Is 24/7 Trading Better?

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ABSTRACT

In a dynamic model of large traders who manage inventory risk, we show that a market closure coordinates liquidity, which can be substantial enough to improve allocative efficiency relative to a market open 24/7, even though trade cannot occur during the closure. If traders have heterogeneous information about the asset value, trade is less aggressive on the whole, but closure can still improve welfare. The improvement in welfare due to closure applies to small markets with infrequent shocks, whereas large markets would benefit from extending trading hours, which has implications for the ongoing debates on a 24/7 market design.

I. Introduction

Since the founding of the NYSE in 1792, trading hours have closely mirrored the conventional workday due to the human involvement that was essential for trading. However, in the 21st century, electronic execution has become the dominant form of trade for standardized products.[1] This transition has helped facilitate faster and more efficient trade (Litzenberger et al., 2012), with little to no needed human interaction. Beyond the electronic execution of trades, the majority of trades are submitted by algorithmic traders.[2] Given that electronic trading and execution have substantially reduced the need for constant human involvement, market participants and regulators are actively discussing whether all markets should be open 24/7. Further, due to the increased globalization of firms and the financial sector, there has been a growing demand from international traders for 24/7 trade.[3] This paper studies the implications for liquidity and welfare due to modifications to the length of daily market closures.[4]

We study a model of large traders who rationally anticipate how their orders affect prices while dynamically managing risky inventory positions. Traders experience shocks to their private valuations of an asset, which randomly liquidates at a future date, motivating trade. Traders optimally balance the benefits of eliminating undesired inventory against the costs of incurring price impact. We study welfare, measured by the allocative efficiency of the market, in equilibria of two market designs: one in which there is a daily closure for a fixed fraction of the day, and another in which closure is eliminated. A daily closure is costly because it eliminates traders' ability to manage their inventory for a fraction of the day, leading traders to arrive at the start of the next day in positions that may be far from desirable. Is there a benefit to daily market closures?

If there is a closure, traders rationally anticipate being unable to directly manage their inventory positions overnight, which incentivizes them to be in a good inventory position by the end of the

[1] For example, out of thirteen registered equity exchanges, the NYSE is the only one that is not 100% electronic.

[2] In 2003, about 15% of the U.S. equity volume consisted of algorithmic trades, whereas in 2023, estimates suggest $70 - 80\%$ of U.S. equity volume is algorithmic trades (SEC, 2020). Brogaard et al. (2025) show that floor brokers still play an important role in trading on the NYSE, though most of their benefits are concentrated at the open, the close, and early in the trading day. Their role could easily still be preserved in an extended trading hour environment by having the closure, for example, be between 2:00-3:00 PM EST.

[3] 24X, backed by Steve Cohen's Point72 Venture fund, was approved in November 2024 by the SEC to launch the first 23/7 equity exchange (`https://www.federalregister.gov/d/2024-28551`). The NYSE polled market participants about 24/7 trading in April 2024 and is moving their Arca exchange's trading hours to 22/5. The Nasdaq and Cboe EDGX Equities Exchange are also extending their equity trading hours to 24/5 and 23/5, respectively. Robinhood and Interactive Brokers already allow trade 24/5 for some stocks and ETFs by internalizing the trades.

[4] While the theory developed in this paper naturally applies to equities, it also covers other asset classes.

trading day. Therefore, all traders trade more aggressively towards a desirable position at the end of the day. In turn, this aggressive trade increases liquidity at the end of the day, which lowers the cost of trade and further incentivizes aggressive trade at the market closure. Therefore, liquidity is coordinated, and "liquidity begets liquidity." Aware that liquidity will be coordinated in the final trading session of the day, agents endogenously postpone trade within the day to the last session of the day. If the liquidation rate or the number of traders are not too high, this incentive to postpone trade within the day can be sufficiently large that there is no trade, or an endogenous "halt" in trade, in the sessions just preceding the final trading session. This result is consistent with empirical evidence that trade at closing auctions is highly concentrated, potentially at the expense of preceding sessions (e.g., AMF (2019)).

The mechanisms of the model can be summarized through the behavior intraday volume. Trade in the model can be decomposed into two components that vary over time: a component that determines the gap a trader faces between their current and desired inventory and a component that determines how aggressively a trader trades to eliminate this gap. At the start of the day, traders face large gaps between their current and desired inventory, as shocks arrive overnight that traders are not able to respond to. This generates a large volume at the start of the day, despite relatively low trade aggressiveness. At the end of the day, traders trade very aggressively to close any gap that remains. So, even though the gap between current and desired might not be very large, this aggressive trade results in large volume. In the middle of the day, traders' gaps between desired and current inventories are not very large, and trade is not particularly aggressive, resulting in low volume relative to other parts of the day. Thus, as in the data (e.g., Chan et al. (1996), Jain and Joh (1988)), intraday volume is U-shaped.

When trade is 24/7, there is no equilibrium in which traders coordinate trade. Since traders rationally anticipate how their demand affects prices and future inventory positions, they break up their orders over time to minimize execution costs, which creates socially inefficient excess inventory costs (Du and Zhu, 2017, Rostek and Weretka, 2015, Vayanos, 1999). Liquidity is spread out, and price impact further increases, which further incentivizes traders to break up their orders. In this market design, liquidity, a public good, is spread relatively thinly throughout the trading day. A market closure can potentially benefit traders by concentrating and coordinating liquidity.[5]

[5] Even in securities that trade 24/7, such as forex, we empirically observe volume spikes coinciding with the opening

Next, we study whether the coordination a market closure provides is enough to offset the costs of strategic delay within the day, along with the cost of arriving at the start of the next day in a worse position. Contrary to the basic intuition that a constraint on trade is purely a cost, the endogenous coordination of trade due to market closure can offset this cost. In particular, in smaller markets, that is, markets where the number of traders and the rate of shocks to private values are small, a closure can improve the allocative efficiency of the market by coordinating trade. Empirically, this portion of the paper points to the heterogeneous cross-section of trading-day lengths across exchanges and security types.[6] In markets with a large number of traders, liquidity is already substantial, minimizing the relative coordinating benefits of trade. In markets in which shocks to private valuations are frequent, the costs of restricting traders' ability to respond to these shocks are high, implying the optimal market structure is one in which the market is always open. Further, if the volatility of shocks or marginal holding costs per unit of time are lower overnight, even with the total volatility across the 24-hour period being constant, welfare increases when there is a closure.

To study the magnitude of the benefits that market closure provides, we analyze welfare under various market designs relative to first-best (efficient) market allocations, which avoid the social costs of strategic trade. We find that the common market design of trading for 6.5 hours a day for equities is less efficient than what is optimal. Particularly when the market is large, traders would benefit from a longer trading day, as liquidity is already fairly concentrated throughout the day, making the costs of market closure relatively large. Moreover, in markets with frequent shocks to private values, a long trading day is optimal.

We also show that this paper's main results are robust to including heterogeneity in beliefs about fundamental asset values through noisy private signals. An interesting implication of this extension is that heterogeneity tends to reduce the aggressiveness of trade overall. Yet, closure still concentrates liquidity, allowing traders to trade very aggressively at the end of the day with minimal price impact and improving welfare in small markets or markets with infrequent shocks to private valuations.

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and closing of other exchanges, such as the NYSE. These volume spikes suggest market closures are even important across asset exchanges.

[6]The NYSE and Nasdaq operate from 9:30 AM - 4:00 PM EST (6.5 hours). The CBOE for index options operates from 3:00 AM - 6:00 PM EST (15 hours). The CME Globex operates 6:00 PM - 5:00 PM EST (23 hours). MarketAxess' SEF operates 2:00 AM - 5:30 PM EST (15.5 hours). GFI operates 7:00 AM - 5:00 PM EST (10 hours). Finally, most forex and cryptocurrency exchanges operate 24 hours a day, while some US Treasuries and stock index futures trade 24/5.

There is an extensive literature empirically documenting intraday and overnight patterns in financial markets.[7] A substantial literature theoretically explains these facts (Hong and Wang, 2000, Slezak, 1994, Foster and Viswanathan, 1993, Brock and Kleidon, 1992, Foster and Viswanathan, 1990, Admati and Pfleiderer, 1989, 1988). However, all of these studies take the existence and length of a nighttime as fixed. This paper considers variations in the existence and length of a nighttime.

This paper also contributes to the literature that studies how common financial market structures interact with strategic trading and their implications for the allocative efficiency of the market. Chen and Duffie (2021), Malamud and Rostek (2017), and Kawakami (2017) study market fragmentation. Fuchs and Skrzypacz (2019), Du and Zhu (2017) and Vayanos (1999) study trading frequency. Antill and Duffie (2020), Duffie and Zhu (2017), and Blonien (2024) examine the addition of a trading session at a fixed price. Fuchs and Skrzypacz (2015) study government market freezes in a dynamic adverse selection model. deHaan and Glover (2024) provide empirical evidence that retail traders achieve better portfolio performance when trading hours decrease. Apart from deHaan and Glover (2024), whose focus is on retail trade, none of these papers study the effects of daily market closures.

The presence of market closures is closely linked to the existence of closing auctions, whose characteristics have been of recent interest. The percentage of daily volume transacted in these special sessions has reached an all-time high in recent years (Bogousslavsky and Muravyev, 2023). Consistent with this, our model generates a substantial fraction of daily volume near the opening and closing. The Autorité des Marchés Financiers (AMF, 2019) has expressed concerns that this increase in concentration at the close may lead to price and liquidity deterioration during trading beforehand. We find that prices during trading are unaffected by the closure. However, liquidity leading up to the closure does suffer as traders delay for the substantial liquidity offered at the close, consistent with AMF's concerns. Despite these concerns, we show that there are instances in which the social costs of strategic delay to the closing auction can be outweighed by the coordination benefits of a closure. Bogousslavsky and Muravyev (2023), Jegadeesh and Wu (2022), and Hu

[7] For example, Bogousslavsky (2021), Hendershott et al. (2020), Lou et al. (2019), Branch and Ma (2012), Kelly and Clark (2011), Cliff et al. (2008), Branch and Ma (2006), Andersen and Bollerslev (1997), Chan et al. (1996), Amihud and Mendelson (1991), Stoll and Whaley (1990), Barclay et al. (1990), Harris (1989, 1988), Amihud and Mendelson (1987), Harris (1986), Fama (1965).

and Murphy (2025) compare the NYSE and Nasdaq closing auctions to study liquidity and price efficiency around the closing auction. This paper speaks to liquidity and allocative efficiency as a function of the length of the trading day.

The paper proceeds as follows. Section II defines the model. Section III defines and solves for the equilibrium and builds intuition for how the traders optimally trade with and without a market closure. Section IV quantifies welfare. Section V extends the model to allow for heterogeneous information. Section VI concludes. The Appendices provide technical details and proofs.

II. The Model

Section IIA introduces a model of strategic trading under imperfect competition with periodic market closures. Section IIIB studies a version of the model without market closure that is equivalent to special cases of the model studied in Du and Zhu (2017).

A. A Model with a Nighttime

Time is continuous and goes from 0 to ∞. We set a unit of clock time to be 24 hours, or a day. Each 24 hour period is divided into K evenly spaced subperiods of length $h := \frac{1}{K}$. Trade occurs the first $T+1$ periods, and no trade is permitted in the last Δ periods. We refer to the fraction of the 24 hours where trade occurs as "day" and the remaining fraction is referred to as "night".

Let us illustrate this setup in the first day, where clock time t is in $[0,1]$. Trade occurs at times $0, h, \ldots, Th$, and the night spans times $(Th, 1)$, which includes times $(T+1)h, (T+2)h, \ldots, (T+\Delta)h$. Note, $(T+\Delta)h = 1-h$. At time 1, the next day starts, and the timing repeats.

There are $N \geq 3$ risk-neutral traders who trade a divisible asset. Traders want to hold the asset because it pays a liquidating dividend of v per unit of inventory held.[8] The time of liquidation is random and exponentially distributed, denoted $\mathcal{T} \sim Exp(r)$ so that the expected time until liquidation is $\frac{1}{r}$.[9] Define trader i's inventory of the asset at time t to be z_t^i, and the average aggregate inventory, $\bar{Z}$, is a constant.

[8] In Section V, we extend the model to where this value is unobserved and stochastic and traders receive signals about it throughout time.

[9] Allowing the dividend to be paid continuously with the common discount rate r leads to an essentially identical model and results. Also, the dividend is liquidating for simplicity. One could also allow discrete dividends to be paid at a rate forever.

Each trading session is modeled as a uniform-price double auction. Each trader i submits a demand schedule $D^i : \mathbb{R} \to \mathbb{R}$ that is a mapping of price to demand, $p \mapsto D^i(p)$. The market clearing price, p_t^*, is the price that sets net demand to be zero,

$$\sum_{i=1}^{N} D^i(p_t^*) = 0. \tag{1}$$

After trade at time t, trader i's inventory moves to $z_t^i + D^i(p_t^*)$. Each trader pays the equilibrium price, p_t^*, times the amount of the asset they were allocated, $D^i(p_t^*)$. If $D^i(p_t^*) < 0$, then trader i receives the equilibrium price times the amount of the asset they were allocated. The modeling of trade as an auction as opposed to a limit-order book provides tractability while maintaining the important economic mechanism of price impact from trade.[10]

Each trader is endowed with some portion of the asset, referred to as the trader's initial inventory. Traders also have a private value per unit for the asset that is realized upon liquidation, denoted $w_{i,\mathcal{T}}$. Their value of the asset at liquidation is $v + w_{i,\mathcal{T}}$. The private value, $w_{i,t}$, is a jump process which has $N(0, \sigma_w^2)$ distributed jumps that arrive at constant rate λ.[11] Exogenous shocks to private values induce continued gains from trade over time and can be motivated by risk management considerations or shocks to preferences.

Finally, each trader incurs a holding cost per unit of time of $\gamma \times (z_t^i)^2$.[12] Chen (2022), Chen and Duffie (2021), Antill and Duffie (2020), Duffie and Zhu (2017), Du and Zhu (2017), Sannikov and Skrzypacz (2016), Rostek and Weretka (2012), Vives (2011) and Blonien (2024) all use a similar quadratic holding cost. This cost can be interpreted as representing inventory costs or collateral requirements.

Since traders can only manage inventory through trade during the day, and private values can be shocked in the day or overnight, the restrictions that market closures impose have obvious costs. If a shock to private values arrives overnight, traders will arrive at the start of the next day at positions that are suboptimal. In the model, traders trade off maintaining suboptimal inventory positions with price impact costs. Therefore, they trade slowly towards their desired inventory

[10] A series of papers (Budish et al., 2024, Aquilina et al., 2021, Budish et al., 2015) recommend that the market be switched from a continuous limit-order book to frequent batch auctions to curtail the socially wasteful arms race for speed.

[11] Results for other exogenous information arrival processes, for instance with shocks every nh units of time for an integer n, are similar.

[12] We can solve the model for any deterministic and time-varying holding costs, γ_t. For simplicity, we focus on constant marginal holding costs throughout time.

position, heightening the costs of a temporary closure. This paper's goal is to study the costs and benefits of daily market closure through the organization of trade it induces and to determine their relative magnitudes.

Now, let us define the traders' value functions. In the following sections, we will study equilibria that are periodic, with a period equal to one day. Therefore, to ease the exposition, we simply focus on $t \in [0,1]$ and note that results at any other time are analogous. Recall trade in the first day occurs at times $0, h, \ldots, Th$. For $t = kh$ in any of these periods apart from the last, denote any trader's value function V_k. The value function is a function of current inventory position z^i, current private value w^i, and aggregate private value $\bar{W} = \frac{1}{N}\sum_{i=1}^{N} w^i$, and satisfies the following Bellman equation:

$$V_k(z^i, w^i, \bar{W}) = \max_{D^i} \Bigg\{ -\underbrace{D^i p_t^*}_{\substack{\text{cost of}\\ \text{trade}}} + \underbrace{(1-e^{-rh})(z^i + D^i)(v + w^i)}_{\substack{\text{expected liquidation}\\ \text{value}}} \\ - \underbrace{\frac{(1-e^{-rh})\gamma}{2r}(z^j + D^i)^2}_{\substack{\text{expected inventory}\\ \text{flow cost}}} + \underbrace{e^{-rh} E_{kh} V_{k+1}(z^i + D^i, w^i_{(k+1)h}, \bar{W}_{(k+1)h})}_{\substack{\text{expected}\\ \text{future value}}} \Bigg\}. \quad (2)$$

The maximum is over demand schedules, not simply realized demands. The first term corresponds to the cost (allocated quantity times market clearing price) of trade incurred in the double auction at time kh. The next term corresponds to the expected payoff if the asset liquidates before the next session times the probability it liquidates before the next session. The third term is the expected holding cost before the next session, which incorporates the probability that the asset might liquidate, after which there is no more holding cost. The last term is the next period's continuation value, assuming the asset does not liquidate before then, times the probability the asset does not liquidate before the next period. As we will show, prices reveal the average private value $\bar{W}$ in equilibrium. Therefore, the value function is a function of $\bar{W}$ insofar as it affects future prices and realized demands, and thus, utility. In the last trading period of the day, that is in the time $T+1$ trading session at clock-time Th, the Bellman equation is modified to the following:

$$V_T(z^i, w^i, \bar{W}) = \max_{D^i} \Bigg\{ - \underbrace{D^i p^*_{Th}}_{\substack{\text{cost of}\\ \text{trade}}} + \underbrace{(1 - e^{-rh(1+\Delta)})(z^i + D^i)(v + w^i)}_{\substack{\text{expected liquidation}\\ \text{value}}} - \underbrace{\frac{(1 - e^{-rh(1+\Delta)})\gamma}{2r}(z^j + D^i)^2}_{\substack{\text{expected inventory}\\ \text{flow cost}}} + \underbrace{e^{-rh(1+\Delta)} E_{Th} V_0(z^i + D^i, w^i_1, \bar{W}_1)}_{\substack{\text{expected}\\ \text{future value}}} \Bigg\}. \quad (3)$$

The terms are modified to reflect the increased likelihood that the asset liquidates before the next trading session, as there are $h(1+\Delta)$ units of clock time between trade instead of h.

III. Equilibrium

A. Model with a Closure

Prior literature (e.g., Antill and Duffie (2020), Du and Zhu (2017), Vayanos (1999)) frequently studies symmetric, linear, and stationary equilibria. That is, the equilibrium demand schedules of each trader are the same linear combination of price and other relevant state variables and inputs. In our model with daily market closures, such an equilibrium will not exist. The trading problem that faces every trader will not be ex-ante identical at each trading session, as the opportunity set changes throughout the day, precluding the existence of stationary equilibria. For instance, as the closure approaches, traders will behave differently as the inability to manage inventory overnight presents a substantial change in the opportunity set.

Therefore, we focus on symmetric, linear, and daily-periodic demand schedules. For example, in equilibrium, all demand schedules submitted at 9:30 AM will be the same function every day, but all traders may use a different demand schedule at 10:00 AM than they did at 9:30 AM. Concretely, we conjecture that the equilibrium demand schedule at trading session $k \in \{0, \ldots, T\}$ is of the following form:

$$D^i_k(z^i, w^i, p) = a_k + b_k p + c_k z^i + f_k w^i, \quad (4)$$

where $b_k \leq 0$. By market clearing, trader i faces the residual supply curve of the other $N-1$ traders and effectively chooses a price and quantity pair. If trader i chooses demand quantity d^i, then by market clearing, the price must solve $d^i + \sum_{j \neq i}(a_k + b_k p + c_k z^j + f_k w^j) = 0$. Therefore, the market clearing price is

$$\Phi_k(d^i, z^i, W^{-i}) := p = -\frac{1}{b_k(N-1)}\big(d^i + (N-1)a_k + c_k(N\bar{Z} - z^i) + f_k W^{-i}\big), \tag{5}$$

where $W^{-i} = \sum_{j\neq i} w^j$. Traders are strategic, and thus, they rationally anticipate and internalize how their demand affects prices due to imperfect competition. As price impact itself is only a wealth transfer between traders, it is the strategic effects of avoiding price impact that can be socially costly by reducing allocative efficiency.

A symmetric (Markov perfect) equilibrium of the above stochastic game is defined by the sequences $(a_k)_{k=0}^T$, $(b_k)_{k=0}^T$, $(c_k)_{k=0}^T$ and $(f_k)_{k=0}^T$. Equilibrium requires that if trader i conjectures the other $N-1$ traders use the linear demand schedule (4), trader i's best response is to use the same demand schedule, and the market clears. We show in the Appendix that this equilibrium exists and is characterized by Proposition 1.

PROPOSITION 1: *If $N > 2$ or rh are sufficiently large, there exists a unique symmetric, linear, and periodic equilibrium with trade in periods $0, \ldots, T$ and which satisfies the following properties:*

1. *The equilibrium quantity traded takes the form*

$$D_k^i(p_k^*) = c_k\left(z_k^i - \big(\frac{r}{\gamma}(w_k^i - \bar{W}_k) + \bar{Z}\big)\right), \tag{6}$$

   where $k = 0, \ldots, T$, for c_k characterized in Appendix A, and satisfies $-1 \leq c_k \leq 0$.

2. *The equilibrium market clearing price is*

$$p_k^* = v + \bar{W}_k - \frac{\gamma}{r}\bar{Z}. \tag{7}$$

3. *Let $\bar{c}$ denote the equilibrium value of c_k if there is no market closure. In two consecutive periods of trade, if $c_{k+1} > \bar{c}$, then $c_k < \bar{c}$. Similarly, if $c_{k+1} < \bar{c}$, then $c_k > \bar{c}$. An analogous pattern applies to $1/b_k$, which determines price impact as $c_k = \frac{r}{\gamma}b_k$.*

Let us discuss these results. First, let us look at the functional form of the allocation, $c_k(z^i - (\frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}))$. The allocation is current inventory net of a measure of desired inventory, which we will define as $\tilde{z}^i := \frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$, scaled by c_k. $\tilde{z}^i$ is the inventory position a trader would reach each period after trade if the market was competitive. We refer to $\frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$ as desired inventory because if $z^i = \frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$ for every trader, then there is no more trade in equilibrium.

Consider the post-trade inventory position,

$$z_{k+1}^i = z_k^i + D_k^i(p_k^*) = (1+c_k)z_k^i - c_k\big(\frac{r}{\gamma}(w_k^i - \bar{W}_k) + \bar{Z}\big) = (1+c_k)z_k^i - c_k\tilde{z}_k^i. \tag{8}$$

Recalling that $c_k \in [-1, 0]$, then c_k is a measure of trade aggressiveness as it tells us what fraction of our new inventory position is made up of the old inventory position, and the remaining fraction is the desired inventory position. Subtracting $\tilde{z}_k^i$ from both sides of equation 8, the gap between next period's inventory and the desired inventory is

$$z_{k+1}^i - \tilde{z}_k^i = (1+c_k)\big(z_k^i - \tilde{z}_k^i\big). \tag{9}$$

As c_k approaches -1, which is the value it would have under perfect competition, this gap approaches zero, and the allocation of the asset becomes more efficient.

The coefficient c_k in the equilibrium allocation is largest, in absolute value, and most negative at the end of the trading day. As the end of the day approaches, traders are aware that they will soon lose the opportunity to manage random shocks in the desired inventory through trade. They all, therefore, have the incentive to enter nighttime in a desirable inventory position. As a result, traders are more willing to incur price impact and temporary trading costs toward the end of the trading day. The old adage of "liquidity begets liquidity" comes into effect; liquidity improves due to the fear of suboptimal inventory positions being exacerbated overnight, so it becomes even cheaper to trade more aggressively now, further encouraging aggressive trade.

This incentive to enter overnight in a good position is strongest in the final period of trade. In fact, by backwards induction, traders know trade will be cheap in the final period. So, traders have an incentive to postpone trade until then. This reduces liquidity in the penultimate period. This explains property 3 of the equilibrium, which formalizes the strategic incentives in the model. Essentially, if trade is aggressive next period, trade is less aggressive this period, as traders postpone to the next period when price impact is lower. Similarly, if trade is less aggressive next period, trade will be more aggressive this period. Thus, trade has some oscillatory properties. In our numerical examples, this oscillation is by far the strongest in the last two periods and decays relatively quickly.

If the incentives to postpone trade are sufficiently strong, the equilibrium with trade every period breaks down, and there is a period of no trade leading up to the closing session. These incentives are strongest when the market is small, i.e. N is small, or liquidation risk before the next trading session is large, i.e. rh is small. If N is small, price impact is generally large, implying the benefits of a liquid final period of trade are substantial. If rh is small, the costs of delaying trade are low as the asset is unlikely to liquidate in the meantime. Empirically, an analog of these results is the fact that in markets with closing auctions, liquidity prior to the closing auction is relatively thin, as trade is delayed due to the coordination in the closing auction.

Below, we provide a result that describes an equilibrium that can arise if the parameters are such that traders are not willing to trade in the period preceding liquid close. In order to obtain this modified equilibrium, we relax the assumption that submitted demand schedules must be downward sloping by allowing traders to not trade for a period.

PROPOSITION 2: *Assume that an equilibrium with trade in every period does not exist. Then, under weaker conditions than the existence of an equilibrium with trade every period, there is a unique equilibrium in which demand schedules are permitted to be uniformly zero for a single period during the trading day. This equilibrium has no trade in period* $T-1$ *and satisfies properties 1, 2, and 3 in the other periods.*

This result implies there are equilibria in which there is a "halt" in trade for a period. It can be generalized to allow for halts in multiple periods. Before moving on to analyzing the model in more detail, we note there is a continuous time version, in which trade occurs in a continuous sequence of uniform-price double auctions for the first $1-\Delta-\epsilon$ units of the day, there is a halt in trade for the next endogenous length ϵ units of time, and a closing auction occurs at time $1-\Delta$. In this version of the model, the length of the halt can be determined analytically, with no parameter restrictions apart from $N>2$. Moreover, prior to the halt, demand schedules are stationary and thus do not depend on time and so do not oscillate. In addition, trade in the final period is more aggressive than trade in the opening sequence of sessions.

It is worth highlighting some of the expressions in the continuous time version of the model, as quantities such as c_k and b_k for the discrete time model, are provided in the Appendix but are not readily interpretable. In the continuous time model, the length of the halt ϵ is

$$\epsilon = \min\left\{1-\Delta, \frac{1}{r}\log\left(\frac{e^{-\Delta r}+(1-e^{-\Delta r})N}{e^{-\Delta r}+(1-e^{-\Delta r})(N-1)}\right)\right\}. \tag{10}$$

For $\epsilon > 1 - \Delta$, the coefficient c_T in the demand function at the close, $1 - \Delta$, is

$$c_T = -\frac{(N-2)(1-e^{-\Delta r})}{e^{-\Delta r} + (1-e^{-\Delta r})(N-1)}, \tag{11}$$

and $c_T = \frac{\gamma}{r} b_T$. It's straightforward to see that ϵ is increasing in Δ (as long as the minimum above does not bind) and decreasing in N, while both c_T and b_T become more negative as N and Δ increase. These comparative statics are analyzed in the discussion surrounding Figure 2 below.

B. *Model of* 24/7 *Trading*

Let us briefly review the solution without market closure and then compare the two models. The model with no nighttime, $\Delta = 0$, is a special case of Du and Zhu (2017) in which there is no adverse selection. We make no other modifications to the model from the previous section other than setting $\Delta = 0$. Once again, we differ from most prior literature by conjecturing linear, symmetric, and periodic equilibria of the same form as Equation 4. Periodicity again requires the demand functions to be periodic functions of time with period 1.

We characterize the equilibrium in Proposition 3.

PROPOSITION 3: *When $\Delta = 0$ and $N > 2$, there exists a unique symmetric, linear, and periodic equilibrium with the following properties:*

1. *The equilibrium quantity traded takes the form*

$$D_k^i(p_k^*) = c\left(z_k^i - \left(\frac{r}{\gamma}(w_k^i - \bar{W}_k) + \bar{Z}\right)\right), \tag{12}$$

where $k = 0, \ldots, T$, and $c \in [-1, 0]$ and is equal to

$$c = \frac{-(N-1)(1-e^{-rh}) + \sqrt{(N-1)^2(1-e^{-rh})^2 + 4e^{-rh}}}{2e^{-rh}} - 1.$$

2. *The equilibrium market clearing price is*

$$p_k^* = v + \bar{W}_k - \frac{\gamma}{r}\bar{Z}. \tag{13}$$

A first point to note is that the equilibrium strategy played is time-invariant. Despite allowing the demand schedules submitted to be periodic across days, the unique equilibrium is constant across time, as in Duffie and Zhu (2017). Therefore, conditional on state variables $z^i, w^i, \bar{W}$, prices and allocations are the same across time. It is worth noting that prices are the same when trade is 24/7. Although liquidity differs across the two market structures, the net quantity demanded, which is a function of the marginal costs and benefits of holding the asset, remains unchanged, as incentives for holding the asset due to liquidity are symmetric across traders in equilibrium. In the model with closure, trade is non-stationary throughout the day. Importantly, this non-stationarity leads to a coordination of liquidity towards the end of the day.

C. Equilibrium Intuition

Next, we compare the equilibrium in Propositions 1 and 2, when there is a market closure, and Proposition 3, where trading occurs 24/7. The introduction of a long pause in trading, lasting $h(1+\Delta)$ units of clock time instead of just h, creates non-stationarity in the equilibrium demand functions. Instead of there being a constant fraction of excess inventory closed at each trading period, as in the 24/7 model, the intensity with which agents trade in the model with a closure, c_k, typically has three distinct periods of behavior. To see this and compare the two models, we will look at an example in Figure 1.

Figure 1 quantifies the magnitude of the coordination and concentration of liquidity when there is a market closure, for various trading frequencies. The y-axis is the expected percentage of excess inventory left for a given trader relative to the start of the day. Mathematically, the y-axis is $\prod_{j=0}^{k}(1+c_j)$, where k is the $(k+1)^{th}$ trading session of the day, which occurs at clock time kh. Recall excess inventory is simply the difference between current inventory, z_t^i, and desired inventory, $\tilde{z}_t^i$, which is closed by $1+c_k$ in trading session k. The figure displays how much of the traders' excess inventory is eliminated assuming there are no shocks to tastes during the day, so that desired inventory is constant throughout the day.

When trade is 24/7, c is constant and between -1 and 0, and traders close $1+c$ percent of the excess inventory per period. Comparing the orange and blue dashed lines, when trading frequency is higher, the strategic effects are amplified as liquidity per trading session is lower, which increases price impact, which further reduces a trader's willingness to trade. Du and Zhu (2017) study the tradeoff between this strategic cost and the ability to react to shocks more quickly.

When we add a closure, the strategic behavior of traders changes the equilibrium trading patterns dramatically. This is reflected in the solid blue lines for a slower market and solid orange lines for a faster market. It is easiest to work backward to understand. Starting at the close, traders rationally anticipate that they will be stuck in an inventory position overnight, which flow cost

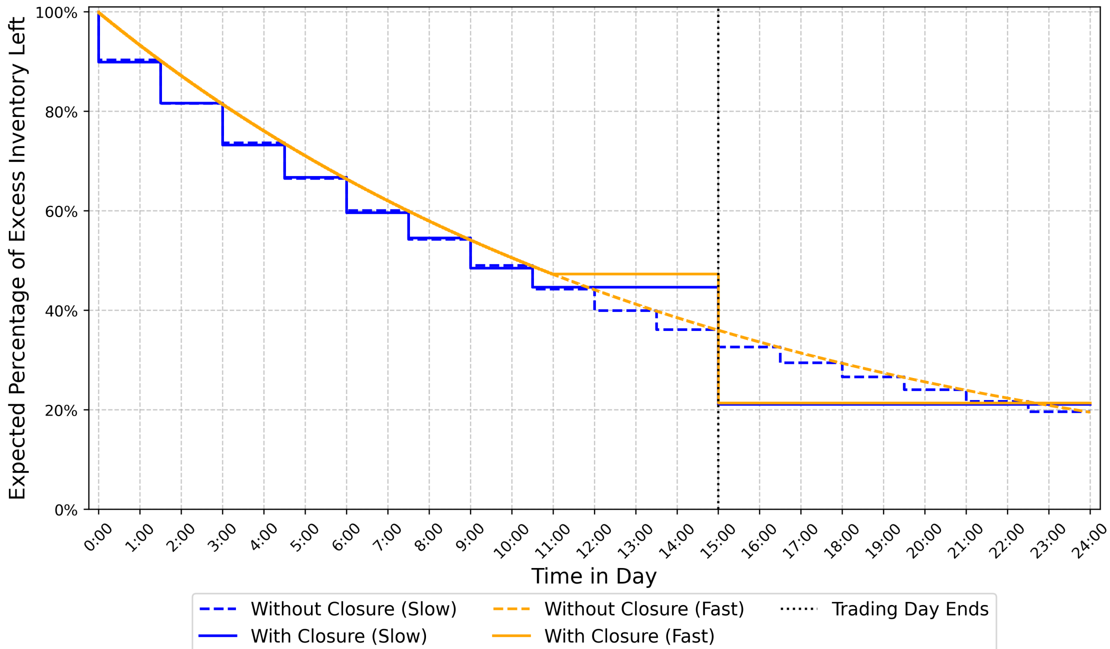


Figure 1. Trading Intensity Throughout the Day

This figure plots trading intensity for various regimes throughout the day. The y-axis is the expected percentage of excess inventory left at point t in the day relative to the position at the start of the day for a given market design. The solid lines are market designs with a closure of 31.25% of the day from Proposition 1, and the dashed lines are market designs without a closure from Proposition 3. The colors map to the trading frequency of the market, with blue being $K = 16$ periods a day and red being $K = 640$ periods a day. The vertical dotted line is when the market closes for trading with respect to Proposition 1. We use $N = 100$ and $r = 1/30$.

overnight irrespective of the shocks to their private value for the asset, and there is some chance the asset will liquidate. Moreover, traders will not be able to react to shocks to taste that occur overnight, making excess inventory at the end of the day even less desirable. These risks increase traders' willingness to incur additional price impact at the end of the day to avoid a worse inventory position overnight. This is a symmetric incentive across all traders. So, as they all become more aggressive, liquidity increases, which decreases price impact for a desired quantity. They, therefore, can become even more aggressive, and this logic repeats. The closure helps traders coordinate their trade that is otherwise very spread out when trade is 24/7. This can be seen in the plot by the large downward jump in the amount of excess inventory held right after the last trading session of the day. It takes almost all of the night for traders to offload as much of their initial excess inventory when trade is 24/7.

Yet, because trade is very efficient at the close and traders are rational and strategic, in periods leading up to the closure, traders know that if they delay trade for a short period, they will be able to trade very cheaply. This incentive to delay trade is so strong that, in the plotted example, there is no trade with downward sloping demand in equilibrium in the periods just preceding the close. Therefore, the only equilibrium that exists is that no trade happens, $b_k = 0$.

When traders are far enough away from the close, the undesired flow costs and liquidation risk throughout the day are sufficiently large that it is worth incurring some price impact to optimize positions, and there is non-zero trade. When trade limits to occur continuously, trade is the same during this time whether there is a closure or not, which can be seen by the solid and dotted orange lines being effectively indistinguishable. When trade is slower, you can see some oscillation in aggressiveness around the level of aggressiveness in the 24/7 trade model (see Property 3 of Proposition 1). If liquidity is better next period, agents are less willing to trade now, which lowers aggressiveness and liquidity this period. If liquidity is poor next period, agents are more willing to trade now and incur price impact. So, the non-stationarity of the trader's problem generates an oscillation that increases in magnitude as the closure approaches. This oscillation can be seen by the dashed blue line flipping alternating being below or above the solid blue line in subsequent periods.

In Figure 2, we study how aggressiveness trade is at the close and the length of the endogenous period of no trade, or "halt", as a function of the size of the market or the length of overnight closure. The lines in the plot are the discrete-trading versions of equations 10 and 11. Panel A studies how these endogenous quantities change as the market grows in size. First, the closer the dotted blue line is to 100%, the closer the model is to perfect competition and the more efficiently the asset is traded as the close. The y-axis is the fraction of excess inventory that is sold at the close. As the market becomes larger, price impact decreases as demand is dispersed across more traders, and very quickly, the majority of the excess inventories are reallocated. The orange line plots the length of the halt prior to the closure. For these parameters, and when the market is small, there actually is no trade except for the closing auction until there are about 75 traders. Then, as the market size increases, the fraction of the day with endogenously no trade decreases towards zero. As the market grows, price impact decreases, making it less costly to trade in any period before close and minimizing the relative benefits of coordinated liquidity at the close. In

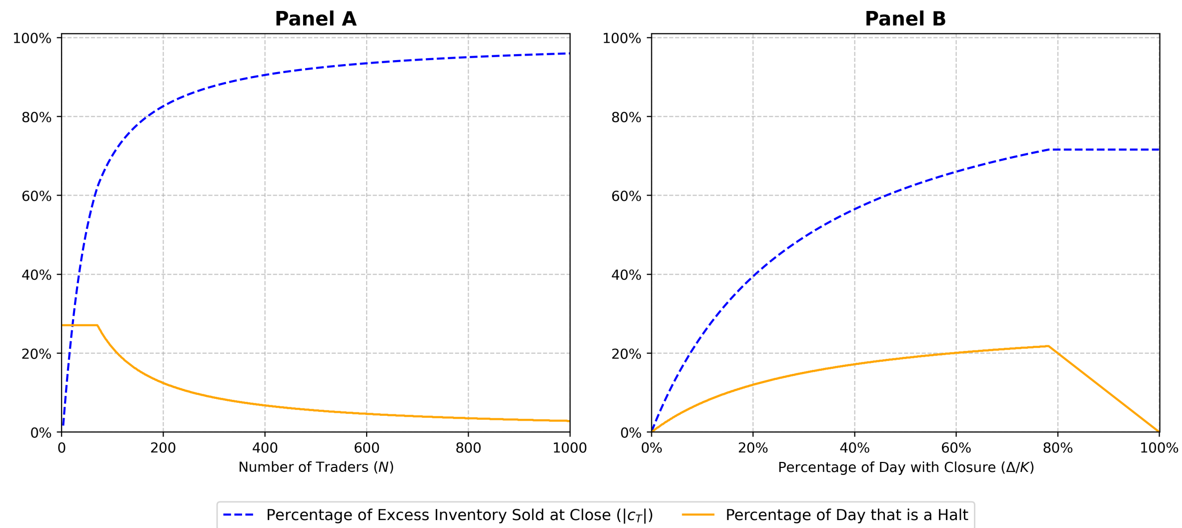
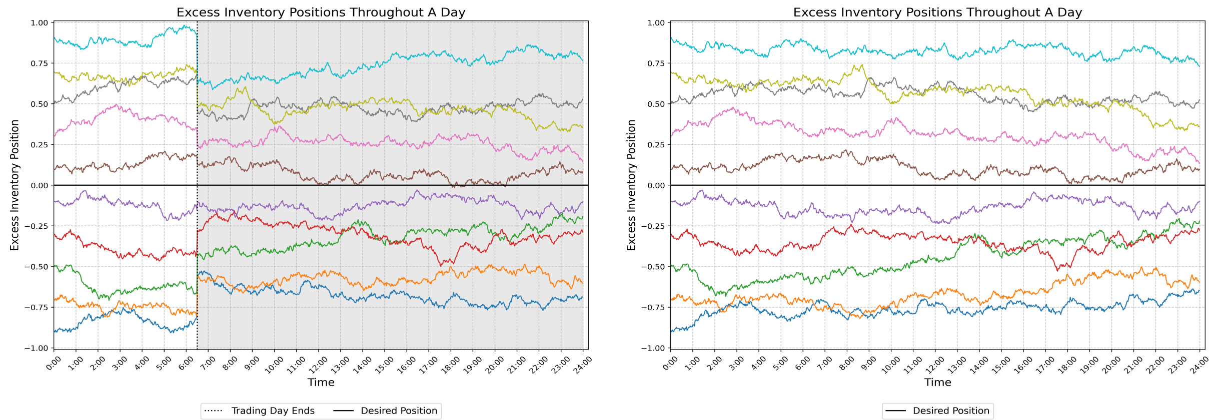


Figure 2. Trading Around the Close
We plot the aggressiveness of traders at the close, $|c_T|$ where the closer to 100% being closer to perfect competition, with a blue dotted line and the percentage of the trading day where no trade endogenously happens leading up to the close with an orange solid line. In Panel A, we plot these two quantities as a function of the size of the market, N. In Panel B, we plot these two quantities as a function of the percentage of the day where the market is closed, Δ/K. The continuous-time version of the blue-dotted line is equation 11, and the continuous time version of the solid-orange line is equation 10. We use $r = 1/30$ and $K = 1,000$ for both plots. In Panel A, we set $\Delta/K = 73\%$, and in Panel B, we set $N = 100$.

Panel B, as the length of closure increases, so does the efficiency of trade at the close. As the costs of closure increase, so does traders' willingness to incur price impact at the close. Eventually, the closure is so long that there is only trade at the close, and the line flattens out. By similar logic, the length of the halt increases as the efficiency of the closing session increases. Eventually, there is only trade at the close, which is mechanically moved towards the open, which results in the line having a slope of -1.

D. Simulating the Models

We simulate a trading day for a market with ten traders. We run a single simulation for two scenarios: first, when trade only occurs for the first 6.5 hours of the day, followed by nighttime and no trade for 17.5 hours, and second, when trade can occur 24/7. Each trader receives the same shocks to their inventory position in the two scenarios. The only difference is how their strategies endogenously change when there is a closure. For the $N = 10$ traders, we set the initial excess



(a) Simulation with $\Delta = 17.5$ hours of night

(b) Simulation with trading 24/7 ($\Delta = 0$)

Figure 3. Simulation With and Without Closure

These figures plot excess inventory paths under the same simulated shocks over a single day for ten traders, $N = 10$, but the left plot has a nighttime of 17.5 hours, and the right plot has trading 24/7. The desired excess inventory position (the solid black line) is zero, and the shocks to traders' private values are the same across plots and occur every period right after trade. The parameters used are $\sigma = 1$, $r = 10\%$, $K = 1,000$, and $\gamma = .4$.

inventory positions to be equally spaced between $-.9$ to $.9$.

The results of these simulations are plotted in Figure 3. Starting with Figure 3(a), while there is noise in the traders' inventory positions during the trading day, at the close, there is a large drop in the amount of excess inventory held across traders. While they slowly trade to eliminate excess inventory early in the day, and there is an endogenous pause of no trade, they become very aggressive at the close. After the trading day ends, traders can no longer control the drift of their inventory positions. Therefore, inventories randomly evolve overnight. Traders dislike that this may leave them in an undesirable position at the start of the next day. Yet, traders incur a much lower amount of undesired flow cost early at night due to the ability to trade cheaply toward their desired inventory positions.

Using the same shocks, we plot how trader's excess inventory position would have endogenously evolved in a model with trade that is 24/7 in Figure 3(b). Without market closure, traders strategically break up their orders over time, spreading out liquidity and trading slowly toward their desired inventory position. Without the coordination of liquidity, traders never substantially close the gap. They do appear to be in better positions by the end of the day, though. From this simulation alone, it is not clear which scenario the traders would prefer ex ante. In Section IV, we

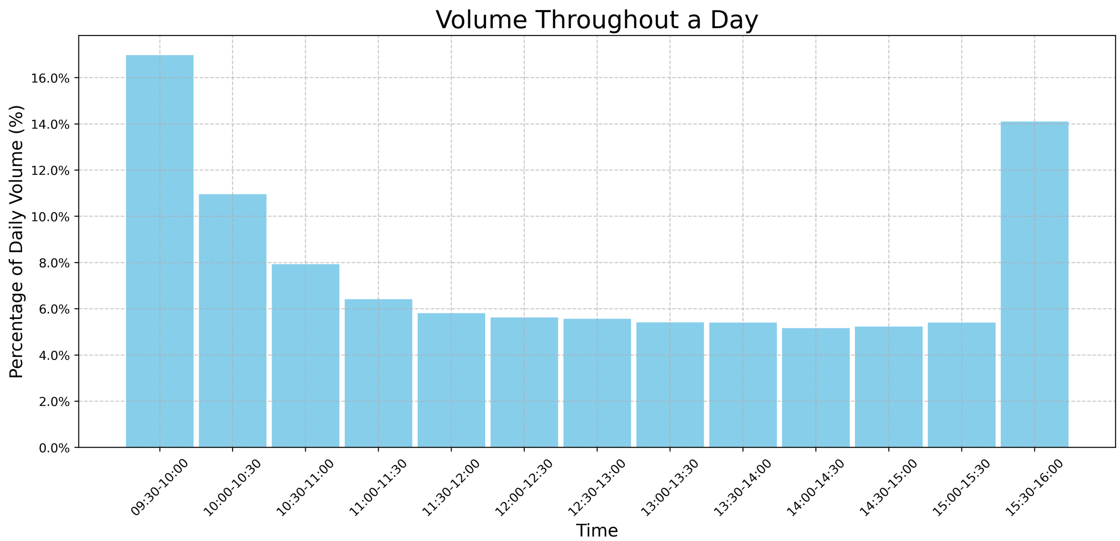


Figure 4. Volume Throughout the Day
This figure is the percentage of the expected daily trading volume in each 30-minute bin when trading occurs for 6.5 hours a day. This example uses $N = 500$, $r = 10\%$, $\sigma_d = \sigma_n$, $K = 1,000$, and $\Delta/K = \frac{17.5}{24}$.

will formally study trader welfare as a function of market structure.

E. Volume

One way to see the implications of the model for intraday trade by looking at volume throughout the day. A strong empirical pattern is the U-shaped (smirk) pattern of trading volume throughout the day (Chan et al., 1996, Jain and Joh, 1988). Due to the non-stationarity of the demand schedule, it is not obvious what volume will look like throughout the day.

Due to the inability to trade overnight, the absolute gap between any trader's current and desired inventory position is expected to grow overnight. Therefore, although trade is not very aggressive in the morning in the sense that traders exchange a small percentage of the gap, due to the large average gap, they still trade a large quantity of the asset. In the middle of the day, they still are not very aggressive nor have a large excess inventory position. Finally, at the close, they become very aggressive and close the gap significantly, resulting in a large increase in trading volume.

Figure 4 demonstrates the above logic. Figure 4 plots the expected fraction of the total daily volume in each 30-minute trading bucket by computing average volume in simulations of the model.

To match the NYSE, we assume the trading day is 6.5 hours. If trade volume was uniformly distributed throughout the day, you would expect about 7.7% of the daily volume in each bin. Yet, we see significantly more near the open and close. About 17% of the daily volume happens in the first 30 minutes, and about 14% happens in the last 30 minutes.

Some markets, such as foreign exchange (FX) markets or cryptocurrencies, already trade 24/7. Yet, volume patterns in these markets are not flat throughout the day, as the equilibrium of Proposition 3 would imply. Empirically, we see spikes in volume in FX markets when either the London or New York stock exchanges first open or close and especially during their overlap in opening hours. Likewise, for cryptocurrencies, we see volume rise when the London Stock Exchange opens and a further increase when the NYSE opens. Allowing for the volatility or frequency of shocks to be a deterministic function of time would likely match these patterns. As volatility increases, so would volume. Therefore, if volatility or rate of shocks spike during market openings in other asset classes, you could see generate spikes in trading volume even for assets that trade 24/7. Introducing another set of traders who only trade between certain hours of the day, such as when another market opens and closes, might also generate similar patterns in volume in markets that are open 24/7. The daily entry and departure of these traders could potentially coordinate trade sufficiently to generate the empirical patterns we observe in markets that do trade 24/7. More generally, modeling the interdependence between exchanges and their hours is well beyond the scope of this paper; to be fully understood, it would require the study of traders' dynamic strategic trade between correlated assets trading on different exchanges.

IV. Welfare

We now formally study if traders are ex-ante better off having a daily closure of some length or being allowed to trade 24/7. We do this by studying the aggregate ex-ante welfare of traders. Specifically, we define welfare as traders' ex-ante expected value of their value functions across all traders in the market. As each trader's value function aggregates their expected profits net of flow costs, the higher the value, the more efficient the market is. In this section, for simplicity, we assume that the initial inventory position for each trader is zero, $z_0^i = 0$, which implies that $\bar{Z} = 0$, and each initial private benefit is $w^i \stackrel{iid}{\sim} N(0, \sigma^2)$.

As a first benchmark, we define the first-best (efficient) welfare as that which continuously and perfectly reallocates each trader's inventory position to the competitive benchmark. This benchmark is what a benevolent social planner would achieve if both frictions in the model were eliminated by making markets perfectly competitive and letting trade occur continuously and 24/7. The efficient welfare is

$$W^e := \sum_{i=1}^{N} \mathbb{E}\big[V^e(z^i = 0, w^i, \bar{W})\big] = \frac{\sigma^2(N-1)(r+\mu)}{2\gamma}. \tag{14}$$

Next, we quantify welfare under the market design that never has a market closure where trade occurs 24/7. This is the welfare from Proposition 3. The 24/7 welfare is

$$W^{24/7} := \sum_{i=1}^{N} \mathbb{E}\big[V(z^i = 0, w^i, \bar{W})\big] = N\alpha_0 + \sigma^2\Big(N\alpha_5 + \alpha_6 + \alpha_9\Big), \tag{15}$$

where the α's are defined by equations 21, 22, 23, and 24 in Appendix A.3. Finally, we quantify the welfare achieved from Proposition 1 and extensions of it that allow for halts in trade, where there is a market closure of Δ periods. Welfare under a market closure of length Δ is

$$W(\Delta) := \sum_{i=1}^{N} \mathbb{E}\left[\frac{1}{T+1}\sum_{k=0}^{T} V_k(z^i = 0, w^i, \bar{W})\right] = \frac{1}{T+1}\sum_{k=0}^{T}\left[N\alpha_0^k + \sigma^2\Big(N\alpha_5^k + \alpha_6^k + \alpha_9^k\Big)\right], \tag{16}$$

where the α's are defined by equations 17, 18, 19, and 20 in Appendix A. Since welfare with a closure is a non-stationary function of time, we compute welfare by averaging across time periods in the trading day. In effect, time is an additional state variable, and, as with w^i and $\bar{W}$, we also randomize over time.

A. *Welfare Comparative Statics*

In Figure 5, we plot the percentage change in welfare from trade with closure for a fraction of the day relative to welfare in a market structure with 24/7 trade. We look at this relationship as we vary the length of the closure. Panel A plots the relationship for two different market sizes, and Panel B plots the relationship for two different shock arrival rates.

In Panel A, we show that as the length of closure increases, traders become relatively worse off. The welfare loss is larger for the larger market. This is because, in larger markets, the strategic costs are lower. Despite trade at the close still being more efficient and a smaller endogenous length of no trade before the close, there isn't as much price impact at any period throughout the day, and, therefore, closure is relatively more costly. In small markets, the benefit of the closing session

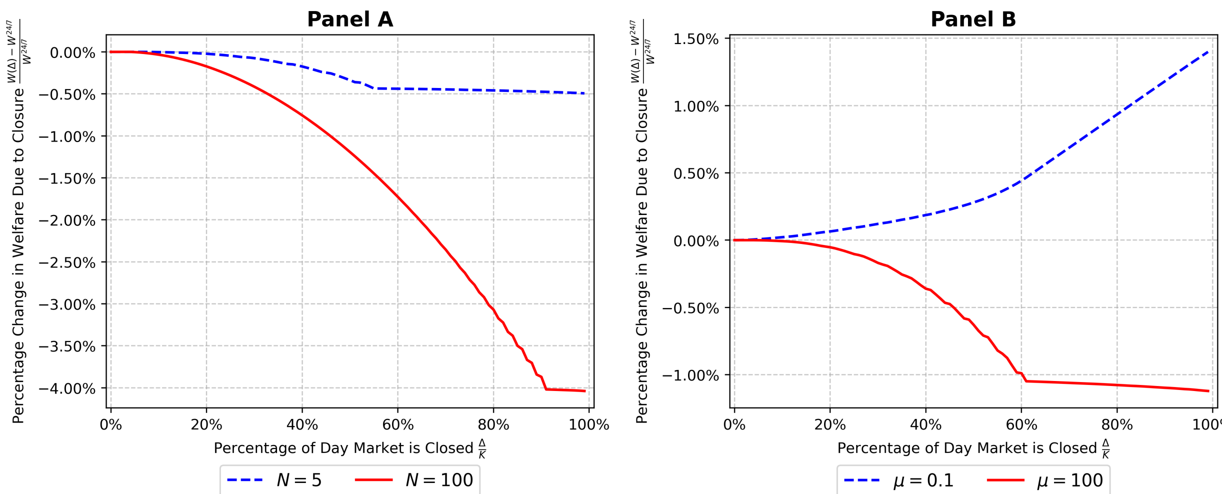


Figure 5. Welfare Comparative Statics

Above is the percent change between welfare under a market closure and welfare under 24/7 trade as we vary the length of the closure. Panel A plots this relationship for two different market sizes. Panel B plots this relationship for two different rates of shocks. Both plots use $K = 100$ and $r = 10\%$. In Panel A, $\mu = 10$. In Panel B, $N = 10$.

offsets more of the cost through the coordination of liquidity that is otherwise spread out thinly throughout the day, and in fact, there is an interior optimal length of closure near 5% of the day. However, as will be noted in the following subsection, the relative welfare differences are small.

In Panel B, welfare differences are displayed for different rates of shocks to private values. If the shocks are infrequent, closure benefits traders. The higher the frequency of shocks, the lower the relative welfare with a closure. This is due to the fact that the average gap generated overnight between current and desired inventory widens as the length of closure increases and as the rate of shocks increases. If there are not any shocks overnight, then the probability that your inventory position, which tends to be good at the close, is near the desired position at the following open is high. But if there are many shocks at night, then the position you start at the beginning of the next day will be suboptimal, which will be costly to slowly correct in the following trading day.

B. *Welfare when Night Characteristics Differ From the Day*

To this point, we have assumed that marginal holding costs and the private value shock process have been the same whether the market is open or closed. However, this is unlikely to be true. In this subsection, we look at the welfare gain (loss) of a short market closure of an hour versus

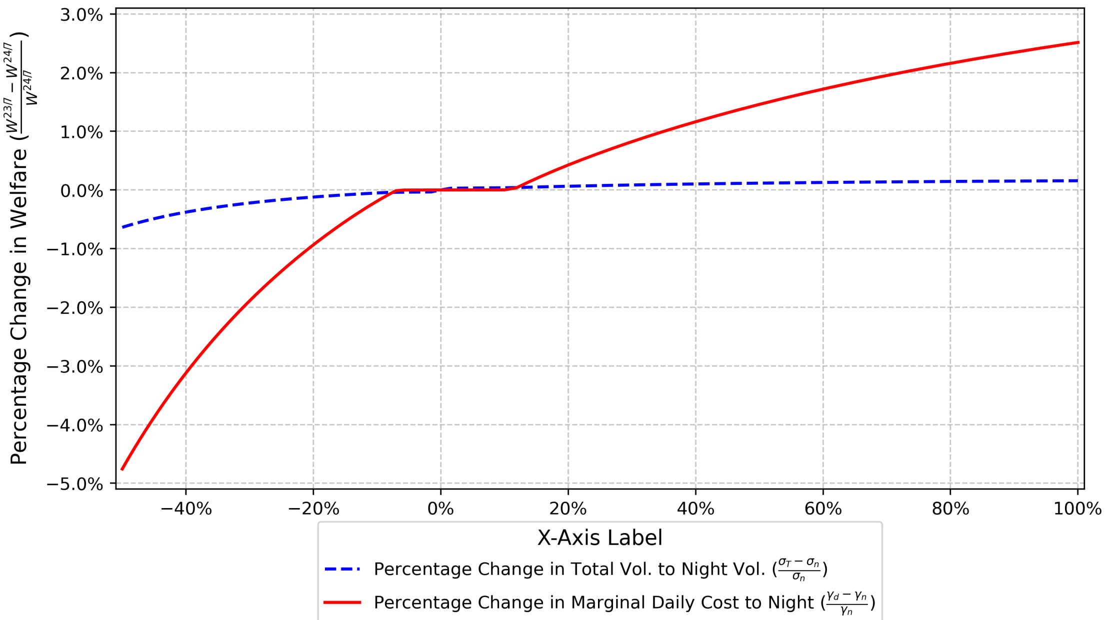


Figure 6. Welfare Change Under Heterogeneity From Day to Night
Above is the percent change between welfare under a market closure of one hour and welfare under 24/7 trade as we vary the marginal holding cost or volatility of the shocks between night to day. The dotted blue line plots the welfare change as a function of marginal holding cost during the day to that of the night. The solid red line plots the welfare change as a function of total volatility, $\sigma_T^2 = (1-\Delta)\sigma_d^2 + \Delta\sigma_n^2$, relative to volatility at night where σ_d solves that equation. Both plots use $K = 100$, $\Delta = 5$, $r = 3.33\%$, $\mu = 10$, $N = 100$, and σ and γ equal 1 unless specified to be different.

24/7 trading when holding costs or shock magnitudes differ from night to day. We choose to focus on the case of a one hour closure as this is the most common closure length of proposed extended hours by the NYSE, Nasdaq, CBOE, and 24X.

Figure 6 plots an example. The blue dotted line varies the volatility of shocks to private values at night while holding the total volatility in a day fixed. Mathematically, $\sigma_d = \sqrt{\frac{\sigma_T^2 - \Delta\sigma_n^2}{1-\Delta}}$. This choice ensure potential gains from trade, which are larger when there are more shocks to private values, are not a function of the length of closure. When volatility at night is less than the total volatility, there is an increase in welfare due to the hour-long closure, and welfare decreases when the night is more volatile. The solid red line plots the change in welfare as a function of the change in the marginal holding cost from day to night. As it becomes cheaper to hold inventory overnight when $\gamma_d > \gamma_n$, there are large welfare gains. When $\gamma_d < \gamma_n$, the short closure rapidly hurts welfare relative to having the market open 24/7. Therefore, any policy recommendation

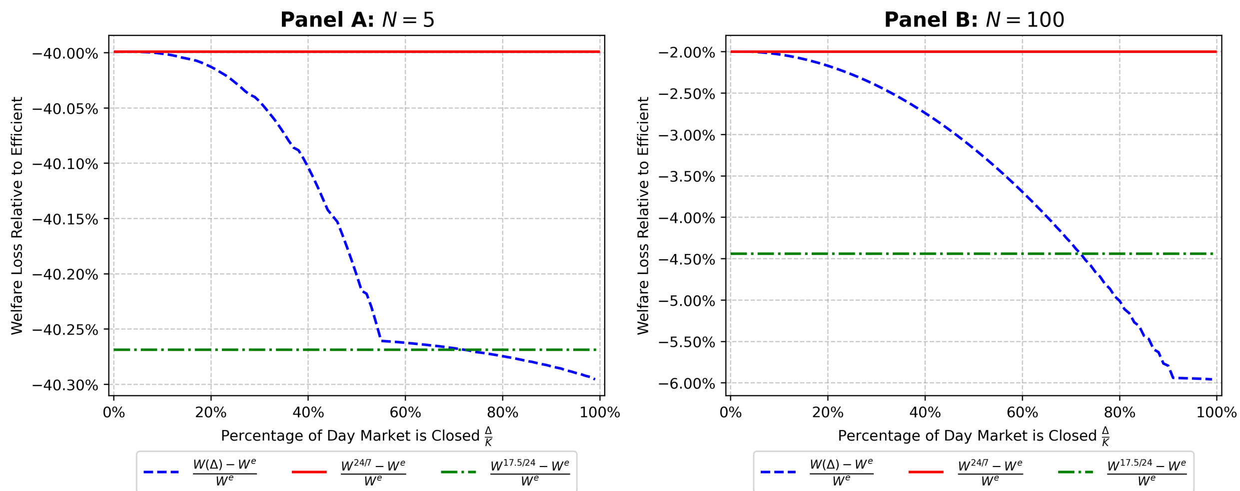


Figure 7. Welfare Loss Relative to Efficient Benchmark
We plot the percent welfare loss under different market designs relative to the first best (efficient) welfare. Panel A plots this loss for a small market, and Panel B plots this loss for a large market. The solid red line is the welfare loss of a market design that is open 24/7 relative to the efficient welfare. The dashed blue line is the welfare loss of a market design that is closed for Δ periods a day relative to the efficient welfare. The dashed-and-dotted green line is the welfare loss of a market design that is closed for 17.5 hours a day, such as many equity exchanges, relative to the efficient welfare. Both plots use $K = 100$, $r = 10\%$, and $\mu = 10$.

should take seriously variation in these parameters, which are also likely endogenous to the length of market closure.

C. The Cost of Imperfect Competition for Differing Closure Lengths

So far, we have focused on comparing welfare under a market structure with 24/7 trade and with a daily closure, ignoring the cost of each relative to the first-best allocation. The first-best allocation would be achieved if there was perfect competition and if trade occurred continuously throughout the day. In this setting, no trader ever holds any undesired inventory. Making comparisons relative to the first-best allocation allows us to better quantify the costs and benefits of market closure.

Figure 7 plots the percentage of welfare loss of different market designs relative to the first-best (efficient) allocations. Panel A is for a small market, and Panel B is for a large market. The solid red line is the welfare loss of a market design with 24/7 trade relative to efficient welfare. The dashed blue line is the welfare loss of a market design that is closed for Δ periods a day relative to efficient welfare. The dashed-and-dotted green line is the welfare loss of a market design that is closed for 17.5 hours a day, such as many equity exchanges, relative to efficient welfare.

As before, the 24/7 market is better for traders than a market with a closure. However, in Panel A, the welfare loss due to closure is very small, less than 1%, relative to the overall welfare loss of imperfect competition, $\approx 40\%$. Take, for example, when $\Delta/K \approx 50\%$. The welfare cost is only an extra 0.18% worse than 24/7 trading, despite only allowing trade for 50% of the day. The endogenous response by traders and coordination of liquidity at the end of the day offset the majority of the extra costs incurred due to the inability to trade at night. The current equity market structure of trading for 6.5 hours a day is associated with about 0.27% extra loss in welfare relative to the efficient benchmark.

In Panel B, when there is a bigger market, closure becomes relatively more costly. Now, trading for 6.5 hours a day has doubled the welfare loss relative to the efficient benchmark. In this larger market, the costs relative to the efficient benchmark are significantly lower, though, due to the imperfect competition friction being less important. Therefore, long closures are fairly costly in these larger and more liquid markets whose liquidity wouldn't endogenously deteriorate too much if trading hours were extended. It is worth noting that these results assume constant volatility and holding costs across the day and night.

Overall, 24/7 trading tends better for traders within the assumptions of the model of this paper. A closure tends to be useful in small markets where shocks are infrequent. Asset classes such as corporate bonds or index CDSs fit this description well. On the other hand, traders in larger markets with frequent shocks to desired positions, such as equities, cryptocurrencies, futures, and foreign exchange markets, are better off in the model with 24/7 trade. However, we caveat that there are other benefits of closure that are not modeled in our paper. Closing prices are heavily used reference prices, and it is not obvious what price to use if there is never a closure. Further, most companies prefer to announce news outside of market hours. Exchanges also need time outside of market hours for updates and maintenance. Finally, managing, collecting, and settling contracts, margin accounts, and collateral need time and a reference close price. These benefits may more than compensate for some small welfare loss in our model from a short closure of an hour.

V. Heterogeneous Information

In this section, we summarize an extension that allows for heterogeneous information regarding the level of the dividend. The main results are analogous to those of previous sections, suggesting our results regarding the effect of a market closure on liquidity and allocative efficiency are robust to the consideration of information problems. The introduction of an information problem is done by adding two components to the model: a stochastic liquidating dividend and private signals regarding its payoff. These components generate a learning problem, discussed below, on top of the inventory management problem discussed in detail in previous sections.

The liquidating dividend is now assumed to evolve according to a jump process. Jumps in the dividend v_t are assumed to coincide with jumps in the taste shocks and are $N(0, \sigma_D^2)$ distributed. Each trader receives private signals about these jumps. If a jump in the dividend level occurs at time t, the signal is given by $\hat{S}_t^i = v_t - v_{t-} + \epsilon$, where $\epsilon \sim N(0, \sigma_\epsilon^2)$. If jumps occurred at dates $t_1 < t_2 < \cdots < t_k$, trader i forms a signal $S^i \equiv \sum_{j=1}^{k} \hat{S}_{t_j}^i$. Assume these normally distributed shocks are all independent of each other and of all shocks in the model. All other aspects of the model are the same as before.

We focus on daily-periodic, linear, and symmetric strategies and conjecture equilibrium demand schedules at time $t = kh + n$ for any integer n take the following form:

$$D_k^i(z^i, w^i, S^i, p) = a_k + b_k p + c_k z^i + f_k(w^i + AS^i).$$

Based on these demand schedules, in equilibrium, any investor will be able to observe $\bar{W} + A\bar{S}$ directly from the price. Note that there is no time dependence in A. This is a technical point, but an important one. If there were time dependence, investors' conditional expectations of the dividend would no longer be a simple function of a few state variables, namely w^i, S^i and $\bar{W} + A\bar{S}$. In particular, time dependence in f would effectively force beliefs to be a state variable of the problem. Any investor i's beliefs would depend on other investors' beliefs, which in turn depend on investor i's beliefs. This loop iterates, leading to an infinite regress of beliefs problem, which the literature has yet to understand how to resolve.

Given the above demand schedules, each investor solves a learning problem. Traders observe w^i, S^i and $\bar{W} + A\bar{S}$, from which they infer the level of the dividend. In particular, conditional beliefs at time $t = kh$ of the value of the dividend are

$$E_t[w_t^i + v_t] = w_t^i + B_1 S_t^i + B_2(\bar{W}_t + A\bar{S}_t),$$

for some constants, B_1, B_2. B_1 and B_2 unsurprisingly depend on A, as the relative weight of the signal from the price on $\bar{Z}$ and $\bar{S}$ affects the learning problem. Conversely, A depends on B_1 and B_2, as optimal demand schedules depend on beliefs. This fixed point problem leads to a straightforward non-linear equation for A.

We provide the solution of this model in the Appendix B. It is fairly straightforward to show that if the learning problem goes away, in the sense that $B_1 = B_2 = 0$, the equilibrium reduces to that described in Proposition 1. Defining $s^i = \frac{1}{\alpha}(w^i + AS^i)$ for a constant α, with a slight relabelling of the demand function, equilibrium demand is given by

$$D_k^i = c_k\Big(z_t^i - \big(\frac{r(N\alpha - 1)}{\lambda(N-1)}(s_t^i - \bar{s}_t) + \bar{Z}\big)\Big).$$

s_k^i is simply a weighted sum of trader i's private taste and their signal. As shown in Figure 8, the main result of this paper still holds when learning is introduced. As the trading day comes to an end, traders trade aggressively towards their desired allocations. As they do so, price impact decreases, further improving liquidity and the incentives to trade aggressively in the final period.

We plot trading intensity and welfare in Figures 8 and 9. We consider the model of this section alongside a model in which σ_ϵ is set to 0 so that information asymmetry is eliminated and alongside a model with information asymmetry but without market closure. In Figure 8, we consider trading intensity by plotting $\prod_{j=0}^{k}(1 + c_j)$ as a function of k. This quantity measures how much of the gap between a trader's initial inventory and initial desired inventory has closed in expectation between the start of the trading day and time t. For both models with closures, trade is most aggressive in the final period. Perhaps unsurprisingly, trading intensity with information asymmetry is slightly slower than without. Traders avoid price impact as doing so increases other's beliefs about the liquidation value, making them even less willing to sell the asset. It is worth noting that this slower trading is due primarily to heterogeneity, not simply uncertainty. If one plots the trading intensity corresponding to a model in which signals are public, it is indistinguishable from the plot in which there is no uncertainty about the dividend.

In the right-hand panel, we see that market closure continues to have consequences for welfare.

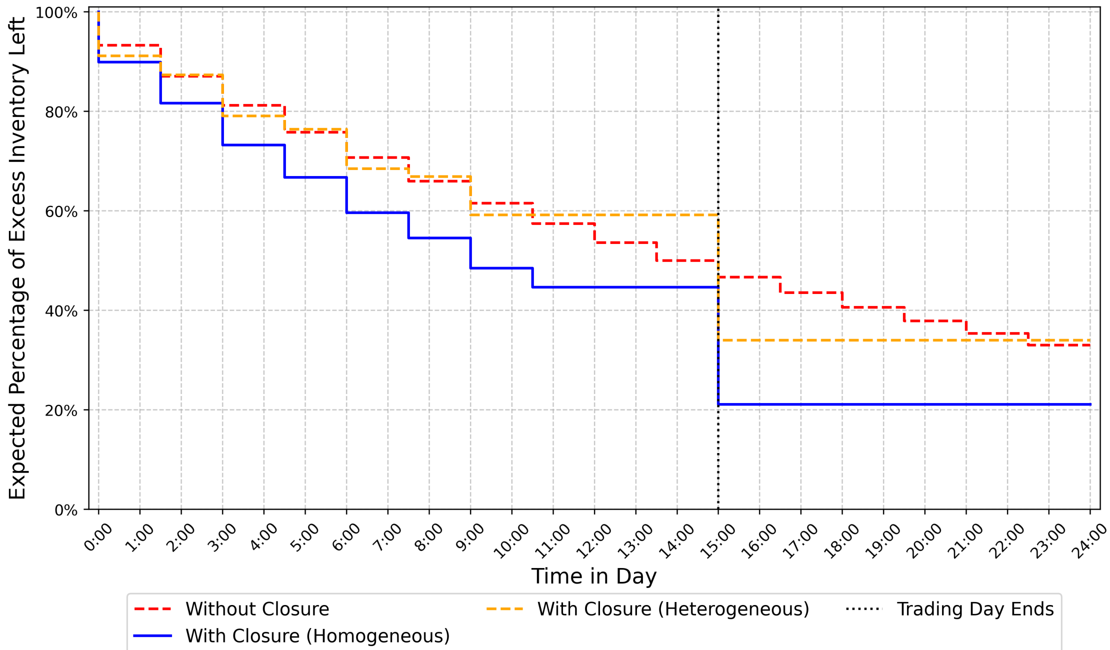


Figure 8. Trading Intensity with Heterogeneous Information
This figure plots trading intensity for various regimes throughout the day. The y-axis is the expected percentage of excess inventory left at point t in the day relative to the position at the start of the day for a given market design. If there is a closure, its length is 31.25% of the day. The parameters are $K = 16$, $N = 100$, and $r = 1/30$. Moreover, $\sigma_D = \sigma_w = 1$, $\sigma_\epsilon = 0.1$, and $\mu = 1$. If information is homogeneous, σ_ϵ is set to 0.

Welfare is better with a sufficiently long closure if the rate of information arrival is sufficiently slow. Moreoever, if the number of traders is sufficiently small, the results of the left panel suggest a closure of roughly 10% of the day is optimal. Relative to Figure 5, welfare with a market closure is slightly better relative to welfare under 24/7 trade when agents have heterogeneous information. This is not particularly surprising since the coordination a closure provides near the end of the trading day is relatively more important when liquidity is already spread thin due to heterogeneous information. Overall, the primary mechanisms of this paper are present when there is heterogeneous information regarding asset values.

Although not the focus of this paper, it is worth discussing any implications the model might have for price efficiency. One can think of price efficiency as the magnitude of a trader's conditional variance of the dividend given their signals and the price, relative to the unconditional variance of the dividend, that is $\frac{\text{Var}_t(v_t)}{\text{Var}(v_t)}$. This value jumps down whenever trading opens, as traders infer information from the price, and increases on average whenever the market closes. Thus, market

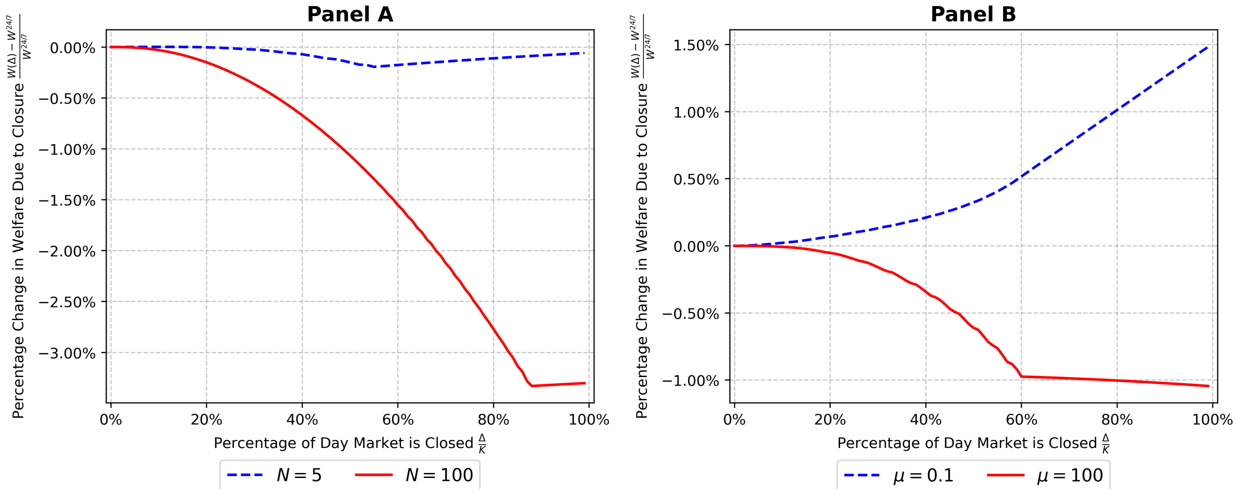


Figure 9. Welfare Comparative Statics with Heterogeneous Information
Above is the percent change between welfare under a market closure and welfare under 24/7 trade as we vary the length of the closure, in the equilibrium with heterogeneous information. Panel A plots this relationship for two different market sizes. Panel B plots this relationship for two different rates of shocks. Both plots assume $\sigma_D = \sigma_w = 1$, $\sigma_\epsilon = 0.1$, $K = 100$ and $r = 10\%$. In Panel A, $\mu = 10$. In Panel B, $N = 10$.

closure hinders price efficiency simply because prices are not observed overnight. Although worth pointing out, this is not a particularly surprising finding, as the information structure we consider is simple enough to make the model tractable. Extensions in which some traders had higher quality signals than others might yield interesting results. For instance, our results suggest that market closure helps coordinate liquidity near the close. If some traders were more informed than others, they might be more willing to trade, improving price efficiency. Can this mechanism outweigh the detrimental effect of closure on price efficiency that we document? More generally, the impact of market closure on the dynamic interaction between allocative efficiency, liquidity, and price efficiency under heterogeneously informed investors promises to yield very interesting research, which we leave to future study.

VI. Conclusion

Despite the rise of electronification in trading and execution, rendering many historical reasons for the existence of market closures obsolete, this paper shows that market closures can still play an important role in the allocative efficiency of market designs. Despite adding a constraint on when trade can occur, we show there exists a market closure length such that trader welfare is higher than a market design where trade can occur 24/7 in small markets with infrequent shocks. This can also be true if marginal holding costs or volatility is lower overnight. This result follows from strategic traders coordinating their aggressive trade at the market closure, which offsets the costs of the inability to trade during the closure itself. We further show that this result is robust to traders having heterogeneous information, which slows down trade but strengthens our main channel.

While our model focuses on the effect of a market's opening hours on allocative efficiency, market closures may play an important role for many other reasons. Closing auction prices are used in the settlement of many derivative contracts, for margin requirements, performance of institutional investors, to price mutual fund shares, and the asset value for ETFs and stock indices. Further, market closures have been used to make announcements without inducing excess short-run volatility in a share price. Both the effect of market closure on the efficiency of closing prices and its interaction with endogenous disclosure decisions are important for policymakers and future research to consider.

Finally, the intersection of market closures and fragmentation is a potentially fruitful path for future research. Traders can effectively trade 24/7 by using other international exchanges, such as the Tokyo or London Stock Exchanges. Yet, as noted in the main text, volume still clusters around the closes and openings of each exchange. Further, there are extended trading hours where trade can occur through electronic communication networks (ECNs), yet these networks suffer from low liquidity and volatile prices. Circumventing closures on one exchange by routing trade through other means may limit the costs of limited trading hours for some securities. Moreover, if aggressive trade due to closure on one exchange leads to spillovers of aggressive trade onto other exchanges, some of the downsides corresponding to markets designed with 24/7 trading may be mitigated.

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Mathematical Appendix

Appendix A. Proofs of Propositions 1-3

This appendix proceeds as follows. First we set up the problem which describes equilibrium. Then, in Appendix A.1 we construct a solution, and describe some of its properties. Appendix A.2 solves the model with a halt in trade, and Appendix A.3 specializes to the case in which $\Delta = 0$, so that there is no overnight period.

Under the assumption of linear demand schedules, and based on the form of the payoffs, the value function will be linear-quadratic:[13]

$$V_k(z^j, w^j, \bar{W}) = a_0^k + a_1^k z^j + a_2^k w^j + a_3^k \bar{W} + a_4^k (z^j)^2 + a_5^k (w^j)^2 + a_6^k (\bar{W})^2 + a_7^k z^j w^j + a_8^k z^j \bar{W} + a_9^k w^j \bar{W}.$$

First, we characterize its solution. The Bellman equation for every time $t = kh$, where $t < T$, is

$$\begin{aligned} V_k(z^j, w^j, \bar{W}) = \max_{D^j} \Bigg\{ & -D^j p_t^* + (1 - e^{-rh})(z^j + D^j)(v + w^j) - \frac{(1 - e^{-rh})\gamma_d}{2r}(z^j + D^j)^2 \\ & + e^{-rh} \Big[a_0^{t+1} + a_1^{k+1}(z^j + D^j) + a_2^{k+1} w^j + a_3^{k+1} \bar{W} \\ & a_4^{k+1}(z^j + D^j)^2 + a_5^{k+1}((w^j)^2 + \sigma^2) + a_6^{k+1}(\bar{W}^2 + \frac{\sigma^2}{N}) \\ & + a_7^{k+1}(z^j + D^j) w^j + a_8^{k+1}(z^j + D^j)\bar{W} + a_9^{k+1}(w^j \bar{W} + \frac{\sigma^2}{N}) \Big] \Bigg\}, \end{aligned}$$

and for the last period, it is

$$\begin{aligned} V_T(z^j, w^j, \bar{W}) = \max_{D^j} \Bigg\{ & -D^j p_T^* + (1 - e^{-rh(1+\Delta)})(z^j + D^j)(v + w) - \frac{(1 - e^{-rh(1+\Delta)})\gamma_n}{2r}(z^j + D^j)^2 \\ & + e^{-rh(1+\Delta)} \big[a_0^0 + a_1^0(z^j + D^j) + a_2^0 w^j + a_3^0 \bar{W} \\ & a_4^0(z^j + D^j)^2 + a_5^0((w^j)^2 + \sigma^2) + a_6^0(\bar{W}^2 + \frac{\sigma^2}{N}) \\ & + a_7^0(z^j + D^j) w^j + a_8^0(z^j + D^j)\bar{W} + a_9^0(w^j \bar{W} + \frac{\sigma^2}{N}) \Big] \Bigg\}. \end{aligned}$$

[13] One can apply a contraction mapping theorem to show uniqueness. First, one can restrict the decision space to a compact subset of the set of linear demand functions. Value iteration will map the set of bounded continuous functions into itself, assuming a Feller-type condition regarding continuity of the conditional expectation of the continuation value, and assuming boundedness is defined using a weighted norm of the form $||f|| = \sup |f(t, z, w, \bar{W}) e^{-||(z,w,\bar{W})||_2^2}|$. Then, using Blackwell's conditions along with the Contraction Mapping Theorem, one gets uniqueness on any compact subset of linear demand functions.

The FOC for optimal demand in the first $T-1$ periods is then

$$0 = -p_t^* - \lambda_k D^j + (1-e^{-rh})(v+w^j) - \frac{(1-e^{-rh})\gamma_d}{r}(z^j + D^j) + e^{-rh}[a_1^{k+1} + 2a_4^{k+1}(z^j+D^j) + a_7^{k+1}w^j + a_8^{k+1}\bar{W}],$$

and in the last trading session of the day

$$0 = -p_T^* - \lambda_T D^j + (1-e^{-rh(1+\Delta)})(v+w^j) - \frac{(1-e^{-rh(1+\Delta)})\gamma_n}{r}(z^j + D^j) + e^{-rh(1+\Delta)}[a_1^0 + 2a_4^0(z^j+D^j) + a_7^0 w^j + a_8^0\bar{W}].$$

where $\lambda_k := \frac{\partial \Phi_t}{\partial d^j} = -\frac{1}{b_k(N-1)}$. Assume

$$D_k^j = a_k + b_k p_t + c_k z^j + f_k w^j.$$

Market clearing implies the equilibrium price is

$$p_t = -\frac{a_k + c_k\bar{Z} + f_k\bar{W}_t}{b_k},$$

and equilibrium demand is

$$D_k^j = c_k(z_t^j - \bar{Z}) + f_k(w_t^j - \bar{W}_t).$$

Substituting these expressions into the FOC,

$$\frac{a_k + c_k\bar{Z} + f_k\bar{W}}{b_k} + \frac{1}{b_k(N-1)}(c_k(z^j-\bar{Z}) + f_k(w^j-\bar{W})) + (1-e^{-rh})(v+w^j) - \frac{(1-e^{-rh})\gamma_d}{r}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W})) + e^{-rh}\left[a_1^{k+1} + 2a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W})) + a_7^{k+1}w^j + a_8^{k+1}\bar{W}\right] = 0,$$

and

$$\frac{a_T + c_T\bar{Z} + f_T\bar{W}}{b_T} + \frac{1}{b_T(N-1)}(c_T(z^j-\bar{Z}) + f_T(w^j-\bar{W})) + (1-e^{-rh(1+\Delta)})(v+w^j) - \frac{(1-e^{-rh(1+\Delta)})\gamma_n}{r}((1+c_T)z^j - c_T\bar{Z} + f_T(w^j-\bar{W})) + e^{-rh(1+\Delta)}\left[a_1^0 + 2a_4^0((1+c_T)z^j - c_T\bar{Z} + f_T(w^j-\bar{W})) + a_7^0 w^j + a_8^0\bar{W}\right] = 0.$$

Grouping common terms,

$$\frac{a_k + c_k\bar{Z}}{b_k} - \frac{c_k\bar{Z}}{b_k(N-1)} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} + e^{-rh}a_1^{k+1} - 2e^{-rh}a_4^{k+1}c_k\bar{Z} = 0,$$

$$\frac{c_k}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d(1+c_k)}{r} + 2e^{-rh}a_4^{k+1}(1+c_k) = 0,$$

$$\frac{f_k}{b_k(N-1)} + (1-e^{-rh}) - \frac{(1-e^{-rh})\gamma_d f_k}{r} + 2e^{-rh}a_4^{k+1}f_k + e^{-rh}a_7^{k+1} = 0,$$

$$\frac{f_k}{b_k} - \frac{f_k}{b_k(N-1)} + \frac{(1-e^{-rh})\gamma_d f_k}{r} - 2e^{-rh}a_4^{k+1}f_k + e^{-rh}a_8^{k+1} = 0,$$

and similarly at time T. Note the SOC is equivalent to ca_3 being positive. We show below that $\alpha_7^k + \alpha_8^k = 1$ and hence $f_k = -b_k$ by the 3rd and 4th FOCs. Then

$$b_k = \frac{r\big(N-2-(N-1)e^{-rh}(1-a_7^{k+1})\big)}{(N-1)(\gamma_d(e^{-rh}-1)+2re^{-rh}a_4^{k+1})},$$

$$c_k = \frac{2+(a_7^{k+1}-1)e^{-rh} - N(1+e^{-rh}(a_7^{k+1}-1))}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))},$$

$$f_k = \frac{r(1+e^{-rh}(a_7^{k+1}-1))c_k}{\gamma_d(e^{-rh}-1)+2re^{-rh}a_4^{k+1}},$$

$$a_k = -\frac{c_k(N-2)\bar{Z}}{N-1} + b_k\Bigg(v(e^{-rh}-1) - e^{-rh}a_1^{k+1} + \frac{c_k\gamma_d(e^{-rh}-1)\bar{Z}}{r} + 2e^{-rh}c_k\bar{Z}a_4^{k+1}\Bigg).$$

The expression for c_k simplifies to

$$c_k = \frac{1}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))} - 1.$$

Thus, given the coefficients describing the value function, the demand functions are known. Let us now characterize the value function. Returning to the Bellman equation, we have

$$\begin{aligned}
V_k = {} & (c_k(z^j-\bar{Z}) + f_k(w^j-\bar{W}))(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{f_k}{b_k}\bar{W}) \\
& + (1-e^{-rh})((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W}))(v+w^j) \\
& - \frac{(1-e^{-rh})\gamma_d}{2r}(((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W})))^2 \\
& + e^{-rh}\Big[a_0^{t+1} + a_1^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W})) + a_2^{k+1}w^j + a_3^{k+1}\bar{W} \\
& a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W}))^2 + a_5^{k+1}((w^j)^2+\sigma^2) + a_6^{k+1}(\bar{W}^2 + \frac{\sigma^2}{N}) \\
& + a_7^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W}))w^j + a_8^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W}))\bar{W} + a_9^{k+1}(w^j\bar{W} + \frac{\sigma^2}{N})\Big]
\end{aligned}$$

Ok, now matching coefficients:

$$a_0^k = -\bar{Z}\frac{c_k a_k + c_k^2 \bar{Z}}{b_k} - c_k(1-e^{-rh})v\bar{Z} - \frac{(1-e^{-rh})\gamma_d}{2r}c_k^2\bar{Z}^2$$
$$+ e^{-rh}a_0^{t+1} - e^{-rh}a_1^{k+1}c_k\bar{Z} + e^{-rh}a_4^{k+1}c_k^2\bar{Z}^2 + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N}$$
$$a_1^k = \frac{c_k a_k + c_k^2\bar{Z}}{b_k} + (1-e^{-rh})(1+c_k)v + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1}$$
$$a_2^k = \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} + (1-e^{-rh})(f_k v - c_k\bar{Z}) + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1}$$
$$- e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$
$$a_3^k = -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})f_k v - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} - e^{-rh}f_k a_1^{k+1} + e^{-rh}a_3^{k+1}$$
$$+ e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z}$$
$$a_4^k = -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2$$
$$a_5^k = (1-e^{-rh})f_k - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k$$
$$a_6^k = -\frac{f_k^2}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k$$
$$a_7^k = (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k)$$
$$a_8^t = \frac{c_k f_k}{b_k} + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k - 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_8^{k+1}(1+c_k)$$
$$a_9^t = \frac{f_k^2}{b_k} - (1-e^{-rh})f_k + \frac{(1-e^{-rh})\gamma_d}{r}f_k^2 - 2e^{-rh}a_4^{k+1}f_k^2 - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1}$$

Simplifications:

Let's simplify some of these recursions by using the FOCs:

$$a_0^k = \bar{Z}c_k\bigg(-\frac{a_k + c_k\bar{Z}}{b_k} - (1-e^{-rh})v - \frac{(1-e^{-rh})\gamma_d}{2r}c_k\bar{Z} - e^{-rh}a_1^{k+1} + e^{-rh}a_4^{k+1}c_k\bar{Z}\bigg)$$

$$+ e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N}$$
$$= -\bar{Z}^2c_k^2\bigg(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\bigg) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N}$$

and

$$
\begin{aligned}
a_1^k &= \frac{c_k a_k + c_k^2 \bar{Z}}{b_k} + (1-e^{-rh})(1+c_k)v \\
&\quad + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= c_k\left(\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})v - \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} - e^{-rh}a_1^{k+1} + 2e^{-rh}a_4^{k+1}c_k\bar{Z}\right) \\
&\quad + (1-e^{-rh})(1+c_k)v + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k^2\bar{Z}}{b_k(N-1)} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - 2e^{-rh}c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k}
\end{aligned}
$$

and

$$
\begin{aligned}
a_2^k &= \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} + (1-e^{-rh})(f_k v - c_k\bar{Z}) + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1} \\
&\quad - e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
&= f_k\left(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right) \\
&\quad - (1-e^{-rh})c_k\bar{Z} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
&= f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}
\end{aligned}
$$

$$
\begin{aligned}
a_3^k &= -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})f_k v - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} - e^{-rh}f_k a_1^{k+1} + e^{-rh}a_3^{k+1} \\
&\quad + e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
&= -f_k\left(\frac{a_k}{b_k} + \frac{c_k\bar{Z}}{b_k} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right) \\
&\quad - \frac{f_k c_k}{b_k}\bar{Z} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
&= -\frac{c_k f_k\bar{Z}}{b_k(N-1)} - \frac{f_k c_k}{b_k}\bar{Z} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}
\end{aligned}
$$

and adding these last two, and using the solution to f/b by adding the last two optimality of demand, $a_2^k + a_3^k = e^{-rh}(a_2^{k+1} + a_3^{k+1})$ which implies $a_2 = -a_3$.

$$a_5^t = (1-e^{-rh})f_k - \frac{(1-e^{-rh})\gamma_d}{2r} f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k$$
$$= (1-e^{-rh})\frac{f_k}{2} - \frac{f_k^2}{2b_k(N-1)} + e^{-rh}a_7^{k+1}\frac{f_k}{2} + e^{-rh}a_5^{k+1}$$

$$a_6^k = -\frac{f_k^2}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r} f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k$$
$$= -\frac{f_k^2 N}{2b_k(N-1)} - e^{-rh}a_8^{k+1}\frac{f_k}{2} + e^{-rh}a_6^{k+1}$$

These two imply $a_5^k - a_6^k = -\frac{f_k}{2} + e^{-rh}(a_5^{k+1} - a_6^{k+1})$.

$$a_7^k = (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k)$$
$$= -\frac{f_k}{b_k(N-1)}(1+c_k)$$

and

$$a_8^k = \frac{f_k(1+c_k)}{b_k(N-1)} - \frac{f_k}{b_k}$$

Adding the equations for a_7, a_8,

$$a_7^k + a_8^k = -\frac{f_k}{b_k} = (1-e^{-rh}) + e^{-rh}(a_7^{k+1} + a_8^{k+1}).$$

$$a_9^k = \frac{f_k^2}{b_k} - (1-e^{-rh})f_k + \frac{(1-e^{-rh})\gamma_d}{r} f_k^2 - 2e^{-rh}a_4^{k+1}f_k^2 - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1}$$
$$= \frac{f_k^2}{b_k} + \frac{f_k^2}{b_k(N-1)} + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1}$$
$$= \frac{2f_k^2}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d f_k^2}{r} + 2e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_9^{k+1}$$
$$= \frac{f_k^2(2+c_k)}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}$$

Therefore, we have

$$a_0^k = -\bar{Z}^2 c_k^2 \left(\frac{1}{b_k(N-1)} + e^{-rh} a_4^{k+1} \right) + e^{-rh} a_0^{t+1} + e^{-rh} a_5^{k+1} \sigma^2 + e^{-rh} a_6^{k+1} \frac{\sigma^2}{N} + e^{-rh} a_9^{k+1} \frac{\sigma^2}{N} \tag{17}$$

$$a_1^k = \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k \bar{Z}}{b_k}$$

$$a_2^k = f_k \frac{c_k \bar{Z}}{b_k(N-1)} - (1 - e^{-rh}) c_k \bar{Z} + e^{-rh} a_2^{k+1} - e^{-rh} a_7^{k+1} c_k \bar{Z}$$

$$a_3^k = -\frac{c_k f_k N \bar{Z}}{b_k(N-1)} + e^{-rh} a_3^{k+1} - e^{-rh} a_8^{k+1} c_k \bar{Z}$$

$$a_4^k = -\frac{(1 - e^{-rh})\gamma_d}{2r}(1 + c_k)^2 + e^{-rh} a_4^{k+1} (1 + c_k)^2$$

$$a_5^k = (1 - e^{-rh}) \frac{f_k}{2} - \frac{f_k^2}{2b_k(N-1)} + e^{-rh} a_7^{k+1} \frac{f_k}{2} + e^{-rh} a_5^{k+1} \tag{18}$$

$$a_6^k = -\frac{f_k^2 N}{2b_k(N-1)} - e^{-rh} a_8^{k+1} \frac{f_k}{2} + e^{-rh} a_6^{k+1} \tag{19}$$

$$a_7^k = -\frac{f_k}{b_k(N-1)}(1 + c_k)$$

$$a_8^k = \frac{c_k f_k}{b_k} - \frac{f_k(N-2)}{b_k(N-1)}(1 + c_k)$$

$$a_9^k = \frac{f_k^2(2 + c_k)}{b_k(1 + c_k)(N-1)} + e^{-rh} a_9^{k+1} \tag{20}$$

for $t < T$. There is an analogous recursion at time T.

Appendix A.1. Construction of solution

In this subsection we describe the solution for a_7. If we have a solution for a_7, that yields solutions for c. This then allows for a solution for a_4, since the recursion is linear. Solutions of a_7, c, a_4 yield solutions for b, f, and the remaining coefficients, which solve linear recursions. We have

$$a_7^k = \frac{1}{(N-1)^2(1 + e^{-rh}(a_7^{k+1} - 1))}$$

for $k = 0, \ldots, T-1$. Then, at time T,

$$a_7^T = \frac{1}{(N-1)^2(1 + e^{-r(1+\Delta)h}(a_7^0 - 1))}$$

Setting $a_7^0 = d$ for some constant d. We can write the solution in terms of a quadratic equation in d. Write $\delta = e^{-rh}$. The constant term in the quadratic equation is

$$
\begin{aligned}
&-2\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&+2\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&+\delta^{T+1+\Delta}\Bigg[\left((-1+\delta)\delta^{-(T+1)} - (-1+\delta)\delta^{-(T+1+\Delta)}\right)(N-1)^2 \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad-\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg) \\
&\quad+\left(\delta^{-(T+1)} - \delta^{-(T+1+\Delta)}\right)\sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)} \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad+\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg)\Bigg].
\end{aligned}
$$

The coefficient on the first order term is

$$
\begin{aligned}
&\delta^{T+1+\Delta}\Bigg[\left(-2\delta^{-T} + (1-\delta)\delta^{-(T+1)} + \delta^{-(T+1+\Delta)}(1+\delta)\right)(N-1)^2 \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad-\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg) \\
&\quad-\left(\delta^{-(T+1)} - \delta^{-(T+1+\Delta)}\right)\sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)} \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad+\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg)\Bigg],
\end{aligned}
$$

and the coefficient on the second order term is

$$
\begin{aligned}
2\delta^{1+\Delta}(N-1)^2\Bigg(&\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&-\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta + (-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg).
\end{aligned}
$$

One can show the discriminant is positive implying one root is positive and one is negative. Note based on the recursions if $a_7^k > 0$ for some k the recursion for a^7 imply all other $a_7^{k'}$ are also positive. Similarly if one of the $a_7^k < 0$ for some k, all the other $a_7^{k'}$ must be negative too. If one were positive, all the subsequent $a_7^{k'}$ would be positive, prohibiting a solution. We show next that only the positive solution can occur in equilibrium.

Characterizing the solution and its existence:

The SOC for demand optimization is given by

$$\frac{1}{b(N-1)} - \frac{(1-e^{-rh})\gamma_d}{r} + 2e^{-rh}a_4^{k+1} < 0$$

First, note since $f/b = -1$, we must have $f > 0$ in equilibrium. By the third FOC above, this fact combined with the SOC implies

$$(1-e^{-rh}) + e^{-rh}a_7^{k+1} > 0.$$

Then, by the expression for f, both c and $\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4^{k+1}$ must have the same sign. Now, the second FOC implies

$$\frac{c_k}{b_k(N-1)} = \frac{(1-e^{-rh})\gamma_d(1+c_k)}{r} - 2e^{-rh}a_4^{k+1}(1+c_k).$$

If $c \leq -1$, the LHS is positive while the RHS is negative. So $c \geq -1$. This fact implies, as hinted at in discussions above, that $a_7 > 0$.

Now since c and $\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4^{k+1}$ have the same sign, we can analyze $\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4^{k+1}$ to determine the sign of c. Let's consider the case $t = 0$. Other cases are similar.

$$\begin{aligned}
a_4^0 &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_0)^2 + e^{-rh}a_4^1(1+c_0)^2 \\
&= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_0)^2 - \frac{(1-e^{-rh})\gamma_d}{2r}e^{-rh}(1+c_0)^2(1+c_1)^2 + e^{-2rh}a_4^2(1+c_0)^2(1+c_1)^2 \\
&= \cdots \\
&= -\frac{(1-e^{-rh})\gamma_d}{2r}\sum_{t=0}^{k} e^{-trh}\prod_{i=0}^{t}(1+c_i)^2 + e^{-(k+1)rh}a_4^{k+1}\prod_{i=0}^{k+1}(1+c_i)^2
\end{aligned}$$

for $k \leq T-1$. $k \geq T$ is similar. In order for positions to be non-explosive functions of past positions, based on the expression for equilibrium demand, we only consider equilibria which imply $\prod_{i=0}^{k}(1+c_i) \to 0$ as $k \to \infty$. Note this also implies, taking the limit in the expansion above, that $a_4^0 < 0$. One can show $a_4^k < 0$ similarly.

This in turn implies $c \leq 0$. It's worth noting that $c \leq 0$ will imply $\prod_{i=0}^{k}(1+c_i) \to 0$.

Thus, we've shown that in equilibrium, a_7 must be positive and c must be between -1 and 0. Thus, the positive solution for a_7 found above is the only candidate. Since

$$c_k = \frac{1}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))} - 1,$$

the candidate a_7 will imply $c \geq -1$. However, $c < 0$ only if a_7 is not too large, or if $(N-1)(1-e^{-rh}) > 1$. If this condition holds, we'll have $c_k < 0$ for all k. Hence, by the expressions for a_4 given above, the solution for a_4 will be negative. Thus, f will be positive and b will be negative, given by the solutions to the first order conditions above. Thus, $(N-1)(1-e^{-rh}) > 1$ is a sufficient condition for trade to occur in every period.

Properties of the solution:

We show four properties of the solution. (1) The first is that if one a_7 is larger than the long-run solution (i.e., the solution in which the market is always open), the "next" one must be smaller. To see this, define

$$f(x) = \frac{1}{(N-1)^2(1-e^{-rh}+e^{-rh}x)}.$$

The long run solution solves the quadratic equation given by $f(x_0) = x_0$. Since for $x > 0$, f is decreasing in x, if $x > x_0$, $y \equiv f(x) < f(x_0) = x_0$. So the next iteration y is less than x_0. The opposite happens if $x < x_0$. So solutions oscillate around the long-run solution when the market is open.

(2) Second, we show the size of the oscillations decrease as one gets further away from the end of trade. To do this, note if $a_7^k = x$, where $k \neq 0, 1$,

$$a_7^{k-2} = f(f(x)).$$

Note the long run solution x_0 solves the quadratic equation $x_0 = f(f(x_0))$. After simplifying, we can write this equation as

$$0 = 1 - (1-e^{-rh})(N-1)^2 x_0 - e^{-rh}(N-1)^2 x_0^2.$$

Note the long-run solution x_0 that we care about is the positive root - it's straightforward to show, like our solution for a_7 above, one root is positive and one is negative, and the quadratic function defined by the right-hand side above is decreasing in positive reals. In particular, if $0 < x < x_0$,

$$1 - (1-e^{-rh})(N-1)^2 x - e^{-rh}(N-1)^2 x^2 > 0,$$

which by reversing the same operations that led us from $f(f(x_0)) = x_0$ to the quadratic equation, implies $f(f(x)) > x$, so that $a_7^{k-2} > a_7^k$. Similarly, if $x > x_0$, then $a_7^{k-2} < a_7^k$. So the oscillations decrease in magnitude as one moves further from the end of trade.

We illustrated these first two properties for a_7. The correspondence between a_7 and c imply analogous results for c.

(3) The third property is that $c_k/f_k = -\gamma/r$. First, recall

$$a_7^k = (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k).$$

Plugging in the expression for f_k derived above, this implies

$$a_7^k = (1 - e^{-rh})(1 + c_k)^2 + e^{-rh} a_7^{k+1} (1 + c_k)^2.$$

Thus, defining $\kappa_k = \frac{2r}{\gamma_d} a_4^k + a_7^k$, we have $\kappa_k = e^{-rh} \kappa_{k+1} (1 + c_k)^2$. for $t < T$, and similarly when $t = T$. This periodic recursion has unique solution $\kappa_k = 0$. Then, the expression for f_k implies $f_k = -\frac{r}{\gamma} c_k$.

(4) The last property is that $a_k/b_k = -v$. Recall the first FOC for optimal demand is

$$\frac{a_k + c_k \bar{Z}}{b_k} - \frac{c_k \bar{Z}}{b_k(N-1)} + (1 - e^{-rh})v + \frac{(1 - e^{-rh})\gamma_d c_k \bar{Z}}{r} + e^{-rh} a_1^{k+1} - 2e^{-rh} a_4^{k+1} c_k \bar{Z} = 0,$$

By the third FOC above, this can be rewritten as

$$0 = \frac{a_k}{b_k} + \frac{c_k}{b_k} \bar{Z} + \frac{c_k \bar{Z}}{f_k} e^{-hr} a_7^{k+1} + (1 - e^{-rh})(v + \frac{c_k \bar{Z}}{f_k}) + e^{-rh} a_1^{k+1}.$$

Then, the recursions for a_1, a_7 imply

$$-\frac{r}{\gamma \bar{Z}} a_1^k + a_7^k = \frac{r}{\gamma \bar{Z}} \left(\frac{a_k}{b_k} + \frac{c_k \bar{Z}}{b_k} \right).$$

Combined, these last two expressions imply

$$-\frac{r}{\gamma \bar{Z}} a_1^k + a_7^k = -\frac{r}{\gamma \bar{Z}} (1 - e^{-rh}) \left(v - \frac{\gamma}{r} \bar{Z} \right) + e^{-rh} (a_7^{k+1} - \frac{r}{\gamma \bar{Z}} a_1^{k+1}).$$

It's straightforward to show this relation also holds when there is no trade, implying $-\frac{r}{\gamma \bar{Z}} a_1^k + a_7^k = -\frac{r}{\gamma \bar{Z}} (v - \frac{\gamma}{r} \bar{Z})$. Plugging this back into the simplified FOC above, we arrive at $\frac{a_k}{b_k} = -v$.

Appendix A.2. Solution when there's a halt in trade for a period

In this subsection, we illustrate how the solution is constructed when there is a halt in trade for a period. If there's a halt in trade of one period before in the penultimate period, then we have

$$a_7^k = \frac{1}{(N-1)^2 (1 + e^{-rh} (a_7^{k+1} - 1))}$$

for $k = 0, \ldots, T-2$. Then, at time T,

$$a_7^T = \frac{1}{(N-1)^2 (1 + e^{-r(1+\Delta)h} (a_7^0 - 1))},$$

and $a_7^{T-1} = (1-e^{-rh}) + e^{-rh}a_7^T$. Setting $a_7^0 = d$ for some constant d. We can write the solution in terms of a quadratic equation in d. The constant term in this quadratic equation is

$$
\begin{aligned}
&2\left(1-\delta^{1+\Delta}\right)(N-1)^2\Bigg[\left((-1+\delta)(N-1)^2-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1} \\
&\quad-\left((-1+\delta)(N-1)^2+\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1}\Bigg] \\
&\quad+(N-1)^2\Big(-\delta(1-\delta)-(N-1)^2(1-\delta)^2(1-\delta^{1+\Delta})\Big) \\
&\quad\Bigg[\left((-1+\delta)(N-1)^2-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1} \\
&\quad-\left((-1+\delta)(N-1)^2+\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1}\Bigg] \\
&\quad-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\Big(-\delta-(N-1)^2(1-\delta)(1-\delta^{\Delta+1})\Big) \\
&\quad\times\Bigg[\left((-1+\delta)(N-1)^2-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1} \\
&\quad+\left((-1+\delta)(N-1)^2+\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1}\Bigg].
\end{aligned}
$$

The coefficient on the second order term is

$$
\begin{aligned}
&(-1+\delta)\,\delta^{1+\Delta}(1+2\delta)(N-1)^4\Bigg[\left((-1+\delta)(N-1)^2-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1} \\
&\quad-\left((-1+\delta)(N-1)^2+\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1}\Bigg] \\
&\quad-\delta^{1+\Delta}(N-1)^3\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\Bigg[\left((-1+\delta)(N-1)^2-\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1} \\
&\quad+\left((-1+\delta)(N-1)^2+\sqrt{(N-1)^2}\sqrt{4\delta+(-1+\delta)^2(N-1)^2}\right)^{T-1}\Bigg].
\end{aligned}
$$

It should be straightforward to see that the product of these two terms is negative, so that there is one positive and one negative root, just as before. For the same reasons as before the positive root will characterize the solution. Moreover, all a_7^k must be positive in equilibrium.

Existence of the solution:

By the arguments above, if the solution doesn't allow for trade in period $T-1$, it must be because the optimal c would be positive. If c were negative, the arguments above imply b would be negative as well, so there would be a solution. We will provide a condition under which the solution with a halt in trade must exist if the solution with trade in every period does not.

We first show a_7^{T-1} and a_7^{T-2} are large enough for trade to occur in periods $T-2$ and $T-3$. Since there's no trade in period $T-1$,

$$a_7^{T-1} = (1 - e^{-rh}) + e^{-rh} a_7^T > 1 - e^{-rh}$$

For trade to exist in period $T-2$, we need $c_{T-2} < 0$. It is sufficient that

$$1 < (N-1)\left(1 - e^{-rh} + e^{-rh}(1 - e^{-rh})\right) = (N-1)(1 - e^{-2rh}).$$

This is weaker than the sufficient condition we postulated for trade to occur in every period. It is straightforward to show via numerical examples that this condition is not meaningless, i.e., there are parameters for which $1 - e^{-2rh} > \frac{1}{N-1} > 1 - e^{-rh}$ and only a halt for exactly one period exists.

Confirming a_7^{T-2} is sufficiently large is slightly less straightforward. First note $c_{T-1} > 0$ if

$$(N-1)(1 + e^{-rh}(a_7^T - 1)) < 1,$$

which simplifies to

$$a_7^T < \frac{1 - (N-1)(1 - e^{-rh})}{e^{-rh}(N-1)}.$$

Then,

$$a_7^{T-1} = (1 - e^{-rh}) + e^{-rh} a_7^T < \frac{1}{N-1}.$$

Using this, c_{T-3} to be negative, it suffices to have

$$\begin{aligned}
1 < (N-1)&\left(1 - e^{-rh} + e^{-rh}\left(\frac{1}{(N-1)^2(1 - e^{-rh} + \frac{e^{-rh}}{N-1})}\right)\right) \\
&= (N-1)(1 - e^{-rh}) + e^{-rh}\frac{1}{(N-1)(1 - e^{-rh}) + e^{-rh}} \\
&\qquad = \frac{(N-1)^2(1 - e^{-rh})^2 + e^{-rh}(N-1)(1 - e^{-rh}) + e^{-rh}}{(N-1)(1 - e^{-rh}) + e^{-rh}}.
\end{aligned}$$

This should rearrange to $(N-1)(N-2)(1 - e^{-rh})^2 > 0$, which holds.

Now, arguments based on the oscillation properties discussed above imply $c_k < 0$ for all periods earlier in the day. Additionally $c_T < 0$ will follow based on the condition $(N-1)(1 - e^{-2rh}) > 1$. Thus $c_k < 0$ is negative in all trade periods, and a repetition of the arguments in the previous section implies the equilibrium exists.

Note, there's can't be an equilibrium with a single halt in an earlier period (period $T-2$ or earlier), unless such an equilibrium involved forgoing trade in periods in which there is a trade equilibrium. In other words, in such an equilibrium, a_7^0, a_7^T would be sufficiently large for trade to occur in periods $T, T-1$. The oscillation properties would imply they would be sufficiently large in any prior period, up to the period in which no trade was allowed. Thus, this equilibrium would enforce no trade even when a trade equilibrium is attainable.

Appendix A.3. 24/7 Trade

It is straightforward to see that when $\Delta = 0$, solutions to the recursions must be constant. The recursions describing the value function reduce to

$$
\begin{aligned}
a_0 &= -\bar{Z}^2 c^2 \left(\frac{1}{b(N-1)} + e^{-rh} a_4 \right) + e^{-rh} a_0 + e^{-rh} a_5 \sigma^2 + e^{-rh} a_6 \frac{\sigma^2}{N} + e^{-rh} a_9 \frac{\sigma^2}{N} && (21)\\
a_1 &= \frac{c(c+1)\bar{Z}}{b(N-1)} - \frac{a + c\bar{Z}}{b} \\
a_2 &= -\frac{c\bar{Z}}{N-1} - (1-e^{-rh})c\bar{Z} + e^{-rh} a_2 - e^{-rh} a_7 c\bar{Z} \\
a_3 &= \frac{cN\bar{Z}}{N-1} + e^{-rh} a_3 - e^{-rh} a_8 c\bar{Z} \\
a_4 &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c)^2 + e^{-rh} a_4 (1+c)^2 \\
a_5 &= (1-e^{-rh})\frac{f}{2} + \frac{f}{2(N-1)} + e^{-rh} \frac{f(1+c)}{2(N-1)} + e^{-rh} a_5 && (22)\\
a_6 &= -\frac{fN}{2(N-1)} - e^{-rh} \frac{f}{2} \left(\frac{N-2}{N-1} - \frac{c}{N-1} \right) + e^{-rh} a_6 && (23)\\
a_7 &= \frac{1+c}{N-1} \\
a_8 &= -c + \frac{N-2}{N-1}(1+c) \\
a_9 &= \frac{cf}{(1+c)(N-1)} + e^{-rh} a_9 && (24)
\end{aligned}
$$

and the equations describing the trade equilibrium reduce to.

$$
b = \frac{r\big(N-2-(N-1)e^{-rh}(1-a_7)\big)}{(N-1)(\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4)},
$$

$$
c = \frac{1}{(N-1)(1+e^{-rh}(a_7-1))} - 1,
$$

$$
f = \frac{r(1+e^{-rh}(a_7-1))c}{\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4},
$$

$$
a = -\frac{c(N-2)\bar{Z}}{N-1} + b\left(v(e^{-rh}-1) - e^{-rh} a_1 + \frac{c\gamma_d(e^{-rh}-1)\bar{Z}}{r} + 2e^{-rh} c\bar{Z} a_4 \right).
$$

Therefore,

$$
c = \frac{-(N-1)(1-e^{-rh}) + \sqrt{(1-e^{-rh})^2(N-1)^2 + 4e^{-rh}}}{2e^{-rh}} - 1.
$$

Given c, we can solve for a_7 and a_4. This yields solutions for b, f, a, and the remaining recursions.

Appendix B. Information problem

This appendix characterizes the solution of the model when agents have heterogeneous asset values. Recall S^j is each trader's total signal (sum of past signals). s^j is each trader's modified signal. Write their expectation of the dividend as

$$w^j + B_1 S^j + B_2 \sum_{i \neq j} \left(w^i + AS^i\right),$$

for some constants B_1, B_2, A. Consistency of the learning problem requires $B_1 = A$. See Du and Zhu (2017) for details. Recall the variance of taste shocks is σ_w^2, of dividend shocks is σ_D^2, and of signal shocks is σ_ϵ^2. Then, Du and Zhu (2017) Lemma 1 gives the conditional expectation of v given w^j, S^j, and $\sum_{i \neq j} \left(w^i + AS^i\right)$ is

$$w^j + \frac{1/(A^2\sigma_\epsilon^2)}{1/(A^2\sigma_D^2) + 1/(A^2\sigma_\epsilon^2) + (n-1)/(A^2\sigma_\epsilon^2 + \sigma_w^2)} S^j + \\ \frac{1/(A^2\sigma_\epsilon^2 + \sigma_w^2)}{1/(A^2\sigma_D^2) + 1/(A^2\sigma_\epsilon^2) + (n-1)/(A^2\sigma_\epsilon^2 + \sigma_w^2)} \frac{1}{A} \sum_{i \neq j} \left(w^i + AS^i\right).$$

B_1 is defined in terms of A by the above. A solves the equation $A = B_1$, and B_2 is then given as a function of A.

Ok, so define

$$s^j = \frac{1}{\alpha}(w^j + B_1 S^j),$$

where

$$\alpha = \frac{A^2\sigma_\epsilon^2 + \sigma_w^2}{NA^2\sigma_\epsilon^2 + \sigma_w^2}.$$

Then, the conditional expectation of v is given by

$$\alpha s^j + \frac{1-\alpha}{N-1} s^{-j} = \frac{N\alpha - 1}{N-1} s^j + \frac{N(1-\alpha)}{N-1} \bar{s}.$$

Guess that the value function is linear-quadratic:

$$V_k(z^j, \bar{Z}, s^j, \bar{s}) = a_0^k + a_1^k z^j + a_2^k s^j + a_3^k \bar{s} + a_4^k (z^j)^2 + a_5^k (s^j)^2 + a_6^k (\bar{s})^2 + a_7^k z^j s^j + a_8^k z^j \bar{s} + a_9^k s^j \bar{s}.$$

$\sigma^2 = \frac{1}{\alpha^2}(\sigma_w^2 + A^2(\sigma_D^2 + \sigma_\epsilon^2))$ is variance of the shock to s^j, and $\sigma_N^2 = \frac{1}{\alpha^2}(\sigma_w^2/N + A^2(\sigma_D^2 + \sigma_\epsilon^2/N))$

is the variance of the shocks to $\bar{s}$. The Bellman equation for every period, except the last, is

$$
\begin{aligned}
V_k(z^j, s^j, \bar{s}) = \max_{D^j} \Bigg\{ & -D^j p_t^* + (1 - e^{-rh})(z^j + D^j) \left(\frac{N\alpha - 1}{N-1} s^j + \frac{N(1-\alpha)}{N-1} \bar{s} \right) - \frac{(1-e^{-rh})\gamma_d}{2r} (z^j + D^j)^2 \\
& + e^{-rh} \Big[a_0^{t+1} + a_1^{k+1}(z^j + D^j) + a_2^{k+1} s^j + a_3^{k+1} \bar{s} \\
& a_4^{k+1}(z^j + D^j)^2 + a_5^{k+1}((s^j)^2 + \sigma^2) + a_6^{k+1}(\bar{s}^2 + \sigma_N^2) \\
& + a_7^{k+1}(z^j + D^j) s^j + a_8^{k+1}(z^j + D^j)\bar{s} + a_9^{k+1}(s^j \bar{s}_t + \sigma_N^2) \Big] \Bigg\},
\end{aligned}
$$

and is similar in the last period. The FOC for optimal demand in the first T periods is then

$$
\begin{aligned}
0 = & -p_t^* - \lambda_k D^j + (1 - e^{-rh}) \left(\frac{N\alpha - 1}{N-1} s^j + \frac{N(1-\alpha)}{N-1} \bar{s} \right) \\
& - \frac{(1-e^{-rh})\gamma_d}{r} (z^j + D^j) + e^{-rh} [a_1^{k+1} + 2a_4^{k+1}(z^j + D^j) + a_7^{k+1} s^j + a_8^{k+1} \bar{s}],
\end{aligned}
$$

where $\lambda_k := \frac{\partial p_t}{\partial D_k^j}$. Assume

$$
D_k^j = a_k + b_k p_t + c_k z^j + f_k s^j.
$$

The equilibrium price is

$$
p_t = -\frac{a_k + c_k \bar{Z} + f_k \bar{s}_t}{b_k}.
$$

The FOC implies

$$
\begin{aligned}
& \frac{a_k + c_k \bar{Z} + f_k \bar{s}}{b_k} + \frac{1}{b_k(N-1)} (c_k(z^j - \bar{Z}) + f_k(s^j - \bar{s})) \\
& + (1 - e^{-rh}) \left(\frac{N\alpha - 1}{N-1} s^j + \frac{N(1-\alpha)}{N-1} \bar{s} \right) - \frac{(1-e^{-rh})\gamma_d}{r} ((1 + c_k) z^j - c_k \bar{Z} + f_k(s^j - \bar{s})) \\
& + e^{-rh} \left[a_1^{k+1} + 2a_4^{k+1}((1 + c_k) z^j - c_k \bar{Z} + f_k(s^j - \bar{s})) + a_7^{k+1} s^j + a_8^{k+1} \bar{s} \right] = 0.
\end{aligned}
$$

Then

$$
b = -\frac{e^{hr} \left(-a_8 + a_7(-2 + N) + (-1 + e^{hr})(-2 + \alpha N) \right) r}{((-1 + a_7 + a_8 + e^{hr})(-1 + N) ((-1 + e^{hr})\gamma - 2a_4 r))}
$$

$$
c = \frac{a_8 - a_7(-2 + N) - (-1 + e^{hr})(-2 + \alpha N)}{a_7(-1 + N) + (-1 + e^{hr})(-1 + \alpha N)}
$$

$$
f = -\frac{\left(-a_8 + a_7(-2 + N) + (-1 + e^{hr})(-2 + \alpha N) \right) r}{(-1 + N) (\gamma - e^{hr}\gamma + 2a_4 r)}
$$

Returning to the Bellman equation, we have

$$
\begin{aligned}
V_k = &(c_k(z^j - \bar{Z}) + f_k(s^j - \bar{s}))(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{f_k}{b_k}\bar{s}) \\
&+ (1 - e^{-rh})((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))\left(\frac{N\alpha - 1}{N - 1}s^j + \frac{N(1 - \alpha)}{N - 1}\bar{s}\right) \\
&- \frac{(1 - e^{-rh})\gamma_d}{2r}(((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})))^2 \\
&+ e^{-rh}\Big[a_0^{t+1} + a_1^{k+1}((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})) + a_2^{k+1}s^j + a_3^{k+1}\bar{s} \\
&a_4^{k+1}((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))^2 + a_5^{k+1}((s^j)^2 + \sigma^2) + a_6^{k+1}(\bar{s}^2 + \sigma_N^2) \\
&+a_7^{k+1}((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))s^j + a_8^{k+1}((1 + c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))\bar{s} + a_9^{k+1}(s^j\bar{s} + \sigma_N^2)\Big]
\end{aligned}
$$

Ok, now matching coefficients:

$$
\begin{aligned}
a_0^k &= -\bar{Z}\frac{c_k a_k + c_k^2\bar{Z}}{b_k} - \frac{(1 - e^{-rh})\gamma_d}{2r}c_k^2\bar{Z}^2 \\
&\quad + e^{-rh}a_0^{t+1} - e^{-rh}a_1^{k+1}c_k\bar{Z} + e^{-rh}a_4^{k+1}c_k^2\bar{Z}^2 + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2 \\
a_1^k &= \frac{c_k a_k + c_k^2\bar{Z}}{b_k} + \frac{(1 - e^{-rh})\gamma_d}{r}(1 + c_k)c_k\bar{Z} + e^{-rh}(1 + c_k)a_1^{k+1} - 2e^{-rh}(1 + c_k)c_k\bar{Z}a_4^{k+1} \\
a_2^k &= \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} - (1 - e^{-rh})c_k\bar{Z}\frac{N\alpha - 1}{N - 1} + \frac{(1 - e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1} \\
&\quad - e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
a_3^k &= -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1 - e^{-rh})c_k\bar{Z}\frac{N(1 - \alpha)}{N - 1} - \frac{(1 - e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} - e^{-rh}f_k a_1^{k+1} + e^{-rh}a_3^{k+1} \\
&\quad + e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
a_4^k &= -\frac{(1 - e^{-rh})\gamma_d}{2r}(1 + c_k)^2 + e^{-rh}a_4^{k+1}(1 + c_k)^2 \\
a_5^k &= (1 - e^{-rh})f_k\frac{N\alpha - 1}{N - 1} - \frac{(1 - e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k \\
a_6^k &= -\frac{f_k^2}{b_k} - (1 - e^{-rh})f_k\frac{N(1 - \alpha)}{N - 1} - \frac{(1 - e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k \\
a_7^k &= (1 - e^{-rh})(1 + c_k)\frac{N\alpha - 1}{N - 1} - \frac{(1 - e^{-rh})\gamma_d}{r}(1 + c_k)f_k + 2e^{-rh}a_4^{k+1}(1 + c_k)f_k + e^{-rh}a_7^{k+1}(1 + c_k) \\
a_8^t &= \frac{c_k f_k}{b_k} + (1 - e^{-rh})(1 + c_k)\frac{N(1 - \alpha)}{N - 1} + \frac{(1 - e^{-rh})\gamma_d}{r}(1 + c_k)f_k - 2e^{-rh}a_4^{k+1}(1 + c_k)f_k + e^{-rh}a_8^{k+1}(1 + c_k) \\
a_9^t &= \frac{f_k^2}{b_k} - (1 - e^{-rh})f_k\frac{N\alpha - 1}{N - 1} + (1 - e^{-rh})f_k\frac{N(1 - \alpha)}{N - 1} + \frac{(1 - e^{-rh})\gamma_d}{r}f_k^2 - 2e^{-rh}a_4^{k+1}f_k^2 \\
&\quad - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1}
\end{aligned}
$$

Simplifications:

Let's simplify some of these recursions by using the FOCs:

$$a_0^k = \bar{Z}c_k\bigg(-\frac{a_k + c_k\bar{Z}}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}c_k\bar{Z} - e^{-rh}a_1^{k+1} + e^{-rh}a_4^{k+1}c_k\bar{Z}\bigg)$$

$$+ e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2$$

$$= -\bar{Z}^2c_k^2\bigg(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\bigg) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2$$

and

$$a_1^k = \frac{c_k a_k + c_k^2\bar{Z}}{b_k}$$

$$+ \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1}$$

$$= c_k\left(\frac{c_k\bar{Z}}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} - e^{-rh}a_1^{k+1} + 2e^{-rh}a_4^{k+1}c_k\bar{Z}\right)$$

$$+ \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1}$$

$$= \frac{c_k^2\bar{Z}}{b_k(N-1)} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - 2e^{-rh}c_k\bar{Z}a_4^{k+1}$$

$$= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k}$$

and

$$a_2^k = \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha - 1}{N-1}$$

$$+ \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1} - e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$

$$= f_k\left(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right)$$

$$- (1-e^{-rh})c_k\bar{Z}\frac{N\alpha - 1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$

$$= f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha - 1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$

$$
\begin{aligned}
a_3^k = &-\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k \bar{Z} \\
&- e^{-rh} f_k a_1^{k+1} + e^{-rh}a_3^{k+1} + e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
&= -f_k\left(\frac{a_k}{b_k} + \frac{c_k\bar{Z}}{b_k} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right) \\
&- \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{1-\alpha}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
&= -\frac{c_k f_k \bar{Z}}{b_k(N-1)} - \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}
\end{aligned}
$$

Then,

$$
\begin{aligned}
a_7^k = (1-e^{-rh})(1+c_k)\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k) \\
= -\frac{f_k}{b_k(N-1)}(1+c_k)
\end{aligned}
$$

and

$$
a_8^k = \frac{c_k f_k}{b_k} + \left(\frac{f_k}{b_k(N-1)} - \frac{f}{b}\right)(1+c_k)
$$

Adding the equations for a_7, a_8,

$$
a_7^k + a_8^k = -\frac{f_k}{b_k} = (1-e^{-rh}) + e^{-rh}(a_7^{k+1} + a_8^{k+1}).
$$

Then,

$$
\begin{aligned}
a_9^k = &\frac{f_k^2}{b_k} - (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} + (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} + \frac{(1-e^{-rh})\gamma_d}{r}f_k^2 \\
&- 2e^{-rh}a_4^{k+1}f_k^2 - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
&= \frac{f_k^2}{b_k} + \frac{f_k^2}{b_k(N-1)} + (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
&= -\frac{(1-e^{-rh})\gamma_d f_k^2}{r} + \frac{2f_k^2}{b_k(N-1)} + 2e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_9^{k+1} \\
&= \frac{2f_k^2}{b_k(N-1)} - \frac{c_k f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1} \\
&= \frac{(2+c_k)f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
a_0^k &= -\bar{Z}^2 c_k^2\left(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\right) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2 \\
a_1^k &= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k} \\
a_2^k &= f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
a_3^k &= -\frac{c_k f_k N\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
a_4^k &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2 \\
a_5^k &= (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k \\
a_6^k &= -\frac{f_k^2}{b_k} - (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k \\
a_7^k &= -\frac{f_k}{b_k(N-1)}(1+c_k) \\
a_8^k &= \frac{c_k f_k}{b_k} - \frac{f_k(N-2)}{b_k(N-1)}(1+c_k) \\
a_9^k &= \frac{(2+c_k)f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}
\end{aligned}$$

Appendix C. Continuous Time Model

We conjecture that the other $N-1$ traders submit demand schedules given by Equation 4. Trade is modeled by a uniform price double auction where the price is the solution to Equation 1. Therefore, the equilibrium price is

$$p_t^* = -\frac{a(t) + c(t)\bar{Z} + f(t)\bar{W}_t}{b(t)}.$$

Note that a, b, and c do not have to be continuous in t at the boundary $1-\Delta$. Given the equilibrium price, the demand schedule evaluated at the equilibrium price, rate allocated in equilibrium, is

$$D_t^i = c(t)\big(z_t^i - \bar{Z}\big) + f(t)(w_t^i - \bar{W}_t).$$

Finally, conjecture the day value function takes the following linear-quadratic form

$$\begin{aligned}
J^d(t, z^i, w^i, \bar{W}) = \alpha_0(t) + \alpha_1(t)z^i + \alpha_2(t)w^i + \alpha_3(t)\bar{W} + \alpha_4(t)(z^i)^2 + \alpha_5(t)(w^i)^2 + \alpha_6(t)(\bar{W})^2 \\
+ \alpha_7(t)z^i w^i + \alpha_8(t)z^i\bar{W} + \alpha_9(t)w^i\bar{W}.
\end{aligned}$$

Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, z^i, W^{-i})$, defined in equation 5. Therefore, the Hamilton-Jacobi-Bellman equation is

$$
\begin{aligned}
rJ^d = \max_{d^i} \big\{ & J_t^d + rz^i(v + w^i) - \Phi(t, d^i, z^i, W^{-i})d^i - \frac{\gamma_d}{2}(z^i)^2 + J_{z^i}^d d^i \\
& + \lambda_d E_t\big[J^d(t, z^i, w^i + \xi^i, \bar{W} + \bar{\xi}) - J^d(t, z^i, w^i, \bar{W})\big]\big\},
\end{aligned}
$$

where $\xi_i \overset{iid}{\sim} N(0, \sigma_d^2)$. Recall $r = -\log(1 - \mathcal{P})$, where $\mathcal{P}$ is the probability of a dividend payment in a given 24 hour period. First, we will solve for the equations that define the α functions and then will add in the optimality of demand constraints. Plugging the conjectured day value function into the HJB equation as well as the equilibrium price and demand schedule, we get

$$
\begin{aligned}
& r(\alpha_0(t) + \alpha_1(t)z^i + \alpha_2(t)w^i + \alpha_3(t)\bar{W} + \alpha_4(t)(z^i)^2 + \alpha_5(t)(w^i)^2 + \alpha_6(t)\bar{W}^2 + \alpha_7(t)z^i w^i + \alpha_8(t)z^i\bar{W} + \alpha_9(t)w^i\bar{W}) \\
&= \alpha_0'(t) + \alpha_1'(t)z^i + \alpha_2'(t)w^i + \alpha_3'(t)\bar{W} + \alpha_4'(t)(z^i)^2 + \alpha_5'(t)(w^i)^2 + \alpha_6'(t)\bar{W}^2 + \alpha_7'(t)z^i w^i + \alpha_8'(t)z^i\bar{W} + \alpha_9'(t)w^i\bar{W} \\
&\quad + z^i r(v + w^i) - \frac{1}{b(t)(N-1)}(c(t)(z^i - \bar{Z}) + f(t)(w^i - \bar{W}))^2 - \frac{\gamma_d}{2}(z^i)^2 \\
&\quad + \lambda_d(\alpha_5(t)\sigma_d^2 + \alpha_6(t)\frac{\sigma_d^2}{N} + \alpha_9(t)\frac{\sigma_d^2}{n}).
\end{aligned}
$$

By matching coefficients, we get that

$$
\begin{aligned}
r\alpha_0(t) &= \alpha_0'(t) - \frac{c(t)^2\bar{Z}^2}{b(t)(N-1)} + \lambda_d(\alpha_5(t)\sigma_d^2 + \alpha_6(t)\frac{\sigma_d^2}{N} + \alpha_9(t)\frac{\sigma_d^2}{N}) \\
r\alpha_1(t) &= \alpha_1'(t) + rv + \frac{2}{b(t)(N-1)}c(t)\bar{Z} \\
r\alpha_2(t) &= \alpha_2'(t) + \frac{2}{b(t)(N-1)}f(t)\bar{Z} \\
r\alpha_3(t) &= \alpha_3'(t) - \frac{2}{b(t)(N-1)}f(t)\bar{Z} \\
r\alpha_4(t) &= \alpha_4'(t) - \frac{\gamma_d}{2} - \frac{c(t)^2}{b(t)(N-1)} \\
r\alpha_5(t) &= \alpha_5'(t) - \frac{f(t)^2}{b(t)(N-1)} \\
r\alpha_6(t) &= \alpha_6'(t) - \frac{f(t)^2}{b(t)(N-1)} \\
r\alpha_7(t) &= \alpha_7'(t) + r - \frac{2f(t)c(t)}{b(t)(N-1)} \\
r\alpha_8(t) &= \alpha_8'(t) + \frac{2f(t)c(t)}{b(t)(N-1)} \\
r\alpha_9(t) &= \alpha_9'(t) + \frac{2f(t)^2}{b(t)(N-1)}.
\end{aligned}
$$

To get the optimality of demand equations, we take the first-order condition of the right side of the HJB equation with respect to d^i. This yields the equation

$$-\Phi - \Phi_{d^i} d^i + J^d_{z^i} = 0.$$

Plugging in the equilibrium expressions for Φ and d^i, we are left with the equations

$$\frac{a(t) + c(t)\bar{Z} + f(t)\bar{W}}{b(t)} + \frac{1}{b(t)(N-1)}(c(t)(z^i - \bar{Z}) + f(t)(w^i - \bar{W})) + \alpha_1(t) + 2\alpha_4(t)z^i + \alpha_7(t)w^i + \alpha_8(t)\bar{W} = 0.$$

Matching coefficients in the above equation gives us three equations that must be satisfied for demand to be optimal,

$$\begin{aligned}
\frac{a(t) + c(t)\bar{Z}}{b(t)} - \frac{1}{b(t)(N-1)}c(t)\bar{Z} + \alpha_1(t) &= 0, \\
\frac{c(t)}{b(t)(N-1)} + 2\alpha_4(t) &= 0, \\
\frac{f(t)}{b(t)(N-1)} + \alpha_7(t) &= 0, \\
\frac{f(t)}{b(t)} - \frac{f(t)}{b(t)(N-1)} + \alpha_8(t) &= 0.
\end{aligned}$$

From optimality of demand, $\alpha_8(t) = -\frac{(N-2)f(t)}{(N-1)b(t)} = (N-2)\alpha_7(t)$. Summing the equations for $\alpha_7(t)$, and $\alpha_8(t)$ we have

$$\alpha_7(t) = A_7 e^{rt} + \frac{1}{N-1}.$$

Plugging this back into the equation for $\alpha_7(t)$,

$$rA_7 e^{\lambda t} + \frac{\lambda}{N-1} = rA_7 e^{rt} + r + 2c\alpha_7(t),$$

so $c(t) = \frac{-r(N-2)}{2(A_7(N-1)e^{rt}+1)}$.

Assume A_4 through A_9 are 0, so α_4 through α_9 are constant too. Then $c = -\frac{r(N-2)}{2}$, and $\alpha_7 = \frac{1}{N-1}$. The equation for α_4 becomes

$$r\alpha_4 = -\frac{\gamma_d}{2} - \alpha_4 r(N-2),$$

so $\alpha_4 = -\frac{2\gamma_d}{r(N-1)}$. This implies

$$b(t) = -\frac{c(t)}{2\alpha_4(N-1)} = -\frac{r^2(N-2)}{2\gamma_d}, \quad \text{and } f(t) = -\alpha_7(N-1)b(t) = \frac{r^2(N-2)}{2\gamma_d}.$$

So, $b(t)$, $c(t)$, and $f(t)$ are all constant between time 0 and $1 - \Delta - \epsilon$. Solving the differential equations for the α's, we get

$$\begin{aligned}
\alpha_0(t) &= \frac{\gamma_d(N-2)\bar{Z}^2}{2r(N-1)} - e^{rt}\int_0^t e^{-rs}\lambda_d\left(\alpha_5(s)\sigma_d^2 + \alpha_6(s)\frac{\sigma_d^2}{N} + \alpha_9(s)\frac{\sigma_d^2}{N}\right)ds + A_0e^{rt} \\
\alpha_1(t) &= A_1e^{rt} + v + \frac{4\gamma_d\bar{Z}}{r^2(N-1)} \\
\alpha_2(t) &= A_2e^{rt} - \frac{2\bar{Z}}{r(N-1)} \\
\alpha_3(t) &= A_3e^{rt} + \frac{2\bar{Z}}{r(N-1)} \\
\alpha_4 &= -\frac{\gamma_d}{2r(N-1)} \\
\alpha_5 &= \frac{r(N-2)}{2\gamma_d(N-1)} \\
\alpha_6 &= \frac{r(N-2)}{2\gamma_d(N-1)} \\
\alpha_7 &= \frac{1}{N-1} \\
\alpha_8 &= \frac{N-2}{N-1} \\
\alpha_9 &= -\frac{r(N-2)}{\gamma_d(N-1)}
\end{aligned}$$

Plugging in α_5, α_6, and α_9 into α_0 and simplifying gives

$$\alpha_0(t) = \frac{\gamma_d(N-2)\bar{Z}^2}{2r(N-1)} - \lambda_d\sigma_d^2\frac{(N-2)}{2\gamma_d N}\left(e^{rt}-1\right) + A_0e^{rt}.$$

Let's add a halt of length ϵ where no trade happens, and then there is a closing auction at time $1-\Delta$. Therefore, the value function right before the halt is

$$\begin{aligned}
J^d(t = 1-\Delta-\epsilon, z^i, w^i, \bar{W}) =& (1-e^{-r\epsilon})\left(z^i(v+w^i) - \frac{\gamma_d}{2r}(z^i)^2\right) \\
&+ e^{-r\epsilon}E_{1-\Delta-\epsilon}\left[J^d(t = 1-\Delta^-, z^i, w^i_{t-\Delta}, \bar{W}_{t-\Delta})\right].
\end{aligned}$$

Now, we move on to the discrete auction at the close, $t = 1-\Delta$. Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, z^i, W^{-i})$, defined in equation 5. Therefore, the Hamilton-Jacobi-Bellman equation is

$$J^d(t = 1-\Delta^-, z^i, w^i, \bar{W}) = \max_{d^i}\{J^n(t = 1-\Delta^+, z^i + d^i, w^i, \bar{W}) - \Phi(1-\Delta, d^i, z^i, W^{-i})d^i\}.$$

We still have the optimality of demand equations that need to be satisfied. To get the optimality of demand equations, we take the first-order condition of the right side of the HJB equation with respect to d^i. This yields the equation

$$-\Phi - \Phi_{d^i} d^i + J^n_{d^i} = 0.$$

Plugging in the equilibrium expressions for Φ and d^i, we are left with the equations

$$\frac{a(1-\Delta) + c(1-\Delta)\bar{Z} + f(1-\Delta)\bar{W}}{b(1-\Delta)} + \frac{1}{b(1-\Delta)(N-1)} d^i + \\ \beta_1(1-\Delta) + 2\beta_4(1-\Delta)(z^i + d^i) + \beta_7(1-\Delta)w^i + \beta_8(1-\Delta)\bar{W} = 0.$$

First, plug in the equilibrium demand for d^i, which gives

$$\frac{a(1-\Delta) + c(1-\Delta)\bar{Z} + f(1-\Delta)\bar{W}}{b(1-\Delta)} + \frac{1}{b(1-\Delta)(N-1)} (c(1-\Delta)(z^i - \bar{Z}) + f(1-\Delta)(w^i - \bar{W})) \\ + \beta_1(1-\Delta) + 2\beta_4(1-\Delta)((1 + c(1-\Delta))z^i - \bar{Z}c(1-\Delta) + f(1-\Delta)(w^i - \bar{W})) \\ + \beta_7(1-\Delta)w^i + \beta_8(1-\Delta)\bar{W} = 0.$$

Matching coefficients in the above equation gives us three equations that must be satisfied for demand to be optimal,

$$\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)} - \frac{1}{b(1-\Delta)(N-1)} c(1-\Delta)\bar{Z} + \beta_1(1-\Delta) - 2\beta_4(1-\Delta)c(1-\Delta)\bar{Z} = 0,$$

$$\frac{c(1-\Delta)}{b(1-\Delta)(N-1)} + 2\beta_4(1-\Delta)(1 + c(1-\Delta)) = 0,$$

$$\frac{f(1-\Delta)}{b(1-\Delta)(N-1)} + 2\beta_4(1-\Delta)f(1-\Delta) + \beta_7(1-\Delta) = 0,$$

$$\frac{f(1-\Delta)}{b(1-\Delta)} - \frac{f(1-\Delta)}{b(1-\Delta)(N-1)} - 2\beta_4(1-\Delta)f(1-\Delta) + \beta_8(1-\Delta) = 0.$$

Now, we move on to the value function at night.

$$r\big(\beta_0(t) + \beta_1(t)z^i + \beta_2(t)w^j + \beta_3(t)\bar{W} + \beta_4(t)(z^i_t)^2 + \beta_5(t)(z^j)^2 + \beta_6(t)\bar{W}^2 + \beta_7(t)z^j w^j + \beta_8(t)z^j\bar{W} + \beta_9(t)w^j\bar{W}\big) \\ = \beta_0'(t) + \beta_1'(t)z^i + \beta_2'(t)w^j + \beta_3'(t)\bar{W} + \beta_4'(t)(z^i)^2 + \beta_5'(t)(w^i)^2 + \beta_6'(t)\bar{W}^2 + \beta_7'(t)z^j w^j + \beta_8'(t)z^j\bar{W} + \beta_9'(t)w^j\bar{W} \\ + r z^i_t(v + w^j) - \frac{\gamma_n}{2}(z^i_t)^2 + \lambda_n(\beta_5(t)\sigma_n^2 + \beta_6(t)\frac{\sigma_n^2}{N} + \beta_9(t)\frac{\sigma_n^2}{N}).$$

By matching coefficients, we get

$$
\begin{aligned}
r\beta_0(t) &= \beta_0'(t) + \lambda_n(\beta_5(t)\sigma_n^2 + \beta_6(t)\frac{\sigma_n^2}{N} + \beta_9(t)\frac{\sigma_n^2}{N}) \\
r\beta_1(t) &= \beta_1'(t) + rv \\
r\beta_2(t) &= \beta_2'(t) \\
r\beta_3(t) &= \beta_3'(t) \\
r\beta_4(t) &= \beta_4'(t) - \frac{\gamma_n}{2} \\
r\beta_5(t) &= \beta_5'(t) \\
r\beta_6(t) &= \beta_6'(t) \\
r\beta_7(t) &= \beta_7'(t) + r \\
r\beta_8(t) &= \beta_8'(t) \\
r\beta_9(t) &= \beta_9'(t)
\end{aligned}
$$

Solving the above ODEs yields the following equations

$$
\begin{aligned}
\beta_0(t) &= -e^{rt}\int_{1-\Delta}^{t} \lambda_n e^{-rs}\left(\beta_5(s)\sigma_n^2 + \beta_6(s)\frac{\sigma_n^2}{N} + \beta_9(s)\frac{\sigma_n^2}{N}\right) ds + B_0 e^{rt} \\
\beta_1(t) &= B_1 e^{rt} + v \\
\beta_2(t) &= B_2 e^{rt} \\
\beta_3(t) &= B_3 e^{rt} \\
\beta_4(t) &= -\frac{\gamma_n}{2r} + B_4 e^{rt} \\
\beta_5(t) &= B_5 e^{rt} \\
\beta_6(t) &= B_6 e^{rt} \\
\beta_7(t) &= 1 + B_7 e^{rt} \\
\beta_8(t) &= B_8 e^{rt} \\
\beta_9(t) &= B_9 e^{rt}
\end{aligned}
$$

Note that $\beta_0(t)$ can be simplified down to

$$\beta_0(t) = e^{rt}\Bigg(B_0 - \lambda_n\sigma_n^2\big(B_5 + \frac{B_6 + B_9}{N}\big)\big(t - (1-\Delta)\big)\Bigg).$$

All that is left now is to use the boundary conditions to solve for the constants in the solutions for the α's and β's. Recall that the two boundary equations are $J^d(t = 1-\Delta, z^i, w^i, \bar{W}) = J^n(t = 1-\Delta, z^i + c(1-\Delta)(z^i - \bar{Z}) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W})$ and $\lim_{t\to 1^-} J^n(t, z^i, w^i, \bar{W}) = \lim_{t\to 1^-} \mathbb{E}\big[J^d(t =$

$0, z^i, w^i, \bar{W})\big|\mathcal{I}_t\big]$. Therefore, after the closing auction, the night value function is actually

$$\begin{aligned}
&J^n(1-\Delta, z^i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W}) \\
&\quad = \beta_0(1-\Delta) + \beta_1(1-\Delta)\bigg(z_t^i + c(1-\Delta)\big(z_t^i - Z_t\big) + f(1-\Delta)(w^i - \bar{W})\bigg) + \\
&\qquad \beta_2(1-\Delta)w^i + \beta_3(1-\Delta)\bar{W} \\
&\quad + \beta_4(1-\Delta)\bigg(z_i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W})\bigg)^2 + \beta_5(1-\Delta)(w^i)^2 + \beta_6(1-\Delta)\bar{W}^2 \\
&+ \bigg(z_t^i + c(1-\Delta)\big(z_t^i - Z_t\big) + f(1-\Delta)(w^i - \bar{W})\bigg)\big(\beta_7(1-\Delta)w^i + \beta_8(1-\Delta)\bar{W}\big) + \beta_9(1-\Delta)w^i\bar{W}.
\end{aligned}$$

Combining like terms gives and subtracting off the costs of the trade gives the value at the start of night of

$$\begin{aligned}
&J^n(1-\Delta, z^i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W}) - \Phi(1-\Delta, d^i, z^i, W^{-i})d^i \\
&= \beta_0(1-\Delta) - \beta_1(1-\Delta)c(1-\Delta)\bar{Z} + \beta_4(1-\Delta)c(1-\Delta)^2\bar{Z}^2 - c(1-\Delta)\bar{Z}\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)} \\
&+ \bigg(\beta_1(1-\Delta)(1 + c(1-\Delta)) - 2\beta_4(1-\Delta)\bar{Z}(1 + c(1-\Delta)) + c(1-\Delta)\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)}\bigg)z^i \\
&\quad + \bigg(\beta_1(1-\Delta)f(1-\Delta) + \beta_2(1-\Delta) - 2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z} - \beta_7(1-\Delta)\bar{Z}c(1-\Delta) \\
&\qquad + f(1-\Delta)\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)}\bigg)w^i \\
&\quad + \bigg(-\beta_1(1-\Delta)f(1-\Delta) + \beta_3(1-\Delta) + 2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z} - \beta_8(1-\Delta)\bar{Z}c(1-\Delta) \\
&\qquad - f(1-\Delta)\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)} - \frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\bar{Z}\bigg)\bar{W} \\
&\qquad + \beta_4(1-\Delta)(1 + c(1-\Delta))^2(z^j)^2 \\
&\qquad + \bigg(\beta_4(1-\Delta)f(1-\Delta)^2 + \beta_5(1-\Delta) + f(1-\Delta)\beta_7(1-\Delta)\bigg)(w^j)^2 \\
&\qquad + \bigg(\beta_4(1-\Delta)f(1-\Delta)^2 - \beta_8(1-\Delta)f(1-\Delta) + \beta_6(1-\Delta) - \frac{f(1-\Delta)^2}{b(1-\Delta)}\bigg)\bar{W}^2 \\
&\qquad + \bigg(2\beta_4(1-\Delta)(1 + c(1-\Delta))f(1-\Delta) + (1 + c(1-\Delta))\beta_7(1-\Delta)\bigg)z^j w^j \\
&\quad + \bigg(-2\beta_4(1-\Delta)(1 + c(1-\Delta))f(1-\Delta) + (1 + c(1-\Delta))\beta_8(1-\Delta) + \frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\bigg)z^j\bar{W} \\
&+ \bigg(-2\beta_4(1-\Delta)f(1-\Delta)^2 - f(1-\Delta)\beta_7(1-\Delta) + f(1-\Delta)\beta_8(1-\Delta) + \beta_9(1-\Delta) + \frac{f(1-\Delta)^2}{b(1-\Delta)}\bigg)w^j\bar{W}
\end{aligned}$$

Finally, the value function before the halt satisfies

$$e^{r\epsilon}J^d(t=1-\Delta-\epsilon,z^i,w^i,\bar{W})-(e^{r\epsilon}-1)\left(z^i(v+w^i)-\frac{\gamma_d}{2r}(z^i)^2\right)$$
$$=J^n(t=1-\Delta^-,z^i,w^i,\bar{W})+e^{r\epsilon}\lambda_d\sigma_d^2\epsilon\frac{r(N-2)}{2\gamma_d N}$$

Let's write out the boundary conditions in more detail. The boundary conditions at $t=1-\Delta$ are

$$e^{r\epsilon}\alpha_0(1-\Delta-\epsilon)-e^{r\epsilon}\lambda_d\sigma_d^2\epsilon\frac{r(N-2)}{2\gamma_d N}$$
$$=\beta_0(1-\Delta)-\beta_1(1-\Delta)c(1-\Delta)\bar{Z}+\beta_4(1-\Delta)c(1-\Delta)^2\bar{Z}^2-c(1-\Delta)\bar{Z}\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_1(1-\Delta-\epsilon)-(e^{r\epsilon}-1)v$$
$$=\beta_1(1-\Delta)(1+c(1-\Delta))-2\beta_4(1-\Delta)\bar{Z}(1+c(1-\Delta))+c(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_2(1-\Delta-\epsilon)$$
$$=\beta_1(1-\Delta)f(1-\Delta)+\beta_2(1-\Delta)-2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z}-\beta_7(1-\Delta)\bar{Z}c(1-\Delta)$$
$$+f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_3(1-\Delta-\epsilon)$$
$$=-\beta_1(1-\Delta)f(1-\Delta)+\beta_3(1-\Delta)+2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z}-\beta_8(1-\Delta)\bar{Z}c(1-\Delta)$$
$$-f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)}-\frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\bar{Z},$$
$$e^{r\epsilon}\alpha_4(1-\Delta-\epsilon)+(e^{r\epsilon}-1)\frac{\gamma_d}{2r}=\beta_4(1-\Delta)(1+c(1-\Delta))^2,$$
$$e^{r\epsilon}\alpha_5(1-\Delta-\epsilon)=\beta_4(1-\Delta)f(1-\Delta)^2+\beta_5(1-\Delta)+f(1-\Delta)\beta_7(1-\Delta),$$
$$e^{r\epsilon}\alpha_6(1-\Delta-\epsilon)=\beta_4(1-\Delta)f(1-\Delta)^2-\beta_8(1-\Delta)f(1-\Delta)+\beta_6(1-\Delta)-\frac{f(1-\Delta)^2}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_7(1-\Delta-\epsilon)-(e^{r\epsilon}-1)=2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta)+(1+c(1-\Delta))\beta_7(1-\Delta),$$
$$e^{r\epsilon}\alpha_8(1-\Delta-\epsilon)$$
$$=-2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta)+(1+c(1-\Delta))\beta_8(1-\Delta)+\frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_9(1-\Delta-\epsilon)$$
$$=-2\beta_4(1-\Delta)f(1-\Delta)^2-f(1-\Delta)\beta_7(1-\Delta)+f(1-\Delta)\beta_8(1-\Delta)+\beta_9(1-\Delta)+\frac{f(1-\Delta)^2}{b(1-\Delta)}.$$

and the boundary conditions at $t=1$ are

$$\alpha_i(0)=\beta_i(1),$$

for $i=0,1\ldots,9$.

Solving these, with the four optimality of demand equations from the closing auction, gives that

$$c(1-\Delta) = -\frac{(N-2)(1-e^{-\Delta r})}{e^{-\Delta r}+(1-e^{-\Delta r})(N-1)},$$

$$\epsilon = \max\Bigg\{0, \min\Bigg\{1-\Delta, \frac{1}{r}ln\Bigg[\frac{(N-1)\big(\gamma_d-\gamma_n(1+c(1-\Delta))^2\big)-e^{-r\Delta}(1+c(1-\Delta))^2(\gamma_d-\gamma_n(N-1))}{\gamma_d(N-2)}\Bigg]\Bigg\}\Bigg\}.$$

Assume that $\bar{Z}=0$ and $v=0$, then A_0 is simply

$$A_0 = \frac{(N-2)\big(e^r\lambda_d\sigma_d^2+e^{r(\Delta+\epsilon)}\lambda_d\sigma_d^2(\epsilon r-1)+\Delta r\lambda_n\sigma_n^2\big)}{2\gamma_d(e^r-1)N}.$$

The value function during that halt is

$$J^d(t,z^i,w^i,\bar{W}) = (1-e^{-r(1-\Delta-t)})\left(r(v+w^i)z^i-\frac{\gamma_d}{2r}\big(z^i\big)^2\right) + e^{-r(1-\Delta-t)}J^n(t=1-\Delta^-,z^i,w^i,\bar{W}) + e^{-r(1-\Delta-t)}\lambda_d\sigma_d^2(1-\Delta-t)e^{r\epsilon}\frac{r(N-2)}{2\gamma_d N}$$

The average welfare, before the close, is

$$\bar{W}(\Delta) := \frac{1}{1-\Delta}\int_0^{1-\Delta}\mathbb{E}\left[J^d(t,0,w^i,\bar{W})\right]dt.$$