha+f(1-\alpha)-pg\left(\theta\left(1-\frac{1-\kappa_h\omega_h(\kappa_h)}{1+\omega_h(\kappa_h)}\beta\right)-\alpha-b(\kappa_h)\right)\\ -p\theta\left(\frac{\beta}{1+\omega_h(\kappa_h)}+\phi\left(\frac{\beta}{1+\omega_h(\kappa_h)}\right)\right)-p\nu b(\kappa_h)\end{array}\right)d\kappa_h$$

or equivalently

$$\begin{aligned}&(1-p)\alpha+f(1-\alpha)-p\left[g(\theta(1-\beta)-\alpha-b)+\theta(\beta+\phi(\beta))+\nu b\right]\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)\\ &-\frac{p}{\kappa_h^{\max}}\int_0^{\overline{\kappa}_h(\alpha,\beta,b)}\left(\begin{array}{c}g\left(\theta\left(1-\frac{1-\kappa_h\omega_h(\kappa_h)}{1+\omega_h(\kappa_h)}\beta\right)-\alpha-b(\kappa_h)\right)\\ +\theta\left(\frac{\beta}{1+\omega_h(\kappa_h)}+\phi\left(\frac{\beta}{1+\omega_h(\kappa_h)}\right)\right)+\nu b(\kappa_h)\end{array}\right)d\kappa_h\end{aligned}$$

where b denotes the bailout if $\kappa_h \geq \overline{\kappa}_h(\alpha, \beta, b)$, i.e., for any κ_h such that $\omega_h = 0$.

By the envelope condition, the effect of α and β through b and $b(\kappa_h)$ can be ignored. Also note from Eq. (A.52) that $g'(\cdot) = \nu$ for every κ_h. The first order condition for α is then

$$\begin{aligned}\frac{\partial}{\partial\alpha} &= 1-p+p\nu-f'(1-\alpha)\\ &-\frac{p\theta}{\kappa_h^{\max}}\int_0^{\overline{\kappa}_h(\alpha,\beta,b)}\left(\nu(1+\kappa_h)-1-\phi'\left(\frac{\beta}{1+\omega_h(\kappa_h)}\right)\right)\frac{\beta}{(1+\omega_h(\kappa_h))^2}\frac{d\omega_h(\kappa_h)}{d\alpha}d\kappa_h\end{aligned}$$

Eqs. (A.50) and (A.52) imply $\frac{d\omega_h(\kappa_h)}{d\alpha} = 0$, i.e., the planner understands that a change in α will change ω_h in Eq. (A.50) for a given b and that any net change will then change b in Eq. (A.52), etc., but in the end $\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b$ will not change in Eq. (A.52). Accordingly, $\frac{\beta}{1+\omega_h}$ cannot change in Eq. (A.50), which means $\frac{d\omega_h(\kappa_h)}{d\alpha}=0$. Therefore,

$$\frac{\partial}{\partial\alpha}=1-p+p\nu-f'(1-\alpha)$$

which means that the choice of α is unaffected by uncertainty in κ_h relative to the case where there are no shadow activities and the planner uses the bailout instrument. If $\nu=\overline{\nu}_0$, then $\frac{\partial}{\partial\alpha}=0$ delivers $\alpha=\widehat{\lambda}$.

Using again the envelope condition and $g'(\cdot)=\nu$, the first order condition for β is

$$\begin{aligned}\frac{\partial}{\partial\beta} &= p\theta\left(\nu-1-\phi'(\beta)\right)\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)\\ &-\frac{p\theta}{\kappa_h^{\max}}\int_0^{\overline{\kappa}_h(\alpha,\beta,b)}\left(\begin{array}{c}\left(\nu(1+\kappa_h)-1-\phi'\left(\frac{\beta}{1+\omega_h(\kappa_h)}\right)\right)\frac{\beta}{(1+\omega_h(\kappa_h))^2}\frac{d\omega_h(\kappa_h)}{d\beta}\\ +\frac{1+\phi'\left(\frac{\beta}{1+\omega_h(\kappa_h)}\right)-\nu(1-\kappa_h\omega_h(\kappa_h))}{1+\omega_h(\kappa_h)}\end{array}\right)d\kappa_h\end{aligned}$$

Again, any change in β cannot lead to a change in $\theta\left(1-\frac{1-\kappa_h\omega_h}{1+\omega_h}\beta\right)-\alpha-b$ after all the feedback between ω_h and b is taken into account. Accordingly, $\frac{\beta}{1+\omega_h}$ cannot change, which means $\frac{d\omega_h(\kappa_h)}{d\beta}=\frac{1+\omega_h(\kappa_h)}{\beta}$. Therefore,

$$\frac{\partial}{\partial\beta}=p\theta\left(\nu-1-\phi'(\beta)\right)\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)-\frac{p\theta\nu}{\kappa_h^{\max}}\int_0^{\overline{\kappa}_h(\alpha,\beta,b)}\kappa_h d\kappa_h$$

or equivalently

$$\frac{\partial}{\partial\beta}=p\theta\left(\nu\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}-\frac{(\overline{\kappa}_h(\alpha,\beta,b))^2}{2\kappa_h^{\max}}\right)-\left(1+\phi'(\beta)\right)\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)\right)$$

Using Eq. (A.49) to substitute out $(1+\phi'(\beta))$ further simplifies the expression to

$$\frac{\partial}{\partial\beta}=p\theta\frac{g(x_\nu)}{x_\nu}\left(\varepsilon_g(x_\nu)\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}-\frac{(\overline{\kappa}_h(\alpha,\beta,b))^2}{2\kappa_h^{\max}}\right)-(1+\overline{\kappa}_h(\alpha,\beta,b))\left(1-\frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)\right)$$

where x_ν is a constant defined by $g'(x_\nu)\equiv\nu$.

Setting $\frac{\partial}{\partial\beta}=0$ gives the quadratic

$$\left(\frac{\varepsilon_g(x_\nu)}{2}-1\right)(\overline{\kappa}_h(\alpha,\beta,b))^2+\left(\varepsilon_g(x_\nu)-1+\kappa_h^{\max}\right)\overline{\kappa}_h(\alpha,\beta,b)-\left(\varepsilon_g(x_\nu)-1\right)\kappa_h^{\max}=0$$

which has a unique solution $\overline{\kappa}_h(\alpha,\beta,b) \in (0,\kappa_h^{\max})$ given by the smaller root, i.e.,

$$\overline{\kappa}_h(\alpha,\beta,b) = \frac{\varepsilon_g(x_\nu) - 1 + \kappa_h^{\max} - \sqrt{\left(1+2\kappa_h^{\max}\right)\left(\varepsilon_g(x_\nu)-1\right)^2 + \left(\kappa_h^{\max}\right)^2}}{2-\varepsilon_g(x_\nu)} \tag{A.53}$$

Evaluating the second order condition to confirm a global maximum, we get

$$\frac{\partial^2}{\partial\beta^2} = -p\theta\frac{g(x_\nu)}{x_\nu}\left(1 - \frac{\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}} + \left(\varepsilon_g(x_\nu)-1\right)\frac{1+\overline{\kappa}_h(\alpha,\beta,b)}{\kappa_h^{\max}}\right)\frac{d\overline{\kappa}_h(\alpha,\beta,b)}{d\beta} < 0$$

where the inequality follows from the fact that Eqs. (A.49) and (A.52) imply

$$\frac{d\overline{\kappa}_h(\alpha,\beta,b)}{d\beta} \equiv \frac{x_\nu}{g(x_\nu)}\phi''(\beta) > 0$$

Substituting Eq. (A.53) into (A.49) then gives β as the solution to

$$1+\phi'(\beta) = \frac{1+\kappa_h^{\max} - \sqrt{\left(1+2\kappa_h^{\max}\right)\left(\varepsilon_g(x_\nu)-1\right)^2 + \left(\kappa_h^{\max}\right)^2}}{2-\varepsilon_g(x_\nu)}\frac{g(x_\nu)}{x_\nu} \tag{A.54}$$

The next step here is to show that Eq. (A.54) delivers $\beta < \widehat{h}$. This requires showing that the right-hand side of (A.54) is less than $g'(x_\nu)$, or equivalently that

$$\frac{1+\kappa_h^{\max} - \sqrt{\left(1+2\kappa_h^{\max}\right)\left(\varepsilon_g(x_\nu)-1\right)^2 + \left(\kappa_h^{\max}\right)^2}}{2-\varepsilon_g(x_\nu)} < \varepsilon_g(x_\nu)$$

If $\varepsilon_g(x_\nu) \neq 2$, then this condition simplifies to $\varepsilon_g(x_\nu) > 1$, which is true. If $\varepsilon_g(x_\nu) = 2$, then use l'Hopital's rule to evaluate the left-hand side of the condition and get $\frac{1+2\kappa_h^{\max}}{1+\kappa_h^{\max}} < 2$, which is also true. Thus, $\beta < \widehat{h}$.

Finally, we show that the right-hand side of Eq. (A.54) is increasing in $\kappa_h^{\max}$. Taking the derivative,

$$\frac{d}{d\kappa_h^{\max}}\left(\frac{1+\kappa_h^{\max} - \sqrt{\left(1+2\kappa_h^{\max}\right)\left(\varepsilon_g(x_\nu)-1\right)^2 + \left(\kappa_h^{\max}\right)^2}}{2-\varepsilon_g(x_\nu)}\right) = \frac{1 - \frac{\left(\varepsilon_g(x_\nu)-1\right)^2+\kappa_h^{\max}}{\sqrt{\left(1+2\kappa_h^{\max}\right)\left(\varepsilon_g(x_\nu)-1\right)^2+\left(\kappa_h^{\max}\right)^2}}}{2-\varepsilon_g(x_\nu)}$$

If $\varepsilon_g(x_\nu) \neq 2$, then this derivative will be positive. If $\varepsilon_g(x_\nu) = 2$, then l'Hopital's rule gives that the derivative equals $\frac{1}{\left(1+\kappa_h^{\max}\right)^2}$, which is also positive. Thus, Eq. (A.54) defines β as an increasing function of $\kappa_h^{\max}$.

Now consider $\kappa_h \to \infty$ and $\kappa_\lambda \sim U(0,\kappa_\lambda^{\max})$. For a given κ_λ, recall from $\mathcal{W}_1$ in the proof of Lemma 9 that welfare when $\omega_h = 0$ and $\omega_\lambda \geq 0$ is

$$(1-p)\frac{\alpha}{1+\omega_\lambda} + f\left(1 - \frac{1+\kappa_\lambda\omega_\lambda}{1+\omega_\lambda}\alpha\right) - pg\left(\theta(1-\beta) - \frac{\alpha}{1+\omega_\lambda} - b\right) - p\theta\left(\beta+\phi(\beta)\right) - p\nu b$$

Define the threshold

$$\overline{\kappa}_{\lambda}(\alpha,\beta,b) \equiv 1-\frac{1}{f'(1-\alpha)}\left(1-p+p\frac{g(\theta(1-\beta)-\alpha-b)}{\theta(1-\beta)-\alpha-b}\right) \tag{A.55}$$

If $\kappa_{\lambda} < \overline{\kappa}_{\lambda}(\alpha,\beta,b)$, then $\omega_{\lambda}(\kappa_{\lambda}) > 0$ solves

$$p\frac{g\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}}-b\right)}{\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}}-b} = (1-\kappa_{\lambda})f'\left(1-\frac{1+\kappa_{\lambda}\omega_{\lambda}}{1+\omega_{\lambda}}\alpha\right)-(1-p) \tag{A.56}$$

If instead $\kappa_{\lambda} \geq \overline{\kappa}_{\lambda}(\alpha,\beta,b)$, then $\omega_{\lambda}=0$.

Without commitment, the planner chooses b at $t=1$ after discovering the value of κ_{λ}. If

$$g'\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}}\right) \geq \nu \tag{A.57}$$

then $b(\kappa_{\lambda}) \geq 0$ solves

$$g'\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}}-b\right) = \nu \tag{A.58}$$

That condition (A.57) is true for all $\kappa_{\lambda} \in [0,\kappa_{\lambda}^{\max}]$ follows from $\nu \leq \overline{\nu}_0$, i.e.,

$$g'\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}}\right) \geq g'(\theta(1-\beta)-\alpha) \geq g'\left(\theta\left(1-\widehat{h}\right)-\widehat{\lambda}\right) \equiv \overline{\nu}_0$$

That Eq. (A.58) holds with strict equality for all $\kappa_{\lambda} \in [0,\kappa_{\lambda}^{\max}]$ follows from $\nu \geq g'(0)$.

The planner chooses α and β at $t=0$ to maximize expected welfare, which is given by

$$\frac{1}{\kappa_{\lambda}^{\max}}\int_0^{\kappa_{\lambda}^{\max}}\left(\begin{array}{c}(1-p)\frac{\alpha}{1+\omega_{\lambda}(\kappa_{\lambda})}+f\left(1-\frac{1+\kappa_{\lambda}\omega_{\lambda}(\kappa_{\lambda})}{1+\omega_{\lambda}(\kappa_{\lambda})}\alpha\right)\\ -pg\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}(\kappa_{\lambda})}-b(\kappa_{\lambda})\right)-p\theta(\beta+\phi(\beta))-p\nu b(\kappa_{\lambda})\end{array}\right)d\kappa_{\lambda}$$

or equivalently

$$\begin{aligned}&\frac{1}{\kappa_{\lambda}^{\max}}\int_0^{\overline{\kappa}_{\lambda}(\alpha,\beta,b)}\left(\begin{array}{c}(1-p)\frac{\alpha}{1+\omega_{\lambda}(\kappa_{\lambda})}+f\left(1-\frac{1+\kappa_{\lambda}\omega_{\lambda}(\kappa_{\lambda})}{1+\omega_{\lambda}(\kappa_{\lambda})}\alpha\right)\\ -pg\left(\theta(1-\beta)-\frac{\alpha}{1+\omega_{\lambda}(\kappa_{\lambda})}-b(\kappa_{\lambda})\right)-p\nu b(\kappa_{\lambda})\end{array}\right)d\kappa_{\lambda}\\ &+[(1-p)\alpha+f(1-\alpha)-pg(\theta(1-\beta)-\alpha-b)-p\nu b]\left(1-\frac{\overline{\kappa}_{\lambda}(\alpha,\beta,b)}{\kappa_{\lambda}^{\max}}\right)-p\theta(\beta+\phi(\beta))\end{aligned}$$

where b now denotes the bailout if $\kappa_{\lambda} \geq \overline{\kappa}_{\lambda}(\alpha,\beta,b)$, i.e., for any κ_{λ} such that $\omega_{\lambda}=0$.

Once again, the effect of α and β through b and $b(\kappa_{\lambda})$ can be ignored by the envelope condition. Also note from Eq. (A.58) that $g'(\cdot)=\nu$ for every κ_{λ}, and from Eqs. (A.56) and (A.58), that $\frac{d\omega_{\lambda}(\kappa_{\lambda})}{d\beta}=0$ and $\frac{d\omega_{\lambda}(\kappa_{\lambda})}{d\alpha}=\frac{(1+\kappa_{\lambda}\omega_{\lambda}(\kappa_{\lambda}))(1+\omega_{\lambda}(\kappa_{\lambda}))}{(1-\kappa_{\lambda})\alpha}$.

The first order condition for β is then

$$\frac{\partial}{\partial\beta} = p\theta\left(\nu - 1 - \phi'(\beta)\right)$$

which means that the choice of β is unaffected by uncertainty in κ_λ relative to the case where there are no shadow activities and the planner uses the bailout instrument. If $\nu = \overline{\nu}_0$, then $\frac{\partial}{\partial\beta} = 0$ delivers $\beta = \widehat{h}$.

The first order condition for α is

$$\frac{\partial}{\partial\alpha} = \left(1 - p + p\nu - f'(1-\alpha)\right)\left(1 - \frac{\overline{\kappa}_\lambda(\alpha,\beta,b)}{\kappa_\lambda^{\max}}\right) - \frac{1-p+p\nu}{\kappa_\lambda^{\max}}\int_0^{\overline{\kappa}_\lambda(\alpha,\beta,b)} \frac{\kappa_\lambda}{1-\kappa_\lambda} d\kappa_\lambda$$

or equivalently

$$\frac{\partial}{\partial\alpha} = (1-p+p\nu)\left(1 + \frac{\ln\left(1-\overline{\kappa}_\lambda(\alpha,\beta,b)\right)}{\kappa_\lambda^{\max}}\right) - f'(1-\alpha)\left(1 - \frac{\overline{\kappa}_\lambda(\alpha,\beta,b)}{\kappa_\lambda^{\max}}\right)$$

Using Eq. (A.55) to substitute out $f'(1-\alpha)$ further simplifies the expression to

$$\frac{\partial}{\partial\alpha} = (1-p+p\nu)\left(1 + \frac{\ln\left(1-\overline{\kappa}_\lambda(\alpha,\beta,b)\right)}{\kappa_\lambda^{\max}}\right) - \frac{1-p+p\frac{g(x_\nu)}{x_\nu}}{1-\overline{\kappa}_\lambda(\alpha,\beta,b)}\left(1 - \frac{\overline{\kappa}_\lambda(\alpha,\beta,b)}{\kappa_\lambda^{\max}}\right)$$

where x_ν is still the constant defined by $g'(x_\nu) \equiv \nu$.

Setting $\frac{\partial}{\partial\alpha} = 0$ gives

$$\frac{\kappa_\lambda^{\max} - \overline{\kappa}_\lambda(\alpha,\beta,b)}{\left(1-\overline{\kappa}_\lambda(\alpha,\beta,b)\right)\left(\kappa_\lambda^{\max} + \ln\left(1-\overline{\kappa}_\lambda(\alpha,\beta,b)\right)\right)} = 1 + \frac{\varepsilon_g(x_\nu) - 1}{1 + \frac{1-p}{p}\frac{x_\nu}{g(x_\nu)}} \tag{A.59}$$

Evaluating the second order condition to confirm a global maximum, we get

$$\frac{\partial^2}{\partial\alpha^2} = -\left(\frac{p\left(g'(x_\nu) - \frac{g(x_\nu)}{x_\nu}\right)}{1-\overline{\kappa}_\lambda(\alpha,\beta,b)}\frac{1}{\kappa_\lambda^{\max}} + \frac{1-p+p\frac{g(x_\nu)}{x_\nu}}{\left(1-\overline{\kappa}_\lambda(\alpha,\beta,b)\right)^2}\left(1 - \frac{\overline{\kappa}_\lambda(\alpha,\beta,b)}{\kappa_\lambda^{\max}}\right)\right)\frac{d\overline{\kappa}_\lambda(\alpha,\beta,b)}{d\alpha} < 0$$

where the inequality follows from the fact that Eqs. (A.55) and (A.58) imply

$$\frac{d\overline{\kappa}_\lambda(\alpha,\beta,b)}{d\alpha} = -\frac{1-p+p\frac{g(x_\nu)}{x_\nu}}{\left(f'(1-\alpha)\right)^2} f''(1-\alpha) > 0$$

Substituting Eq. (A.59) into (A.55) then gives α as the solution to

$$\frac{1 + \left(\kappa_\lambda^{\max} - 1\right)\frac{f'(1-\alpha)}{1-p+p\frac{g(x_\nu)}{x_\nu}}}{\kappa_\lambda^{\max} - \ln\left(\frac{f'(1-\alpha)}{1-p+p\frac{g(x_\nu)}{x_\nu}}\right)} = 1 + \frac{\varepsilon_g(x_\nu) - 1}{1 + \frac{1-p}{p}\frac{x_\nu}{g(x_\nu)}}$$

and normalizing $\kappa_{\lambda}^{\max} = 1$ delivers a closed-form solution for $f'(1-\alpha)$, namely

$$f'(1-\alpha) = \left(1 - p + p\frac{g(x_\nu)}{x_\nu}\right) \exp\left(\frac{1}{1 + \frac{1 + \frac{1-p}{p}\frac{x_\nu}{g(x_\nu)}}{\varepsilon_g(x_\nu) - 1}}\right) \tag{A.60}$$

The last step is to show that Eq. (A.60) delivers $\alpha < \widehat{\lambda}$. This requires showing that the right-hand side of (A.60) is less than $1 - p + pg'(x_\nu)$, or equivalently that

$$\frac{1 - p + p\frac{g(x_\nu)}{x_\nu}}{1 - p + pg'(x_\nu)} \exp\left(\frac{p\left(g'(x_\nu) - \frac{g(x_\nu)}{x_\nu}\right)}{1 - p + pg'(x_\nu)}\right) < 1$$

Defining $z \equiv \frac{1-p+p\frac{g(x_\nu)}{x_\nu}}{1-p+pg'(x_\nu)} \in (0,1)$, this condition reduces to $z \exp(1-z) < 1$, which is true within the unit interval. Thus, $\alpha < \widehat{\lambda}$. ■