Complementarity of Bank and Market Debt in Dynamic Debt Structure[†]

Jessica Shi Li[‡]

Booth School of Business, University of Chicago

This Version: April 14, 2025

Abstract

Firms often issue both bank debt and market debt. Using firm-level data, I show that greater proportion of bank debt in the debt structure predicts higher subsequent market debt issuance, suggesting a dynamic complementarity between bank and market debt. The pattern is more pronounced for smaller, less profitable and riskier firms as well as during periods of stress. These empirical findings shed new light on debt structure dynamics, which existing theories are ill-equipped to explain. I develop a dynamic model of firm debt structure. While market debt investors are arm's-length, the bank lender forms a strategic bilateral relationship with the firm. Thus, while market debt is issued or repurchased at competitive prices, bank debt issuances or repurchases (e.g., those occurring upon maturities) are outcomes of bank-firm bargaining. When the firm has a high (low) proportion of bank debt, bank-firm negotiation would result in bank debt repurchase (issuance), leading to higher (lower) market debt value and more (less) market debt issuance. This effect gives rise to strategic complementarity between bank and market debt. Moreover, repurchases of market debt is possible. The strategic complementarity of bank and market debt is robust to the allocation of bargaining power between the bank lender and the firm. However, the allocation of bargaining power has efficiency implications.

Keywords: Debt Structure, Debt Complementarity, Dynamics, Bargaining, Relationship

---

[‡]5807 S Woodlawn Ave, Chicago, IL 60637. E-mail: jessicali@chicagobooth.edu.

1 Introduction

Firms obtain debt financing through two main sources: bank debt and market debt. A firm's debt choice is not a binary decision. Instead, the vast majority of firms issue both bank debt and market debt. For example, Rauh and Sufi (2010) document that 70% of firm-year observations in their sample have debt structures comprising at least two different types of debt instruments, with bank debt and corporate bonds being the two most commonly used debt types. Despite the prevalence of debt structures that include both bank debt and market debt, a satisfactory theory of debt composition has yet to be fully developed. Existing theories typically focus on the choice between bank debt and market debt rather than how firms combine them in their debt structures. A few models that address the latter either are static in nature or rule out incremental adjustment in debt amounts over time (e.g., Park (2000), Hackbarth et al. (2007), Crouzet (2018)).

I begin my analysis by documenting a novel empirical pattern. Using granular firm-level data, I show that greater proportion of bank debt in the debt structure predicts higher rate of market debt issuance in the subsequent year, suggesting a complementarity between bank debt and market debt. This pattern remains robust to alternative empirical specifications as well as using time-series aggregate-level balance sheet data. Moreover, I show that the degree of complementarity between bank and market debt varies across firm characteristics and time periods. The pattern is more pronounced for smaller, less profitable and riskier firms as well as during periods of market stress. These empirical observations shed new light on firm debt structures and their dynamics. However, existing theories are ill-equipped to reconcile with these findings, underscoring the need for a new theory of firm debt structure, particularly one that accounts for leverage and debt structure dynamics.

I proceed to develop a dynamic model of firm debt structure. In reality, bank debt and market debt could be different on many dimensions such as monitoring incentives, market segmentation, and contract flexibility. Nonetheless, my model is deliberately parsimonious and abstracts from such dimensions. The model follows a canonical trade-off framework, where

the benefit of debt arises from the preferential tax treatment of debt versus equity, and the cost of debt stems from the deadweight loss incurred in the event of default. There is one unique difference between bank debt and market debt in the model. Market debt is issued to or repurchased from arm's-length market debt investors at competitive prices. By contrast, bank debt is obtained through a strategic bilateral relationship between the firm and a bank lender, and thus bank debt issuances or repurchases are outcomes of periodic bank-firm negotiations. In practice, the bank lender and the firm often renegotiate when a bank debt matures and is being rolled over. The renegotiation leads to changes in the firm's amount of bank debt outstanding under negotiated terms.

The model builds on DeMarzo and He (2021) and assumes that the firm cannot commit to future debt choices, including both leverage and debt structure policies. Their setting corresponds to a special case of this model when bank debt is unavailable or when the firm is able to commit to a fixed level of bank debt and thus the bank lender and the firm never negotiate bank debt issuance or repurchase after it is initially put in place. In this special case, the firm always issues market debt to the point that the tax benefit is fully dissipated, a result analogous to the Coase (1972) conjecture of durable goods monopoly. However, as I show in this paper, periodic bank-firm negotiations over bank debt issuances or repurchases lead the firm to issue more or less market debt, depending on whether the amount of bank debt is expected to decrease or increase.

Specifically, when the firm has a high proportion of bank debt in its debt structure, negotiation between the bank lender and the firm would result in repurchase of the bank debt. Reduction in the amount of bank debt boosts the value of market debt, leading to greater current capacity for the market debt and higher rate of market debt issuance. The opposite is true when the firm has a low proportion of bank debt in its debt structure. In this case, bank-firm negotiation would lead to bank debt issuance, which depresses the market debt value and slows market debt issuance. Consequently, the firm tends to issue more market debt when it has a greater amount of bank debt in its debt structure. That is, the effect on the firm's

current market debt issuance decision due to future bargaining outcomes of bank debt issuances or repurchases gives rise to strategic complementarity between bank debt and market debt.

The prediction that bank debt and market debt are complementary has important implications for financial stability and policy-making. For instance, consider the following question. How does a shock to the banking sector affect firms' debt financing and investments? The shock to the banking sector strains banks' balance sheets, causing them to decrease lending to firms. If bank debt and market debt are substitutable, the ability to substitute from bank debt to market debt would mitigate the aggregate impact of the bank credit supply shock. By contrast, the strategic complementarity of bank and market debt in this model would instead amplify the shock. This is because although banks have pulled back from lending, the expectation that bank debt level will increase when the banking sector recovers will constrain market debt capacity today.

Moreover in this model, market debt repurchases are possible. This is in contrast to the leverage ratchet effect of Admati et al. (2018) and DeMarzo and He (2021), which states that the firm never voluntarily repurchases its debt. With only market debt, the firm wishing to buy back debt can always benefit from postponing the repurchase. However, this is no longer the case when the bank lender and the firm periodically negotiate bank debt issuances or repurchases. Postponing the market debt repurchase is not necessarily optimal, as it would change the outcome of the bargaining problem between the bank lender and the firm.

Another implication of the model is that bank debt is senior to market debt in equilibrium.[1] If bank debt is senior, when the bank lender and the firm negotiate to issue or repurchase bank debt, they choose a transaction quantity such that the post-transaction amount of bank debt is positive. This result comes from the "sandwich" capital structure where the senior bank lender and the junior equity holders negotiate to extract from the mezzanine market debt investors. Given the firm's earnings and market debt outstanding, borrowing bank debt leads to higher leverage and earlier default by the firm, which also hurts the bank lender and the equity holders.

---

[1]While other mechanisms may also account for the seniority of bank debt, the model offers another plausible explanation.

However, because bank debt is senior, bank debt issuance dilutes the market debt more, thus enabling the bank lender and the equity holders to capture value from market debt investors. Notably, when bank debt is not senior, the second effect is absent and the firm never issues any bank debt.

The pattern of strategic complementarity between bank debt and market debt is robust to how bargaining power is allocated between the bank lender and the firm. When the market debt investors are absent and there are only the bank lender and the firm, it is a well-known result that the allocation of bargaining power does not affect the total value. However, in the presence of market debt investors as a silent non-bargaining party, this result no longer holds. This is because outcome of the bargaining game between the bank lender and the firm affects the market debt investors' value. Assigning bargaining power to the firm not only weakens the complementarity of bank and market debt, but also reduces the total firm value. Hence, the allocation of bargaining power to claimants of different seniorities has efficiency implications.

The rest of the paper is organized as follows. Section 2 documents the novel empirical pattern of dynamic complementarity between bank and market debt. The empirical findings provide motivation for the dynamic model developed in this paper. Section 3 describes the setup of the general model. Section 4 provides an explicit solution for the case with a strong bank lender. The strong bank lender has all the bargaining power and can be thought to make a take-it-or-leave-it offer to the firm when negotiating bank debt issuances or repurchases with the firm. The case allows for several closed-form results, thus better illustrating the model mechanisms. Section 5 extends the model by introducing a bank lending shock modeled as a shock to the bank lender's discount rate. Section 6 considers the effects of firm bargaining power on the strategic complementarity of bank and market debt as well as the total enterprise value. Section 7 concludes.

4

2 Empirical Evidence

In this section, I document an empirical pattern of complementarity between bank and market debt: a higher proportion of bank debt in the firm's debt structure predicts greater future issuances of market debt. The pattern is robust to various alternative specifications as well as using time-series aggregate data. Furthermore, the degree of complementarity between bank and market debt varies across firms and time, and tends to be weaker for larger and more profitable firms, firms with high credit ratings, dividend-paying firms, as well as during favorable economic conditions.

2.1 Firm-Level Data and Sample Construction

For my main empirical analysis, I rely on firm-level data from multiple data sources. I obtain annual firm characteristics and accounting data from Compustat for the period between 2002 and 2024. I restrict the sample to non-financial firms by excluding all firms whose NAICS codes start with 52, which correspond to the finance and insurance sector. I end up with a sample containing 144,016 firm-year observations involving 15,383 unique firms. I then supplement the Compustat data with entity-level credit ratings obtained from the S&P Credit Ratings dataset, which provides a complete history of rating changes by Standard & Poor's (S&P). In particular, I focus on the local currency LT credit rating assigned by S&P. To facilitate the analysis, I give a numeric value to each notch of the rating, with 1, 2, 3, 4, ..., denoting AAA, AA+, AA, AA−, ..., respectively.

I obtain debt structure data from the Capital Structure dataset of Capital IQ, and then merge the data into the Compustat dataset. The Capital Structure dataset categorizes firms' total debt into seven mutually exclusive debt types: commercial paper, drawn credit lines, term loans, senior bonds and notes, subordinated bonds and notes, capital leases, and other debt. The dataset allows me to compute firm-year level observations of outstanding bank debt and market debt respectively. In the baseline, I follow Becker and Ivashina (2014) and define bank debt as

term loans, while market debt includes senior bonds and notes as well as subordinated bonds and notes. For robustness, I also account for short-term credit by expanding the definition of bank debt to include drawn credit lines and the definition of market debt to include commercial paper.

After removing any observation with missing debt structure information from the sample, I get 126,227 firm-year observations involving 14,592 unique firms. Following the literature, I further eliminate any firm-year observation with less than $10 million in assets. The ending sample contains 108,046 firm-year observations and 12,608 unique firms. Among the firm-year observations, 32% have both term loans and senior or subordinated bonds and notes, while 42% have both term loans or drawn credit lines and bonds/notes or commercial paper at the same time. Since I am interested in firms' debt structures and the complementarity between bank and market debt, I focus on firms that have access to both types of debt financing. Accordingly, I restrict the sample to observations with non-zero amounts of both bank debt and market debt outstanding.

Table 1 reports summary statistics of key financial variables for the estimation sample (columns 2-4), and for the subsample of firm-year observations with available S&P credit ratings (columns 5-7). Market value refers to the market value of equity, and is calculated as the product of the closing share price and the total number of shares outstanding. Book leverage is the sum of current and long-term debt divided by total book assets. Debt-to-EBITDA is another measure of firm leverage, and is the ratio of total debt to operating income before depreciation and amortization. Tangibility is measured as net property, plant, and equipment scaled by total assets. Turnover is calculated as total sales divided by total assets. Net profit margin is net income divided by total sales. Return on assets (ROA) is defined as operating income before depreciation and amortization divided by total assets. Market-to-book is computed as the sum of the market value of equity, total debt, and preferred stock, divided by total assets. Dividend payout is the fraction of net income distributed to shareholders as dividends. Bank debt share is the amount of bank debt divided by the sum of bank and market debt, and captures a firm's debt structure. All continuous financial variables are winsorized at the 1st and 99th percentiles.

Table 1: Summary Statistics of Key Financial Variables

	All Observations ($N = 34,424$)			Rated Only ($N = 17,506$)		
	Mean	Median	Std. Dev	Mean	Median	Std. Dev
Book Assets	12,000	2,362	29,414	22,317	6,731	43,065
Total Sales	7,147	1,255	17,755	13,175	3,720	26,400
Market Value	9,556	1,531	23,905	17,577	4,626	34,998
Book Leverage	0.421	0.379	0.258	0.423	0.386	0.223
Debt-to-EBITDA	3.434	2.989	9.723	4.204	3.414	6.059
Tangibility	0.337	0.275	0.259	0.361	0.316	0.249
Turnover	0.786	0.624	0.649	0.736	0.602	0.554
Net Profit Margin	-0.252	0.032	2.168	0.036	0.051	0.205
ROA	0.072	0.098	0.164	0.116	0.109	0.069
Market-to-Book	1.352	1.071	1.026	1.232	1.054	0.689
Dividend Payout	0.290	0.000	1.139	0.385	0.138	1.344
Bank Debt Share	0.399	0.347	0.309	0.305	0.238	0.266
Entity Rating				11.05	11.00	3.627

Note: The table presents summary statistics of key financial variables for all firm-year observations in the estimation sample, and those observations with available S&P credit ratings. Market value equals the product of the closing share price and the total number of shares outstanding. Book leverage is total debt divided by total book assets. Debt-to-EBITDA is the ratio of total debt to operating income before depreciation and amortization. Tangibility is net property, plant, and equipment divided by total assets. Turnover is total sales divided by total assets. Net profit margin is net income divided by total sales. ROA is operating income before depreciation and amortization divided by total assets. Market-to-book is the sum of the market value of equity, total debt, and preferred stock, divided by total assets. Dividend payout is the fraction of net income distributed as dividends. Bank debt share is the amount of bank debt divided by the sum of bank and market debt. All continuous financial variables are winsorized at the 1st and 99th percentiles. The estimation sample is constructed using data from Compustat, S&P Credit Ratings, and Capital IQ for the period between 2002 and 2024.

Among the firm-year observations with available S&P credit ratings, the mean entity rating is 11 (or BB+). Firms in the rated-only sample tend to be larger (e.g., higher book assets, total sales, and market value) and more profitable (e.g., higher net profit margin and ROA), have higher dividend payout ratio, slightly greater leverage and increased asset tangibility. The mean bank debt share is 0.305 for the rated-only sample, lower than the 0.399 in the full sample. That is, firms with available S&P credit ratings tend to have less bank debt in their debt structures on average.

2.2 Complementarity of Bank and Market Debt

To investigate the complementarity between bank and market debt, I begin by plotting the relationship between bank debt share and market debt net issuance rate in the following year. Bank debt share, as previously defined, equals a firm's bank debt outstanding divided by the

total amount of bank and market debt. Market debt net issuance rate is the net increase in market debt scaled by the total amount of bank and market debt. I first sort the observations within each year into 20 equal-sized bins based on the distribution of bank debt share within the year. Each bin thus groups observations with similar values of bank debt share within the given year. Then for each bin-year, I compute the mean bank debt share and the mean market debt net issuance rate. I then average these bin-year mean values across all years. Finally, I plot the resulting binned means into a two-dimensional binscatter graph, with bank debt share on the horizontal axis and market debt net issuance rate on the vertical axis.

Figure 1: Binscatter Plots of Bank Debt Share and Subsequent Debt Issuances
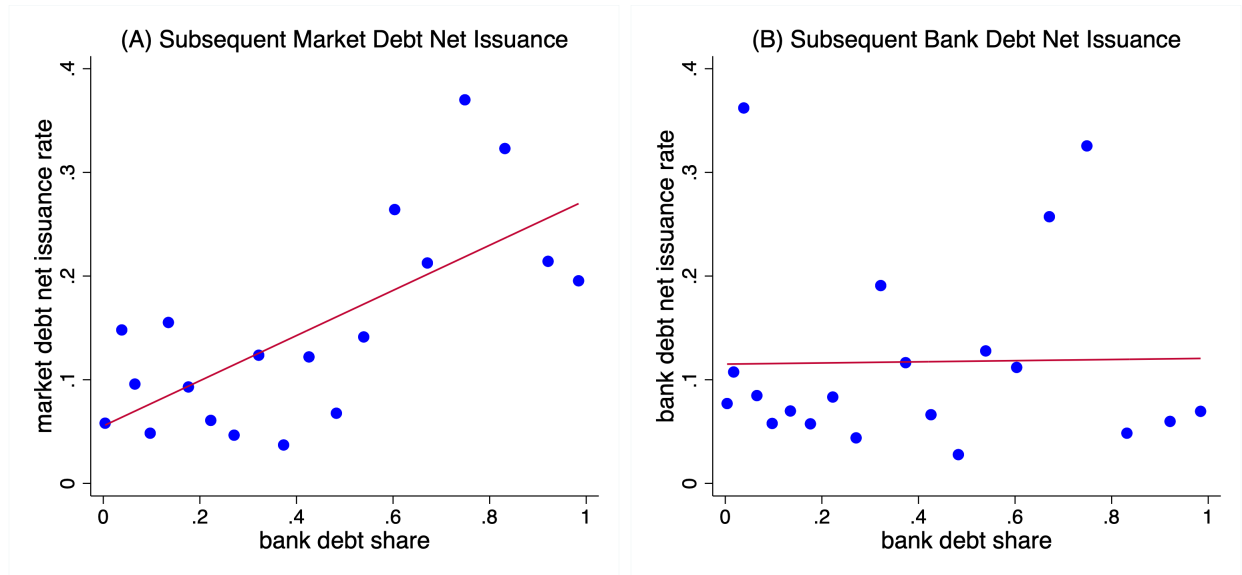
Note: Panel A of the figure plots market debt net issuance rate (defined as the net increase in market debt outstanding during the following year scaled by the total amount of bank and market debt) against the bank debt share (defined as the amount of bank debt divided by the sum of bank and market debt) in a binscatter graph with 20 equal-sized bins. Panel B of the figure plots bank debt net issuance rate (defined as the net increase in bank debt outstanding during the following year scaled by the total amount of bank and market debt) against the bank debt share in a binscatter graph with 20 equal-sized bins. The estimation sample is constructed using data from Compustat, S&P Credit Ratings, and Capital IQ for the period between 2002 and 2024.

The plot is shown in Panel A of Figure 1. It shows a positive relationship between the market debt net issuance rate in the following year and the prevailing bank debt share. In other words, a higher proportion of bank debt in the debt structure predicts greater subsequent market debt issuance, suggesting a complementarity between bank debt and market debt. Although the

plot offers suggestive evidence that bank debt and market debt are complementary, a couple of main concerns need to be addressed.

First, the observed positive relationship between bank debt share and subsequent market debt net issuance may simply reflect firms rebalancing toward an optimal debt composition. That is, firms with higher-than-optimal bank debt shares accelerate market debt issuance to bring their debt structures back toward the optimal mix. However, the same logic implies that such firms should also reduce their bank debt issuance, leading to a negative relationship between bank debt share and subsequent net issuance of bank debt. Panel B of Figure 1 shows that this is not the case, suggesting that the observed positive relationship between bank debt share and subsequent market debt net issuance rate cannot be simply explained by mean-reversion to a target debt structure.

Another concern is that the bank debt reported by Capital IQ may include bridge loans, which are intended to be replaced by bond issuances in the near term. To finance investment projects or acquisitions, firms often initially turn to bank lenders for debt commitments in the form of bridge loans. Bridge loans are short-term facilities that provide interim financing until long-term securities, such as bonds, can be issued to permanently fund the transactions. However, if the observed increase in market debt issuance following a rise in bank debt share is primarily driven by firms replacing bridge loans with bonds, one would expect to see a corresponding decrease in bank debt net issuance. This is inconsistent with the observed pattern shown in Panel B of Figure 1.

Lastly, the positive relationship between bank debt share and subsequent market debt net issuance rate could be a mechanical artifact of secular trends in debt composition over the sample period. For instance, if bank debt share were in steady decline, earlier years would naturally exhibit higher bank debt shares alongside faster market debt issuance in subsequent periods. The within-year binning and averaging procedure employed in constructing the binscatter addresses this concern. To further mitigate this concern, I plot the time series of firm-averaged

9

bank debt shares by year in Figure 2. Bank debt shares during the sample period appear relatively stable, fluctuating around an average of about 0.4.

Figure 2: Average Bank Debt Share Over Time
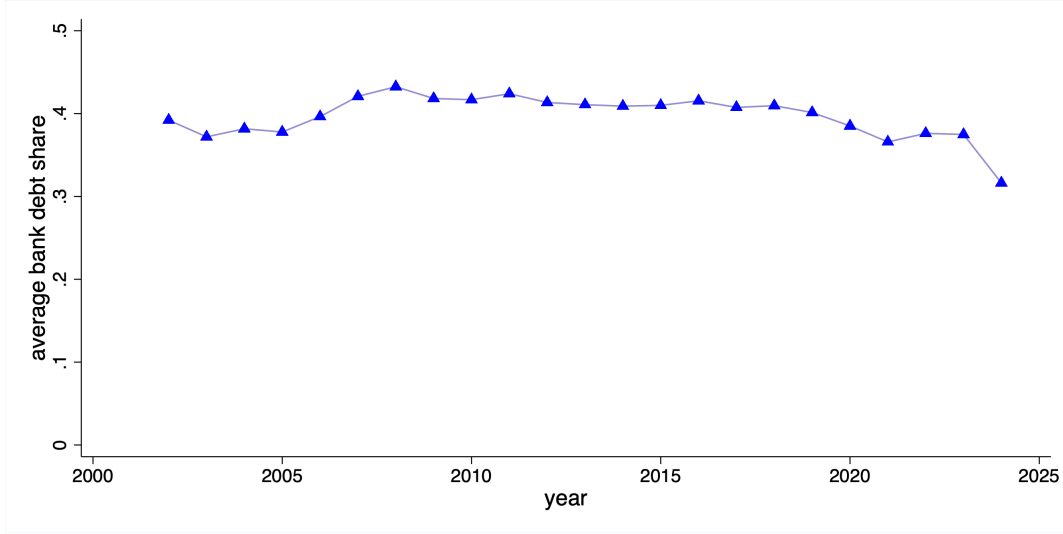
Note: The figure plots the time series of firm-averaged bank debt shares by year, using data from Compustat and Capital IQ for the period between 2002 and 2024. Average bank debt share is computed by taking the average of bank debt shares across all firms within a year, where bank debt share is defined as the amount of bank debt divided by the sum of bank and market debt.

2.3 Regression Analysis

The pattern shown in Figure 1 is correlational. Several confounding factors can distort the interpretation of the positive relationship between bank debt share and subsequent market debt net issuance rate. A major concern is that firm characteristics may affect both the debt structure and market debt issuance behavior. For instance, firms with more tangible assets to pledge as collateral may enjoy better access to both bank debt financing and market debt financing. As a result, these firms may have higher proportions of bank debt in their debt structures and simultaneously issue more market debt, mechanically generating a positive correlation between the two. To address such omitted variable bias, I estimate the following regression specification

$$Market\ Debt\ Net\ Issuance\ Rate_{it} = \beta Bank\ Debt\ Share_{it} + \gamma X_{it} + \alpha_i + \alpha_t + \epsilon_{it}, \quad (2.1)$$

10

where i indexes firm and t indexes year. *Market Debt Net Issuance Rate$_{it}$* is the net increase in firm i's market debt outstanding from year t to $t+1$ divided by the firm's total amount of bank and market debt in year t. *Bank Debt Share$_{it}$* equals firm i's bank debt outstanding in year t scaled by its total amount of bank and market debt in year t. α_i is firm fixed effect which controls for time-invariant characteristics within a firm. α_t is year fixed effect which controls for time-specific shocks common to all firms. Moreover, X_{it} is a set of controls at the firm-year level. Standard errors are double-clustered at the firm and year levels.

The coefficient of interest is β. A positive β means that an increase in the prevailing bank debt share is associated with a higher rate of market debt net issuance in the following year, after controlling for various confounding factors. I start by including the log of book assets, tangibility, ROA, book leverage, and debt-to-EBITDA in the time-varying firm-level controls (X_{it}). The regression results are shown in Table 2. Column 1 is unconditional on market debt net issuance or repurchase by a firm in a given year, while column 3 is conditional on non-zero market debt changes. In both columns, the coefficient estimates on bank debt share are positive and highly statistically significant.

Another potential issue concerns the timing of bank debt and market debt issuances. Firms with better investment opportunities may issue both bank debt and market debt to finance these opportunities. If bank debt is easier or quicker to obtain than market debt (e.g., due to time required for bond roadshows), firms may initially rely more on bank debt before subsequently ramping up market debt issuance. Accordingly, I include several proxies for investment opportunities as additional control variables. These include market-to-book ratio, stock return, and sales growth. I also include dividend payout as a control variable, since distribution of dividend may also affect a firm's demand for credit. The results are shown in column 2 (unconditional on market debt net issuance or repurchase) and in column 4 (conditional on non-zero market debt issuance or repurchase).

Inclusion of the additional control variables reduces the magnitude of the coefficient estimates on bank debt share. However, the estimates remain positive and highly statistically significant.

Table 2: Panel Regressions: Complementarity of Bank and Market Debt

	Market Debt Net Issuance Rate					
	(1)	(2)	(3)	(4)	(5)	(6)
Bank Debt Share	0.925***	0.791***	0.998***	0.819***	0.391**	0.411**
	(0.229)	(0.163)	(0.248)	(0.174)	(0.144)	(0.152)
Log(Assets)	-0.253***	-0.210***	-0.247***	-0.218***	-0.299	-0.339
	(0.048)	(0.068)	(0.051)	(0.072)	(0.174)	(0.198)
Tangibility	0.515	0.283	0.376	0.0397	0.255	0.208
	(0.382)	(0.502)	(0.341)	(0.388)	(0.257)	(0.289)
ROA	-0.113	-0.551**	-0.205	-0.628**	-0.383	-0.384
	(0.266)	(0.233)	(0.308)	(0.269)	(0.647)	(0.783)
Book Leverage	-1.612***	-1.615***	-1.657***	-1.702***	-1.206**	-1.320**
	(0.287)	(0.408)	(0.287)	(0.392)	(0.506)	(0.578)
Debt-to-EBITDA	0.001	0.000	0.001	0.000	0.001	0.001
	(0.002)	(0.001)	(0.002)	(0.001)	(0.001)	(0.001)
Dividend Payout		-0.007		0.005	0.000	0.001
		(0.008)		(0.009)	(0.003)	(0.003)
Market-to-Book		0.205***		0.226***	0.232**	0.250*
		(0.048)		(0.054)	(0.111)	(0.122)
Stock Return		-0.011		-0.009	-0.005	-0.005
		(0.009)		(0.010)	(0.007)	(0.007)
Sales Growth		-0.022		-0.037	0.118	0.129
		(0.064)		(0.072)	(0.104)	(0.120)
Sample	All	All	All	All	Rated	Rated
Conditional on Issuance	No	No	Yes	Yes	No	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.150	0.141	0.203	0.204	0.097	0.103
Observations	28,293	21,762	25,008	19,201	12,036	10,619

Standard errors in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The table presents results of the regression specification (2.1) using data from Compustat, S&P Credit Ratings, and Capital IQ for the period 2002-2024. The outcome variable is market debt net issuance rate defined as the net increase in market debt outstanding during the following year scaled by the total amount of bank and market debt. The coefficients of interest are the coefficients on bank debt share defined as the amount of bank debt divided by the sum of bank and market debt. Columns 1-4 use all firm-year observations from the estimation sample. Columns 5-6 restrict to observations with available S&P credit ratings. Columns 1, 2 ad 5 are unconditional on non-zero market debt changes while columns 3, 4 and 6 are conditional on non-zero market debt changes. In columns 1 and 3, the firm-year level controls include the log of book assets, tangibility, ROA, book leverage, and debt-to-EBITDA. Columns 2 and 4-6 also include dividend payout, market-to-book, stock return, and sales growth as additional control variables. Standard errors are double-clustered at the firm and year levels.

A 1% increase in the prevailing bank debt share is associated with 79-82 basis points higher net issuance rate of market debt in the subsequent year. Finally, I re-estimate the specification with the full set of control variables for the rated-only subsample. The coefficient estimates on bank debt share are positive and statistically significant, though their magnitudes are about half of those in the full estimation sample. Taken together, the regression results in Table 2 are consistent with the pattern documented in Figure 1. In sum, higher bank debt share is associated with greater subsequent net issuance of market debt, implying that bank debt and market debt are complementary.

In Appendix A.1, I provide additional robustness tests. In the first set of robustness tests, I apply alternative definitions for bank debt and market debt. In the second set of robustness tests, I use a matched-sample regression specification by including industry-year-size-leverage fixed effects as the main controls. Finally, I document complementarity between bank and market debt using time-series aggregate balance sheet data obtained from the Flow of Funds.

2.4 Heterogeneity Across Firms and Time

How does the degree of complementarity between bank and market debt vary depending on firm characteristics and time periods? To answer this question, I introduce various dummy variables capturing firm characteristics and market conditions. For example $Large\ Size_{it}$ is a firm-year level indicator variable equal to one if the amount of book assets is above the sample median and zero otherwise. $High\ Sales_{it}$ and $High\ ROA_{it}$ are analogously defined from total sales and ROA respectively. $Div\ Payer_{it}$ is an indicator variable equal to one if firm i pays dividend in year t. $IG\ Rating_{it}$ is an indicator variable equal to one if the entity credit rating assigned by S&P is BBB- or above. $Crisis_t$ indicates crisis periods (including the GFC and COVID-19), and equals one for years 2008, 2009 and 2020. I then interact these variables with $Bank\ Debt\ Share_{it}$ and re-estimate the regression (2.1) with the full set of controls, unconditional on market debt net issuance or repurchase.

The coefficient estimates on the interaction terms are presented in Table 3. The degree of complementarity between bank and market debt tends to be lower for larger firms (e.g., firms with more assets and higher sales), for more profitable firms (e.g., firms with higher ROA), and for firms with investment-grade (IG) credit ratings. Moreover, dividend-paying firms exhibit weaker complementarity between bank and market debt. On the other hand, the degree of complementarity between bank and market debt tends to be amplified during crisis periods such as the GFC and COVID-19.

Table 3: Panel Regressions: Complementarity of Bank and Market Debt

	Market Debt Net Issuance Rate					
	(1)	(2)	(3)	(4)	(5)	(6)
Bank Debt Share×Large Size	-0.667***					
	(0.221)					
Bank Debt Share×High Sales		-0.931***				
		(0.299)				
Bank Debt Share×High ROA			-0.426**			
			(0.204)			
Bank Debt Share×Div Payer				-0.402**		
				(0.162)		
Bank Debt Share×IG Rating					-0.339*	
					(0.173)	
Bank Debt Share×Crisis						0.619**
						(0.274)
FirmFE	Yes	Yes	Yes	Yes	Yes	Yes
YearFE	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.141	0.141	0.141	0.141	0.141	0.141
Observations	21,762	21,762	21,762	21,762	21,762	21,762

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The table regresses market debt net issuance rate in the following year (defined as the net increase in market debt scaled by the total amount of bank debt and market debt) on the interaction terms between the bank debt share variable and various dummy variables capturing firm characteristics and time periods, along with firm-year level controls and firm and year fixed effects. $Large\ Size_{it}$ is a firm-year level indicator variable equal to one if the amount of book assets is above the sample median. $High\ Sales_{it}$ and $High\ ROA_{it}$ are analogously defined from total sales and ROA respectively. $Div\ Payer_{it}$ is an indicator variable equal to one if firm i pays dividend in year t. $IG\ Rating_{it}$ is an indicator variable equal to one if the entity credit rating assigned by S&P is BBB- or above. $Crisis_t$ indicates crisis periods (including the GFC and COVID-19), and equals one for years 2008, 2009 and 2020. Data are from Compustat, S&P Credit Ratings, and Capital IQ for the period 2002-2024. Standard errors are double-clustered at the firm and year levels.

Several potential explanations may account for this heterogeneity across firms and time periods, depending on the underlying mechanisms that drive the complementarity between bank and market debt. One possible explanation is that the degree of complementarity between bank and market debt is affected by the amount of bargaining power the firm has over its bank lender. For a firm that maximizes its shareholders' value, its bargaining power affects the marginal benefit of market debt issuance to shareholders, thereby influencing its market debt issuance behavior. The results in Table 3 seem to be consistent with this explanation. Larger and more profitable firms, as well as dividend-paying firms (which are typically larger and more mature) tend to possess greater bargaining power. Firms with little default risk and firms operating under favorable economic conditions are also likely to enjoy stronger bargaining positions. The coefficient estimates in the table thus suggest that the complementarity between bank and market debt decreases as firm bargaining power increases.

Another potential explanation is that the bank lender's monitoring capability or the contractual flexibility of bank debt generates positive spillover effects on market debt, thereby enhancing the firm's ability to raise market debt financing. To assess the validity of this explanation, I re-estimate the specification (2.1) using only observations in the bottom quintiles of book leverage and market leverage (defined as total debt divided by the sum of the market value of equity, total debt, and preferred stock). These observations have very low financial leverage, with a subsample mean book leverage ratio of 0.08 and a subsample mean market leverage ratio of 0.05. I also run the same specification for observations with entity credit ratings of A− or above, which account for 16% of the rated-only sample and 5% of the total estimation sample. In Table 4, I present the coefficient estimates on the bank debt share variable.

Firms with low leverage and high credit ratings face minimal default risk. As a result, the bank lender's ability to monitor and the contractual flexibility of bank debt (such as the ability to renegotiate terms during distress) offer limited additional value. Hence, if the observed pattern of complementarity between bank and market debt were driven by these features, the pattern should be absent for firms with very low leverage and high credit ratings. However, all coefficient estimates in Table 4 are positive and statistically significant. This suggests that the

Table 4: Complementarity of Bank and Market Debt for Low-Risk Firms

	Market Debt Net Issuance Rate					
	Low Book Lev		Low Mkt Lev		High Ratings	
	(1)	(2)	(3)	(4)	(5)	(6)
Bank Debt Share	1.601***	1.439***	1.460**	1.292**	0.305***	0.277***
	(0.514)	(0.457)	(0.559)	(0.507)	(0.099)	(0.097)
Conditional on Issuance	No	Yes	No	Yes	No	Yes
FirmFE	Yes	Yes	Yes	Yes	Yes	Yes
YearFE	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.374	0.383	0.360	0.369	0.065	0.066
Observations	4,430	3,928	4,116	3,577	1,998	1,929

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The table presents coefficient estimates on bank debt share by estimating the regression specification (2.1), using data from Compustat, S&P Credit Ratings, and Capital IQ for the period 2002-2024. The outcome variable is market debt net issuance rate defined as the net increase in market debt outstanding during the following year scaled by the total amount of bank and market debt. Bank debt share is the amount of bank debt divided by the sum of bank and market debt. Columns 1-2 use observations with book leverage in the bottom quintile. Columns 3-4 use observations with market leverage in the bottom quintile. Columns 5-6 use observations with S&P entity credit ratings of A− and above. All regressions include firm and year fixed effects, and the full set of firm-year level control variables including the log of book assets, tangibility, ROA, book leverage, debt-to-EBITDA, dividend payout, market-to-book, stock return, and sales growth. Standard errors are double-clustered at the firm and year levels.

pattern of complementarity between bank and market debt cannot be primarily attributed to bank monitoring or contractual flexibility, as the pattern is evident even among low-risk firms for which bank monitoring and contractual flexibility are of limited importance.

2.5 Discussion

Note that the empirical specifications employed in this section are dynamic in nature. By examining how current debt structure affects future market debt issuance decisions, the empirical approach captures the evolving nature of debt structure adjustments over time. By contrast, most existing theories are static and are thus not well-suited to explain the dynamic patterns documented in the data. That being said, it is still useful to examine how the empirical patterns documented in this section align with existing theories.

Two leading explanations for capital structure decisions are the trade-off theory and the pecking-order theory. The trade-off theory (e.g., Miller (1977), Leland (1994)) treats bank debt and market debt as substitutes. When a firm carries a high bank debt balance, the trade-off

theory suggests that it should de-lever by issuing less of both types of debt to restore its optimal leverage. Hence, subsequent market debt issuance should be negatively related to current level of bank debt, which contrasts with the observed pattern. The pecking-order theory states that a firm opts for the least costly form of debt financing before turning to more expensive options. The theory, emphasizing a hierarchical substitution among financing sources, also appears inconsistent with the observed complementarity between bank and market debt.

Why does higher proportion of bank debt in the debt structure beget more market debt issuances in subsequent periods? Why is this pattern more pronounced for smaller, less profitable and riskier firms as well as during periods of market stress? To answer these questions, a dynamic theory of firm debt structure is called for.

3   The Model

I develop a model of dynamic debt structure. In the model, a firm is run by its equity holders and borrows from both a relationship bank lender and arm's-length market debt investors. The firm cannot commit to its future debt choices. Instead, the firm dynamically issues or repurchases both its bank debt and its market debt, leading to changes in not only the total firm leverage but also the composition of debt over time.

3.1   General Setup

In the model, all agents are risk-neutral and discount the future at a constant rate $r > 0$. A firm is owned and managed by its equity holders. Let Y_t denote the firm's instantaneous earnings before interest and taxes (EBIT) at time t. I assume that Y_t follows the geometric Brownian motion

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dZ_t, \tag{3.1}$$

17

where the drift μ and the diffusion σ are positive constants, and dZ_t is the increment of a standard Brownian motion. I assume that $r > \mu$.

The firm borrows from both a relationship bank lender and arm's-length market debt investors. The key distinction between the bank lender and the market debt investors is that the former maintains a long-standing financing relationship with the firm. Through this relationship, the bank acquires specialized knowledge of the firm, which causes the bank debt to be illiquid but grants the bank lender bargaining power in negotiations with the firm. By contrast, the market debt is held by dispersed institutional investors such as mutual funds, insurance companies, pension funds and so on. As a result, the market debt investors are arm's-length and competitive.

The firm's equity holders choose when to default. Upon default, the firm is assumed to have liquidation value $\alpha Y^\star$, where $Y^\star$ is the firm's EBIT at default and $0 < \alpha < \frac{1-\pi}{r-\mu}$. That is, the liquidation value is a fraction of the unlevered firm value at default. The assumption thus implies that there is a deadweight cost of default, upon which a portion of the firm's value is lost. If the firm defaults, the equity holders forfeit their equity ownership and receive nothing in recovery. Meanwhile, the liquidation value is divided by the bank lender and the market debt investors subject to the absolute priority rule. The firm is assumed to pay corporate taxes on its earnings after interest at a constant rate $\pi \in (0,1)$. The benefit of debt arises from the tax deductibility of interest expenses.

A. Market Debt

The firm issues a homogeneous class of market debt. The market debt is modeled as a consol with fixed coupon rate $c > 0$ and no stated maturity. The firm may continuously adjust its market debt outstanding by issuing additional market debt or repurchasing existing market debt. Let M_t denote the firm's market debt outstanding at time t. The firm is assumed to issue market debt at an endogenous rate g_t^m such that the firm's market debt outstanding increases by $g_t^m M_t dt$ over the time interval dt. The firm optimally chooses the market debt issuance rate such that the equity holders' value is maximized. Note that g_t^m may be positive or negative,

with $g_t^m > 0$ denoting market debt issuance and $g_t^m < 0$ denoting market debt repurchase. The market debt is priced rationally by the market debt investors, and issuances or repurchases of the market debt occur at the prevailing competitive prices.

B. Bank Debt

The bank debt is borrowed from the bank lender through a bilateral financing relationship that is inherently strategic. Hence, issuances or repurchases of bank debt are outcomes of negotiations between the bank lender and the firm. Specifically, I assume that both the price and the quantity of bank debt issuance (or repurchase) are determined by Nash bargaining between the bank lender and the firm, where the firm has bargaining power $\theta \in [0, 1]$. Let B_t denote the firm's face amount of bank debt at time t. The bank debt is also modeled as a consol with fixed coupon rate c. However, the bank debt's face amount is periodically adjusted through issuance or repurchase transactions bilaterally negotiated between the bank lender and the firm. I model the periodic adjustments in bank debt outstanding by assuming that the bank lender and the firm negotiate bank debt issuances or repurchases at Poisson arrival times with intensity $\lambda > 0$.

An empirically relevant interpretation of λ is the Poisson intensity of bank debt maturities. The bank debt thus has an expected time to maturity of $1/\lambda$. The bank debt can be viewed as periodically maturing and being rolled over. When rolling over the bank debt, the bank lender and the firm may renegotiate the terms of the bank debt. Renegotiations lead to changes in the face amount of bank debt through issuance or repurchase, as well as changes in the pricing through negotiated transfers between the bank lender and the firm.

3.2 Value Functions

If the firm declares default, the equity holders lose their ownership stake and the liquidation value is distributed among the bank lender and the market debt investors. If the firm is not in default, the firm produces $(1 - \pi)\big[Y_t - c(B_t + M_t)\big]dt$ in net income after interest and taxes

from time t to $t + dt$. At Poisson arrival times with intensity λ, the bank lender and the firm negotiate bank debt issuances or repurchases. Furthermore, the firm continuously issues market debt at the endogenous rate g_t^m. Hence, over the time interval dt, the net cash flow available to the firm's equity holders is

$$NC_t = \underbrace{(1-\pi)\big[Y_t - c(B_t + M_t)\big]dt}_{\text{net income after interest and taxes}} + \underbrace{p_t^b d\Gamma_t^b + p_t^m d\Gamma_t^m}_{\text{debt issuance/repurchase}} . \qquad (3.2)$$

Γ_t^b and Γ_t^m are cumulative bank debt and market debt issuances over time, and p_t^b and p_t^m are issuance (or repurchase) prices of the bank debt and the market debt respectively. From time t to $t + dt$, the quantity of bank debt issuance is $d\Gamma_t^b = Q_t^b$ if bank-firm negotiation of bank debt issuance or repurchase takes place, and zero otherwise. The quantity Q_t^b and the corresponding issuance (or repurchase) price p_t^b are determined by Nash bargaining, where the firm is assumed to have bargaining power θ. If $Q_t^b > 0$, the firm issues new bank debt. If $Q_t^b < 0$, the firm repurchases existing bank debt. On the other hand, the firm adjusts its market debt outstanding continuously and $d\Gamma_t^m = g_t^m M_t dt$. In the rest of the paper, I refer to debt issuance and repurchase collectively as *debt issuance*, since debt repurchase can be thought of as negative debt issuance.

The equilibrium concept in the model is that of Markov perfect equilibrium. There are three payoff-relevant states, namely the firm's EBIT Y_t, the face amount of market debt M_t, and the face amount of bank debt B_t. Agents' values are thus functions of Y_t, M_t, and B_t. The firm is run by its self-interested equity holders, who choose the optimal default time $\tau^\star$ and the market debt issuance rate g_t^m to maximize their own value. Let $V^e(Y_t, M_t, B_t)$ denote the equity holders' total value, then

$$V^e(Y_t, M_t, B_t) = \max_{\tau^\star, g_t^m} \mathbb{E}_t\Big[\int_t^{\tau^\star} e^{-r(\tau-t)} NC_\tau\Big]. \qquad (3.3)$$

The equity holders' value function (3.3) can be expressed recursively through the Hamilton-Jacobi-Bellman (HJB) equation. Insofar as the firm is not in default, the equity holders' total

value $V^e(Y_t, M_t, B_t)$ must satisfy the following HJB equation

$$rV^e(Y_t, M_t, B_t) = \max_{g_t^m} \left\{ (1-\pi)\left[Y_t - c(B_t + M_t)\right] + p_t^m g_t^m M_t + g_t^m M_t V_M^e(Y_t, M_t, B_t) \right.$$
$$+ \mu Y_t V_Y^e(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^e(Y_t, M_t, B_t) \tag{3.4}$$
$$\left. + \lambda\left[V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t)\right] \right\}.$$

The left-hand side of (3.4) is the required return of the equity holders. The first two terms on the right-hand side are the net income flow after interest and taxes, and the cash flow from market debt issuance. V_M^e is the first-order partial derivative of the value function with respect to M_t, while V_Y^e and V_{YY}^e are the first-order and the second-order partial derivatives with respect to Y_t. Evolutions of the state variables, as captured by the remaining terms on the right-hand side, give rise to changes in value for the equity holders. In particular, the last line on the right-hand side is the change in the equity holders' value due to negotiations of bank debt issuances, which occur with Poisson intensity λ. Taking the first-order condition of (3.4) with respect to the rate of market debt issuance, one can see that the equity holders' optimal market debt issuance rate g_t^m must be such that

$$p_t^m = -V_M^e(Y_t, M_t, B_t). \tag{3.5}$$

That is, the firm issues market debt until the marginal devaluation of the equity holders' total value due to market debt issuance equals the price at which the market debt is issued.

Similarly, let $V^b(Y_t, M_t, B_t)$ denote the bank lender's value function. From time t to $t + dt$, the bank lender receives $cB_t dt$ in coupon payment as long as the firm is not in default. With Poisson intensity λ, the bank lender and the firm negotiate to issue bank debt at the negotiated price p_t^b. Hence,

$$V^b(Y_t, M_t, B_t) = \mathbb{E}_t\left[\int_t^{\tau^\star} e^{-r(\tau-t)}cB_\tau d\tau - e^{-r(\tau-t)}p_\tau^b d\Gamma_\tau^b\right]. \tag{3.6}$$

21

As long as the firm is not in default, the bank lender's value function $V^b(Y_t, M_t, B_t)$ must satisfy the HJB equation

$$
\begin{aligned}
rV^b(Y_t, M_t, B_t) =& cB_t + g_t^m M_t V_M^b(Y_t, M_t, B_t) \\
& + \mu Y_t V_Y^b(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^b(Y_t, M_t, B_t) \\
& + \lambda\big[V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{3.7}
$$

The first term on the right-hand side of (3.7) is the flow coupon payment on the bank debt. The bank lender's value changes due to evolutions of the state variables Y_t, M_t, and B_t. The last line on the right-hand side is the change in the bank lender's value resulting from negotiations of bank debt issuances, which occur with Poisson intensity λ. By lending an additional Q_t^b in face amount to the firm, the bank lender makes a transfer of $p_t^b Q_t^b$ to the firm's equity holders, while its value changes from $V^b(Y_t, M_t, B_t)$ to $V^b(Y_t, M_t, B_t + Q_t^b)$ reflecting the new amount of bank debt outstanding post-issuance.

The firm continuously issues market debt at the prevailing market debt price. In equilibrium, the market debt price p_t^m is a function of the state variables. Let $p^m(Y_t, M_t, B_t)$ denote the equilibrium market price of the market debt. Without loss of generality, I normalize the face value of each unit of market debt to one. Outside default, a unit of market debt pays cdt in coupon over the time interval dt. Market debt investors are competitive, then

$$
p^m(Y_t, M_t, B_t) = \mathbb{E}_t\left[\int_t^{\tau^\star} e^{-r(\tau-t)}cd\tau\right].
\tag{3.8}
$$

When the firm is not in default, the equilibrium market debt price must thus satisfy the following HJB equation

$$
\begin{aligned}
rp^m(Y_t, M_t, B_t) =& c + g_t^m M_t p_M^m(Y_t, M_t, B_t) \\
& + \mu Y_t p_Y^m(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^m(Y_t, M_t, B_t) \\
& + \lambda\big[p^m(Y_t, M_t, B_t + Q_t^b) - p^m(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{3.9}
$$

22

3.3 Bargaining and Debt Issuances

When the bank lender and the firm negotiate the issuance of the bank debt, they Nash bargain over the issuance quantity Q_t^b and the issuance price p_t^b. Suppose that $\{p_t^b, Q_t^b\}$ are the proposed terms in the bargaining problem. If the firm and the bank lender agree on the terms, the firm's bank debt outstanding becomes $B_t + Q_t^b$ through the issuance of Q_t^b in face amount of bank debt at the negotiated price p_t^b. Accordingly, the equity holders' value and the bank lender's value become $V^e(Y_t, M_t, B_t + Q_t^b)$ and $V^b(Y_t, M_t, B_t + Q_t^b)$ respectively, and the equity holders receive a transfer of $p_t^b Q_t^b$ in issuance proceeds from the bank lender. By contrast, if no agreement is reached between the firm and the bank lender, the equity holders' value and the bank lender's value remain unchanged at $V^e(Y_t, M_t, B_t)$ and $V^b(Y_t, M_t, B_t)$ respectively.

Let $\{\bar{p}_t^b, \bar{Q}_t^b\}$ be the outcome of the Nash bargaining problem between the bank lender and the firm, where the firm has bargaining power θ. Then

$$\{\bar{p}_t^b, \bar{Q}_t^b\} = \underset{p_t^b, Q_t^b}{\arg\max} \ \left[V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t)\right]^\theta$$

$$\left[V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t)\right]^{1-\theta}, \tag{3.10}$$

subject to $V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t) \geq 0$ and $V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t) \geq 0$. The bracketed terms on the right-hand side of (B.99) capture the trade surpluses for the equity holders and the bank lender respectively from the bank debt issuance.

The solution to the Nash bargaining problem (B.99) must satisfy two conditions. First, the transaction terms $\{\bar{p}_t^b, \bar{Q}_t^b\}$ are bilaterally Pareto optimal. Intuitively, if the transaction terms are inefficient, one party can always propose an offer that benefits themselves without making the other party worse off. Second, the transaction terms $\{\bar{p}_t^b, \bar{Q}_t^b\}$ are such that the firm receives θ fraction of the joint surplus, while the bank lender receives $1 - \theta$ fraction of the joint surplus. The solution to the bargaining problem is summarized in the following proposition, with derivations contained in Appendix B.1.

Proposition 1 (Bargaining and Bank Debt Issuance)**.** *Let $\{\bar{p}_t^b, \bar{Q}_t^b\}$ be the bargaining outcome where $\bar{p}_t^b$ is the bank debt issuance price and $\bar{Q}_t^b$ is the issuance quantity. Then*

$$\bar{Q}_t^b = \arg\max_{Q_t^b} \left[V^e(Y_t, M_t, B_t + Q_t^b) + V^b(Y_t, M_t, B_t + Q_t^b) \right], \tag{3.11}$$

and

$$\bar{p}_t^b = \frac{\theta[V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - V^b(Y_t, M_t, B_t)]}{\bar{Q}_t^b} - \frac{(1-\theta)[V^e(Y_t, M_t, B_t + \bar{Q}_t^b) - V^e(Y_t, M_t, B_t)]}{\bar{Q}_t^b}. \tag{3.12}$$

Let $V(Y_t, M_t, B_t)$ denote the sum of the equity holders' total value and the bank lender's value, that is $V(Y_t, M_t, B_t) \equiv V^e(Y_t, M_t, B_t) + V^b(Y_t, M_t, B_t)$. The condition (3.11) thus says that the negotiated amount of bank debt issuance $\bar{Q}_t^b$ must maximize the total joint value of the equity holders and the bank lender. By substituting the condition (3.12) into the trade surpluses for the equity holders and the bank lender, one can see that the equity holders capture a fraction θ of the joint surplus and the bank lender captures a fraction $1 - \theta$ of the joint surplus from bargaining. That is, at $\{\bar{p}_t^b, \bar{Q}_t^b\}$,

$$V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + \bar{p}_t^b \bar{Q}_t^b - V^e(Y_t, M_t, B_t) = \theta\left[V(Y_t, M_t, B_t + \bar{Q}_t^b) - V(Y_t, M_t, B_t)\right], \tag{3.13}$$

and

$$V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - \bar{p}_t^b \bar{Q}_t^b - V^b(Y_t, M_t, B_t)$$
$$= (1-\theta)\left[V(Y_t, M_t, B_t + \bar{Q}_t^b) - V(Y_t, M_t, B_t)\right]. \tag{3.14}$$

By substituting (3.13) into the equity holders' HJB equation (3.4) and using the first-order condition (3.5), one can then rewrite the HJB equation as

$$
\begin{aligned}
rV^e(Y_t, M_t, B_t) =& (1-\pi)\big[Y_t - c(B_t + M_t)\big] \\
&+ \mu Y_t V_Y^e(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^e(Y_t, M_t, B_t) \\
&+ \theta\lambda\big[V(Y_t, M_t, B_t + \bar{Q}_t^b) - V(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{3.15}
$$

Substituting (3.14) into the bank lender's HJB equation (3.7) yields

$$
\begin{aligned}
rV^b(Y_t, M_t, B_t) =& cB_t + g_t^m M_t V_M^b(Y_t, M_t, B_t) \\
&+ \mu Y_t V_Y^b(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^b(Y_t, M_t, B_t) \\
&+ (1-\theta)\lambda\big[V(Y_t, M_t, B_t + \bar{Q}_t^b) - V(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{3.16}
$$

The equilibrium market debt price $p^m(Y_t, M_t, B_t)$ is determined by the first-order condition (3.5). As shown in Appendix B.2, the equity holders' optimal market debt issuance rate g_t^m can be obtained by differentiating (3.15) with respect to M_t and adding up the resultant HJB with the HJB equation for the equilibrium market debt price (3.9). The result is given in the proposition below.

Proposition 2 (Market Debt Issuance). *The optimal rate of market debt issuance chosen by the firm's equity holders is given by*

$$
\begin{aligned}
g_t^m =& \frac{\pi c}{-M_t p_M^m(Y_t, M_t, B_t)} + \frac{\lambda[p^m(Y_t, M_t, B_t + \bar{Q}_t^b) - p^m(Y_t, M_t, B_t)]}{-M_t p_M^m(Y_t, M_t, B_t)} \\
&+ \frac{\theta\lambda[V_M(Y_t, M_t, B_t + \bar{Q}_t^b) - V_M(Y_t, M_t, B_t)]}{-M_t p_M^m(Y_t, M_t, B_t)}.
\end{aligned}
\tag{3.17}
$$

The first term on the right-hand side of (3.17) corresponds to the market debt issuance rate if the bank lender and the firm do not negotiate issuances of the bank debt once the bank debt is in place, that is $\lambda = 0$. In this case, the amount of bank debt remains constant. This expression also appears in DeMarzo and He (2021). It suggests that if the firm can commit

to a fixed level of bank debt by never engaging in renegotiations of the bank debt terms, the firm issues market debt at the rate such that the benefit of market debt issuance (here the tax benefit) is fully dissipated by the negative valuation effect on the market debt investors due to an increase in leverage.

The next two terms represent the effect on the equilibrium rate of market debt issuance due to negotiations of bank debt issuances between the bank lender and the firm. This effect can be further decomposed into two components. The first component, as captured by the second term on the right-hand side of (3.17), reflects the expected impact of negotiated bank debt issuance on the market debt price. For more clarity, consider the limit case with a strong bank lender that holds all the bargaining power (that is $\theta = 0$). The equity holders do not receive any surplus when the bank lender and the firm negotiate to issue the bank debt. Hence, the negotiated bank debt issuance does not affect the equity holders' value. However, an issuance of the bank debt changes the firm's leverage and affects the market debt price. In equilibrium, the price impact on market debt must be offset by the valuation effect of market debt issuance on the market debt investors, resulting in an upward or downward adjustment to the market debt issuance rate. For instance, if the negotiated bank debt issuance is expected to negatively impact the market debt price, the benefit of market debt is reduced, resulting in a lower rate of market debt issuance.

The second component, as captured by the last term on the right-hand side of (3.17), reflects the expected effect of negotiated bank debt issuance due to its impact on equity holders' bargaining surplus. In the limit with $\theta = 0$, the firm's equity holders receive no bargaining surplus and the second component is zero. When $\theta > 0$, a bank debt issuance that changes the firm's leverage affects not only the market debt price but also the equity holders' value through its impact on the joint value of the equity holders and the bank lender. In equilibrium, the firm issues market debt at a rate such that the equity holders' expected bargaining surplus from negotiated bank debt issuance is offset by the valuation effect of market debt issuance on the market debt investors.

3.4 Comparison with Competitive Bank Debt

To highlight the role of bargaining over bank debt issuances, I compare the result in Proposition 2 to that of a benchmark case where the firm issues bank debt to competitive bank lenders. To facilitate comparison, I assume that the firm is able to issue (or repurchase) bank debt at Poisson arrival times with intensity λ, matching the intensity of negotiated bank debt issuances in the main model. However, the bank lenders in the benchmark case are assumed to be competitive. Hence, the bank debt is issued at its prevailing competitive price, similar to the market debt. The firm takes the bank debt price as given and chooses the optimal issuance quantity such that its equity holders' value is maximized.

Let $V^{e,c}(Y_t, M_t, B_t)$ denote the equity holders' total value in the benchmark case with competitive bank lenders. When the firm is not in default, the equity holders' total value must satisfy the HJB equation

$$
\begin{aligned}
rV^{e,c}&(Y_t, M_t, B_t) \\
&= \max_{g_t^{m,c}} \left\{ (1-\pi)\big[Y_t - c(B_t + M_t)\big] + p_t^{m,c}g_t^{m,c}M_t + g_t^{m,c}M_t V_M^{e,c}(Y_t, M_t, B_t) \right. \\
&\qquad + \mu Y_t V_Y^{e,c}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,c}(Y_t, M_t, B_t) \\
&\qquad \left. + \lambda \max_{Q_t^{b,c}} \big[V^{e,c}(Y_t, M_t, B_t + Q_t^{b,c}) + p_t^{b,c}Q_t^{b,c} - V^{e,c}(Y_t, M_t, B_t)\big] \right\},
\end{aligned}
\tag{3.18}
$$

where $g_t^{m,c}$ is the optimal market debt issuance rate chosen by the firm in the benchmark, $Q_t^{b,c}$ is the optimal bank debt issuance amount, while $p_t^{m,c}$ and $p_t^{b,c}$ are the competitive market debt price and the competitive bank debt price respectively. As shown in Appendix B.3, the rate of market debt issuance is given by

$$
g_t^{m,c} = \frac{\pi c}{-M_t p_M^{m,c}(Y_t, M_t, B_t)}.
\tag{3.19}
$$

Comparing (3.17) and (3.19), one can see that there are additional terms on the right-hand side of (3.17) due to negotiations of bank debt issuances. Notably, these terms are absent in the case with competitive bank debt, even if bank debt issuances occur at Poisson times. This is because although market debt issuance impacts the market debt price, this impact is offset by its effect on the equity holders' marginal value (due to the envelope theorem).[2] Hence when bank lending is competitive, the firm's market debt issuance rate is such that the tax benefit of market debt issuance is fully dissipated by its negative valuation effect on the market debt investors, regardless of the frequency of bank debt issuances.

4 Explicit Solution with a Strong Bank

I begin by solving the case with a strong bank lender who holds all the bargaining power, that is $\theta = 0$. The bank lender can be thought to make a take-it-or-leave-it offer to the firm when they negotiate bank debt issuances. By making a take-it-or-leave-it offer, the bank lender extracts all the joint surplus from bargaining. This is because in the one-shot bargaining game, the equity holders are willing to accept any offer that makes them at least as well off as the status quo. Hence, the bank lender proposes the terms that make the equity holders indifferent between accepting and rejecting the offer. This case allows me to derive several closed-form results, thus better facilitating the understanding of the model's key mechanisms.

4.1 Equity and Market Debt

The equation (3.15) implies that when $\theta = 0$, the equity holders' total value can be solved as if the firm issues neither market debt nor bank debt. If the firm never adjusts its debt

---

[2]The envelope theorem says that a change in an external parameter affects the objective function of a maximizing agent only directly, because any indirect effect through the optimal choice is second-order. However, when the outcome is jointly determined by Nash bargaining between two agents (as in the case where bank debt issuances are negotiated) instead of a single maximizing agent, a change in the external parameter affects the size and the division of surplus, and the indirect effect becomes first-order in contrast to standard optimization problems.

outstanding, the market debt and the bank debt both remain constant over time. Thus, the only state variable is the firm's EBIT Y_t. Conditional on the face amount of market debt $M_t = M$ and the face amount of bank debt $B_t = B$, let $V^{e,0}(Y_t; M, B)$ denote the equity holders' *no-issuance value*, that is the equity holders' total value when there is never any debt issuance. $V^{e,0}(Y_t; M, B)$ must then satisfy the following HJB equation

$$
\begin{aligned}
rV^{e,0}(Y_t; M, B) =& (1 - \pi)\big[Y_t - c(B + M)\big] + \mu Y_t V^{e,0\prime}(Y_t; M, B) \\
& + \frac{1}{2}\sigma^2 Y_t^2 V^{e,0\prime\prime}(Y_t; M, B).
\end{aligned}
\tag{4.1}
$$

This is a second-order linear ordinary differential equation (ODE), and it has closed-form solution with a well-known form. The equity holders' no-issuance value can be obtained by solving the ODE subject to the no-bubble condition when $Y_t \to \infty$ as well as the value matching condition and the smooth pasting condition at the default boundary $Y^\star(M, B)$ (that is, the level of the firm's EBIT at which the equity holders choose to default). In equilibrium, the equity holders' total value equals the equity holders' no-issuance value. That is, $V^e(Y_t, M_t, B_t) = V^{e,0}(Y_t; M_t, B_t)$. In the following proposition, I provide the closed-form expression for the equity holders' total value when $\theta = 0$. The detailed derivations are delegated to Appendix B.4.

Proposition 3 (Equity Holders' Total Value). *The equity holders' total value is given by*

$$
\begin{aligned}
V^e(Y_t, M_t, B_t) =& \frac{1 - \pi}{r - \mu}Y_t - \frac{(1 - \pi)c}{r}(B_t + M_t) \\
& + \frac{1}{1 + \gamma}\frac{(1 - \pi)c}{r}\left[\frac{\gamma}{1 + \gamma}\frac{(r - \mu)c}{r}\right]^\gamma (B_t + M_t)^{1+\gamma}Y_t^{-\gamma}.
\end{aligned}
\tag{4.2}
$$

where

$$
-\gamma \equiv -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2},
\tag{4.3}
$$

29

and the default boundary is given by

$$Y^\star(M_t, B_t) = \frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}(B_t + M_t).\tag{4.4}$$

From (4.2), it is easy to see that the equity holders' total value depends on the total amount of debt outstanding $B_t + M_t$ but not the composition of debt. In other words, the bank debt and the market debt are perfect substitutes from the equity holders' point of view. This result is special and holds in the case with $\theta = 0$. As in DeMarzo and He (2021), and in the spirit of the Coase (1972) conjecture, the equity holders derive no marginal gain from market debt issuance. On the other hand, although there is gain from trade when the firm and the bank lender negotiate bank debt issuances, the firm captures no surplus from the take-it-or-leave-it offer proposed by the bank lender. Consequently, the equity holders are indifferent toward the composition of debt. However, despite the equity holders' indifference to the debt structure, the firm's market debt issuance decision does depend on it. Thus, small differences in the initial choices of debt composition between the market debt and the bank debt can lead to very different debt structure and leverage dynamics.

The equilibrium market debt price satisfies the first-order condition as given by (3.5), that is $p^m(Y_t, M_t, B_t) = -V_M^e(Y_t, M_t, B_t)$. Differentiating (4.2), one can immediately obtain the equilibrium market debt price. The market debt issuance rate is given by (3.17) of Proposition 2. The market debt issuance rate depends on the state variables Y_t, M_t, and B_t, thus $g_t^m \equiv g^m(Y_t, M_t, B_t)$. In the following proposition, I provide the closed-form expressions for the equilibrium market debt price and the market debt issuance rate. Derivations can be found in Appendix B.5.

Proposition 4 (Market Debt Price and Issuance Rate). *In equilibrium, the market debt price is given by*

$$p^m(Y_t, M_t, B_t) = \frac{(1-\pi)c}{r}\left\{1 - \left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^\gamma\left(\frac{B_t + M_t}{Y_t}\right)^\gamma\right\},\tag{4.5}$$

and the market debt issuance rate is given by

$$
\begin{aligned}
g_t^m \equiv & g^m(Y_t, M_t, B_t) \\
= & \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{B_t + M_t} \right]^\gamma \frac{B_t + M_t}{M_t} \\
& + \frac{\lambda}{\gamma} \left[1 - \left(\frac{B_t + M_t + \bar{Q}_t^b}{B_t + M_t} \right)^\gamma \right] \frac{B_t + M_t}{M_t},
\end{aligned}
\tag{4.6}
$$

where the constant γ is given by (4.3).

The equilibrium market debt price is strictly decreasing in $(B_t + M_t)/Y_t$, which is the firm's total leverage ratio. This is because in the case with $\theta = 0$, the bank debt and the market debt are perfectly substitutable from the equity holders' perspective. The equity holders are indifferent toward the choice between bank debt and market debt, and thus the equilibrium market debt price does not depend on the composition of debt. By contrast, the equilibrium rate of market debt issuance g_t^m depends on not only the firm's total leverage but also the firm's debt structure. This is because the market debt issuance rate hinges on the outcomes of negotiations between the firm and the bank lender over bank debt issuances, which in turn depend on the prevailing debt composition. Hence, the current debt structure affects the firm's market debt issuance decision.

To illustrate how negotiations of bank debt issuances between the bank lender and the firm affect the firm's market debt issuances, it may be more transparent to examine $\tilde{g}_t^m \equiv g_t^m M_t/(B_t + M_t)$, which is the amount of market debt issuance scaled by the total debt outstanding. $\tilde{g}_t^m$ thus measures how fast the firm is shifting toward market debt financing relative to its total debt level. From (4.6),

$$
\tilde{g}_t^m = \underbrace{\frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{B_t + M_t} \right]^\gamma}_{\text{no negotiation of bank debt issuance}} + \underbrace{\frac{\lambda}{\gamma} \left[1 - \left(\frac{B_t + M_t + \bar{Q}_t^b}{B_t + M_t} \right)^\gamma \right]}_{\text{effect of negotiated bank debt issuance}} .
\tag{4.7}
$$

31

In the case with a strong bank lender, the rate $\tilde{g}_t^m$ can be decomposed into two components. The first component is captured by the first term on the right-hand side of (4.7), and reflects the firm's market debt issuance decision if the bank lender and the firm never negotiate bank debt issuances once the bank debt is in place. That is, the firm commits to a fixed level of bank debt. As shown in Appendix B.6, the first component also corresponds to the case where the firm only has access to market debt financing. This component is strictly decreasing in the amount of bank debt outstanding B_t. That is, in the absence of periodic negotiations over bank debt issuances, the firm issues more market debt when the firm has less bank debt in its debt structure. Thus, bank debt and market debt appear to be substitutes.

The second component is an adjustment term to the firm's market debt issuance decision due to periodic negotiations of bank debt issuances. The component is captured by the second term on the right-hand side of (4.7). When the bank-firm negotiation results in an issuance of the bank debt, that is $\bar{Q}_t^b > 0$, the second component is negative and the firm slows down its market debt issuance. The intuition is as follows. When the bank lender and the firm negotiate to issue more bank debt, the value of the market debt investors is diluted but the equity holders do not capture any surplus from the issuance transaction. The negative valuation effect on market debt investors diminishes the benefit of market debt, causing the firm to issue less of it. The opposite is true when the bank-firm negotiation results in a repurchase of the bank debt, that is $\bar{Q}_t^b < 0$. This is because bank debt repurchase boosts the market debt value without affecting the equity value. The positive valuation effect on market debt investors adds to the benefit of market debt, causing the firm to speed up its market debt issuance.

While the first component is decreasing in the amount of bank debt outstanding B_t, the second component is increasing in B_t.[3] Hence, periodic negotiations of bank debt issuances between the bank lender and the firm introduce an element of strategic complementarity between the bank debt and the market debt. Through this complementarity, the firm's bank debt tends to have a positive impact on its borrowing capacity for market debt. Intuitively, when the firm has a high level of bank debt, negotiation between the bank lender and the firm

---

[3]See Appendix B.5 for proof.

would result in bank debt repurchase. A reduction in the bank debt outstanding boosts the market debt value, thus increasing the benefit of market debt and leading the firm to issue market debt at a faster pace. When the second component is large enough in magnitude relative to the first component, the bank debt and the market debt become complements in the sense that the firm's market debt issuance increases with its bank debt balance.

4.2 Bank Debt

The firm's scale invariance in the setting helps reduce the dimensionality of state variables. Specifically, the bank lender's value $V^b(Y_t, M_t, B_t)$ is homogeneous of degree one. That is,

$$V^b(Y_t, M_t, B_t) = V^b\Big(\frac{Y_t}{M_t}, 1, \frac{B_t}{M_t}\Big)M_t \equiv v^b(y_t, b_t)M_t, \tag{4.8}$$

where $y_t \equiv Y_t/M_t$ is the firm's EBIT-to-market-debt ratio and $b_t \equiv B_t/M_t$ is the bank debt ratio which reflects the firm's debt structure. The value function $v^b(y_t, b_t)$ is the bank lender's value scaled by the firm's market debt outstanding. For brevity, I refer to $v^b(y_t, b_t)$ as the *bank debt value.*

The bank debt value is obtained by substituting (4.8) and the condition (3.11) into the bank lender's HJB equation (3.16). When $\theta = 0$ and as I show in Appendix B.7, the bank debt value satisfies the following partial differential equation (PDE)

$$
\begin{aligned}
(r - g_t^m)v^b(y_t, b_t) =&cb_t - g_t^m b_t v_b^b(y_t, b_t) + (\mu - g_t^m)y_t v_y^b(y_t, b_t) + \frac{1}{2}\sigma^2 y_t^2 v_{yy}^b(y_t, b_t) \\
&+ \lambda\Big[\max_{q_t^b} v(y_t, b_t + q_t^b) - v(y_t, b_t)\Big],
\end{aligned} \tag{4.9}
$$

where $v(y_t, b_t) \equiv v^e(y_t, b_t) + v^b(y_t, b_t)$ is the joint equity and bank debt value, with $v^e(y_t, b_t) \equiv V^e(Y_t, M_t, B_t)/M_t$ being the equity holders' total value scaled by the amount of market debt outstanding. The bank debt value can be solved numerically. Notice that the equity holders choose to default when the firm's EBIT-to-market-debt ratio $y_t < y^\star(b_t)$, where the default

33

boundary $y^\star(b_t)$ depends on the bank debt ratio b_t. To facilitate numerical solution, it is more intuitive to express the value functions in terms of the firm's distance to default $y_t - y^\star(b_t)$ along with the bank debt ratio b_t. The firm defaults when its distance to default reaches zero from above, regardless of its debt structure. More detailed discussion of the numerical procedure can be found in the appendix.

I consider two cases. I first consider the case where the bank debt is senior to the market debt, and provide numerical solutions to the model under the assumption that $\theta = 0$. I then consider the case where the bank debt is pari-passu with the market debt, and show that the firm never issues bank debt in this case.

A. Senior Bank Debt

If the bank debt is senior to the market debt, then the bank lender's recovery value upon default is $\min\{\alpha Y^\star, B_{\tau^\star}\}$. That is, if the firm's liquidation value is enough to cover the principal amount of the bank debt at the time of default, that is $\alpha Y^\star \geq B_{\tau^\star}$, the bank lender receives the face amount of the bank debt and the bank debt is defeased. If the firm's liquidation value is below the principal amount of the bank debt at the time of default, that is $\alpha Y^\star < B_{\tau^\star}$, the bank lender gets all the liquidation value according to the absolute priority rule and the market debt investors receive nothing in recovery. Let $v^{b,\star}$ denote the bank debt value upon default, then

$$v^{b,\star} = \frac{\min\{\alpha Y^\star, B_{\tau^\star}\}}{M_{\tau^\star}} = \min\{\alpha y^\star, b_{\tau^\star}\}. \tag{4.10}$$

The bank debt value can be obtained by solving the PDE (4.9) subject to the value-matching condition at the default boundary.

I present the numerical solutions with the following baseline parameter values: $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = c(1 - \pi) = 0.05$, $\lambda = 1$, and $\alpha = 2.33$. The parameter values are taken from the literature (e.g., DeMarzo and He (2021), Greenwood et al. (2023)). $\lambda = 1$ corresponds to negotiations of bank debt at annual frequencies. $\alpha = 2.33$ implies a liquidation value that is

10% of the unlevered firm value at default. Panel A of Figure 3 plots the bank debt value v^b as a function of the firm's distance to default $\tilde{y}_t \equiv y_t - y^\star(b_t)$ and its bank debt ratio b_t within the no-default region. Panel B plots the bank debt value against $\tilde{y}_t$ for fixed levels of b_t, while Panel C plots the bank debt value against b_t for fixed levels of $\tilde{y}_t$.

Figure 3: Bank debt value $v^b(\tilde{y}_t, b_t)$
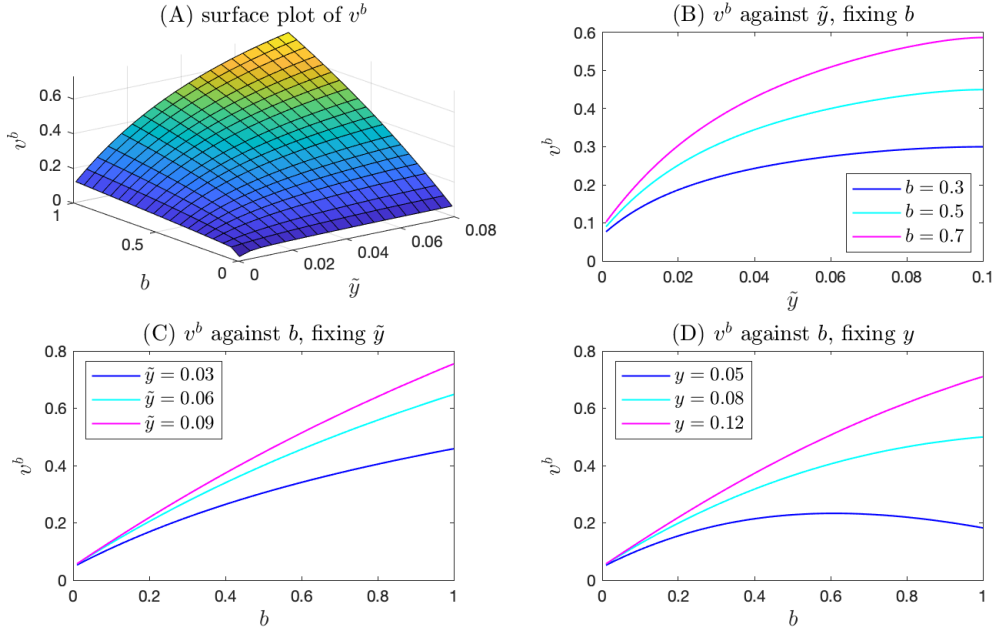
Note: This figure plots the bank debt value v^b as a function of the firm's distance to default $\tilde{y}_t \equiv y_t - y^\star(b_t)$ and the bank debt ratio b_t within the no-default region. Panel A is a two-dimensional surface plot. Panel B plots v^b against $\tilde{y}_t$ for fixed levels of b_t, while Panel C plots v^b against b_t for fixed levels of $\tilde{y}_t$. Panel D plots v^b against b_t for fixed levels of the EBIT-to-market-debt ratio y_t. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = c(1 - \pi) = 0.05$, $\lambda = 1$, and $\alpha = 2.33$.

The figure shows that given the bank debt ratio b_t, the bank debt value is increasing in the distance to default $\tilde{y}_t$. Similarly, given the distance to default $\tilde{y}_t$, the bank debt value is increasing in the bank debt ratio b_t. Hence, fixing the firm's EBIT Y_t and its market debt outstanding M_t, increasing the face amount of bank debt B_t has two countervailing effects on the bank debt value. On the positive side, the increase in the bank debt outstanding leads to a larger bank debt ratio b_t and thus higher bank debt value. On the negative side, the increase in the bank debt outstanding leads to higher total leverage and earlier default and thus reduces the firm's distance to default $\tilde{y}_t$, which tends to lower the bank debt value. Thus for given Y_t

and M_t, the bank debt value is generally non-monotonic in b_t. I illustrate this non-monotonicity in Panel D of Figure 3 by plotting the bank debt value as a function of the bank debt ratio b_t for various fixed levels of EBIT-to-market-debt ratio $y_t \equiv Y_t/M_t$. The bank debt value is increasing in b_t for small values of b_t but decreasing in b_t when b_t becomes relatively large.

B. Bank Debt Issuances

When the bank lender and the firm negotiate bank debt issuance, they choose the issuance quantity that maximizes their joint surplus. Hence, the issuance quantity scaled by the amount of market debt $\bar{q}_t^b \equiv \bar{Q}_t^b/M_t$ must be such that the joint equity and bank debt value $v \equiv v^e + v^b$ is maximized. In Panel A of Figure 4, I plot $\bar{q}_t^b$ against the firm's bank debt ratio b_t for fixed levels of the EBIT-to-market-debt ratio y_t. $\bar{q}_t^b$ is decreasing in b_t for all levels of y_t. When the firm has little bank debt in its debt structure, the bank-firm negotiation leads to issuance of additional bank debt. When the firm has a lot of bank debt, the bank-firm negotiation results in repurchase of existing bank debt.

Figure 4: Negotiated bank debt issuance $\bar{q}_t^b$
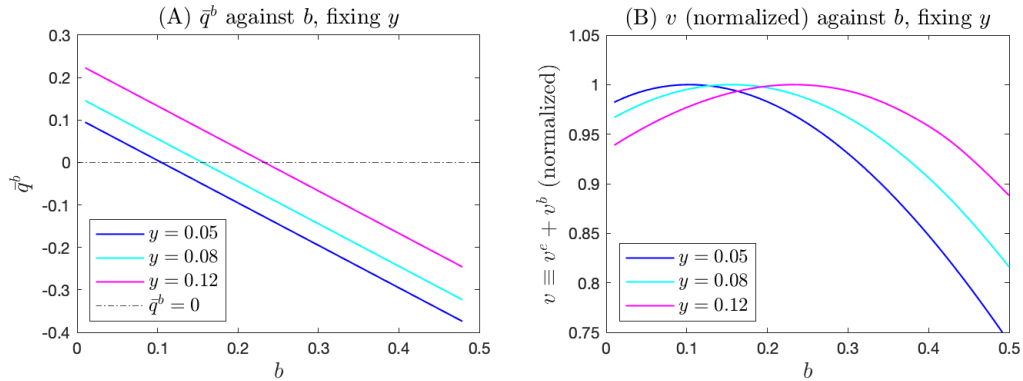
Note: Panel A of the figure plots the negotiated amount of bank debt issuance scaled by the amount of market debt $\bar{q}_t^b \equiv \bar{Q}_t^b/M_t$ as a function of the bank debt ratio b_t for various levels of the EBIT-to-market-debt ratio y_t. Panel B plots the normalized sum of the equity value and the bank debt value $v \equiv v^e + v^b$ against b_t for various y_t. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = c(1 - \pi) = 0.05$, $\lambda = 1$, and $\alpha = 2.33$.

To verify the optimality of the issuance policy $\bar{q}_t^b$, I plot the (normalized) joint equity and bank debt value $v \equiv v^e + v^b$ against the bank debt ratio b_t for different levels of y_t.[4] The results

---

[4]I normalize the joint equity and bank debt value by applying the formula $v^{norm} = \frac{v - v^{min}}{v^{max} - v^{min}}$ where v^{min} and v^{max} are the minimum and the maximum values of v in the given support.

are shown in Panel B of Figure 4. When the bank debt is senior, the joint equity and bank debt value is non-monotonic and concave in the bank debt ratio b_t. Thus, when the bank lender and the firm negotiate bank debt issuance, they choose an issuance quantity such that the post-issuance amount of bank debt is positive. This result comes from the "sandwich" capital structure where the senior claimants (the bank lender) negotiate with the junior claimants (the equity holders) to extract from the mezzanine claimants (the market debt investors).

Given the firm's EBIT Y_t and the market debt outstanding M_t, borrowing from the bank lender leads to higher leverage and earlier default by the firm. This is the *leverage effect* of bank debt issuance, and it tends to decrease the joint equity and bank debt value. However, because the bank debt is senior to the market debt, bank debt issuance dilutes the market debt investors more. The market debt is thus more sensitive than the bank debt to an increase in leverage due to bank debt issuance. Hence, the market debt acts as a buffer to the bank debt, and bank debt issuance enables the bank lender and the equity holders to capture value from the market debt investors. I refer to this effect as the *dilution effect* of bank debt issuance. When the bank lender and the firm negotiate bank debt issuance, they trade off the dilution effect against the leverage effect, leading to positive amount of post-issuance bank debt outstanding.

In Figure 5, I plot the optimal bank debt ratio $\bar{b}_t \equiv b_t + \bar{q}_t^b$ which results from bank-firm negotiations of bank debt issuances for given levels of the bank debt ratio b_t. Panel A plots $\bar{b}_t$ as a function of the firm's distance to default pre-issuance at prevailing states Y_t, M_t and B_t. Panel B plots plots $\bar{b}_t$ as a function of the firm's distance to default post-issuance after taking into account the bank debt issuance's effect on the firm's distance to default. In both panels, the relationship between the optimal bank debt ratio and the distance to default measure are hump-shaped. When the firm is far from default, it is in the best interest of the equity holders and the bank lender to keep the bank debt ratio low and rely heavily on market debt financing. As the firm gets closer to default, the optimal bank debt ratio increases. However, when default is imminent, the bank lender and the firm find it optimal to reduce the amount of bank debt in order to avoid default.

**Figure 5:** Post-issuance optimal bank debt ratio $\bar{b}_t \equiv b_t + \bar{q}_t^b$
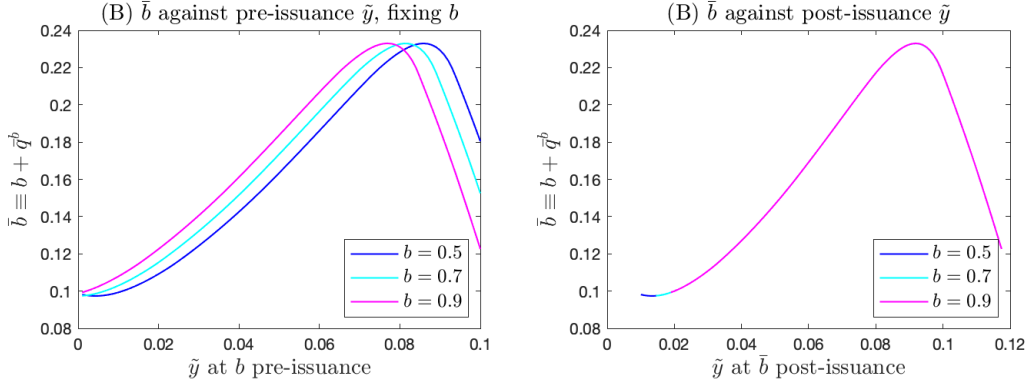
Note: The figure plots the optimal bank debt ratio $\bar{b}_t \equiv b_t + \bar{q}_t^b$ which results from bank-firm negotiations of bank debt issuances for given levels of the bank debt ratio b_t. Panel A plots $\bar{b}_t$ as a function of the firm's distance to default $\tilde{y}_t$ pre-issuance at prevailing states. Panel B plots $\bar{b}_t$ as a function of the firm's distance to default $\tilde{y}_t$ post-issuance, evaluated at the post-issuance bank debt ratio $\bar{b}_t$. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = c(1 - \pi) = 0.05$, $\lambda = 1$, and $\alpha = 2.33$.

C. Complementarity of Bank and Market Debt

How does the firm's market debt issuance decision depend on the proportion of bank debt in its debt structure? To answer this question, I plot the firm's debt growth due to market debt issuance $\tilde{g}_t^m$, as given by (4.7), against the bank debt ratio b_t for fixed levels of the EBIT-to-market-debt ratio y_t. In Panel A of Figure 6, I plot $\tilde{g}_t^m$ against b_t when $\lambda = 1$. In Panel B of Figure 6, I plot the same object for a small λ, namely $\lambda = 0.02$. For comparison, the dashed lines represent $\lambda = 0$ which is the case where the firm commits to a fixed level of bank debt and bank-firm negotiation of bank debt issuance never takes place. As discussed earlier, this scenario is also equivalent to one where only market debt financing is available.

In both panels, the dashed lines are downward-sloping, implying that more bank debt is associated with less market debt issuance absent strategic bank-firm negotiations of bank debt issuances. On the other hand, the solid lines may be upward-sloping or downward-sloping depending on the magnitude of λ. As shown in Panel B, when bank-firm negotiations of bank debt issuances occur infrequently (i.e., when λ is small), the firm's debt growth from market debt issuance $\tilde{g}_t^m$ still decreases with the bank debt ratio b_t, albeit at a slower rate than in the $\lambda = 0$ case. In stark contrast, when the bank lender and the firm negotiate bank debt issuances

Figure 6: Debt growth due to market debt issuance $\tilde{g}_t^m$
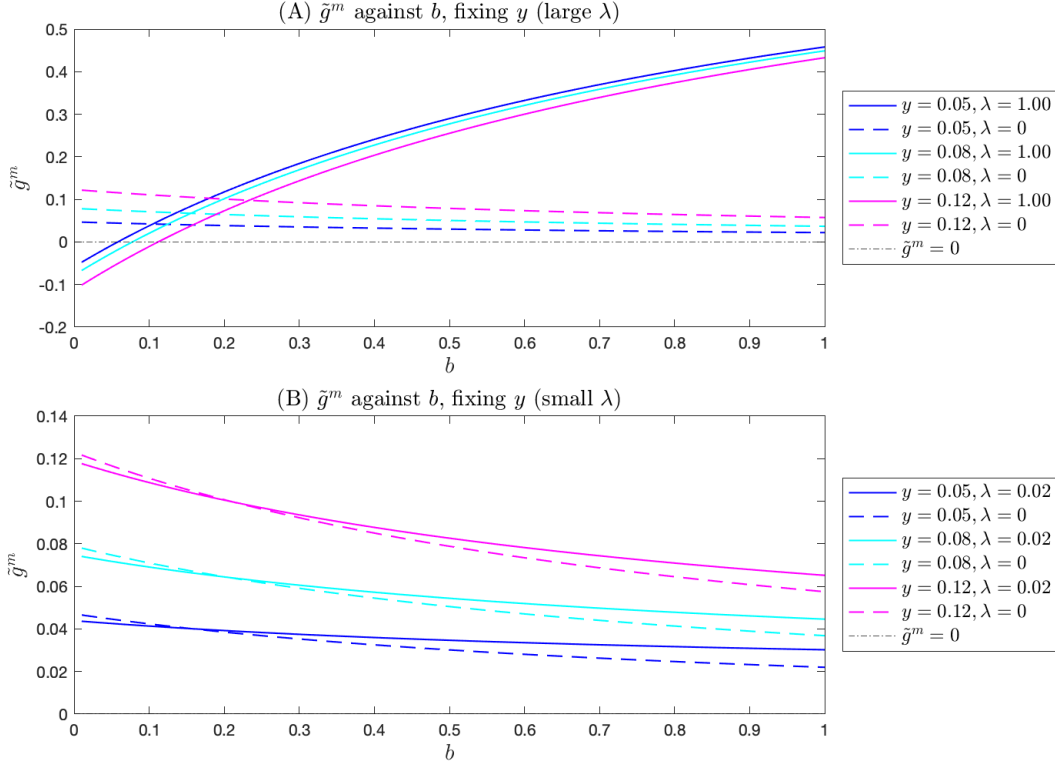
(A) $\tilde{g}^m$ against b, fixing y (large λ)

(B) $\tilde{g}^m$ against b, fixing y (small λ)

Note: This figure plots the rate of total debt growth due to the market debt issuance $\tilde{g}_t^m$ as a function of the bank debt ratio b_t for various levels of the EBIT-to-market-debt ratio y_t. Panel A plots this relationship for $\lambda = 1$, while Panel B plots this relationship for $\lambda = 0.02$. The other parameters take the values: $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = (1-\pi)c = 0.05$, and $\alpha = 2.33$.

relatively frequently (e.g., when $\lambda = 1$ as shown in Panel A), $\tilde{g}_t^m$ is instead increasing in b_t and the firm issues market debt at a faster rate when it has more bank debt in its debt structure.

As is clear from (4.7), $\tilde{g}_t^m$ is increasing in b_t when the second component dominates in magnitude. Since the first component does not depend on λ, the second component tends to dominate when λ is large. Intuitively, bank-firm negotiations of bank debt issuances, as captured by the second component, introduces an element of strategic complementarity between the bank debt and the market debt. When the firm has a high balance of bank debt relative to its total debt outstanding, negotiation between the bank lender and the firm would result in bank debt repurchase. Reduction in the amount of bank debt boosts the market debt value, leading to greater current capacity of market debt and faster market debt issuance. The opposite is true when the firm has a low balance of bank debt in its debt structure, as bank-firm negotiation

39

would result in bank debt issuance, which dilutes the market debt investors and slows market debt issuance by the firm. When negotiations between the bank lender and the firm occur frequently (i.e., when λ is large), this strategic complementarity that result from bank-firm negotiations becomes more pertinent.

Moreover, while $\tilde{g}_t^m$ is always positive when $\lambda = 0$, it can be negative with periodic negotiations of bank debt issuances between the bank lender and the firm. This is in contrast to the leverage ratchet effect of Admati et al. (2018) and DeMarzo and He (2021), which states that the firm never voluntarily repurchases its debt. In the current setting, if the firm can commit to a fixed level of bank debt ($\lambda = 0$), the firm wishing to buy back the market debt can always benefit from postponing the repurchase. However, this is no longer the case when the bank lender and the firm periodically negotiate bank debt issuances. Postponing the market debt repurchase is not necessarily optimal, as it would change the outcome of the bargaining problem between the bank lender and the firm.

D. Issuance Prices and Credit Spreads

While issuances or repurchases of the market debt occur at the prevailing competitive prices, issuances or repurchases of the bank debt take place at negotiated prices determined by bargaining between the bank lender and the firm. The market debt price p_t^m is given by (4.5) of Proposition 4. The negotiated issuance price of the bank debt $\bar{p}_t^b$ can be obtained from the condition (3.12) of Proposition 1. In Appendix B.8, I derive the bank debt issuance price for the case with $\theta = 0$. Panel A of Figure 7 plots the negotiated issuance price of the bank debt $\bar{p}_t^b$ and the competitive issuance price of the market debt p_t^m as functions of the firm's distance to default $\tilde{y}_t$, for different levels of the bank debt ratio b_t. Panel B of Figure 7 plots the corresponding credit spreads, $\bar{cs}_t^b$ and cs_t^m, respectively defined as

$$\frac{c}{r + \bar{cs}_t^b} \equiv \bar{p}_t^b \quad \text{and} \quad \frac{c}{r + cs_t^m} \equiv p_t^m. \tag{4.11}$$

Figure 7: Issuance prices and credit spreads of bank debt and market debt
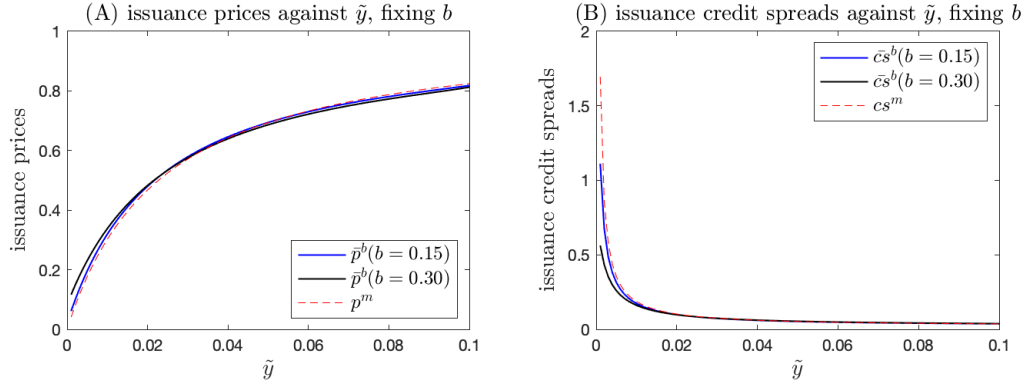
Note: The figure plots the issuance prices (panel A) and the corresponding credit spreads (panel B) of the bank debt and the market debt against the firm's distance to default $\tilde{y}_t$, for bank debt ratio $b_t = 0.15$ and $b_t = 0.30$. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = (1 - \pi)c = 0.05$, $\lambda = 1$, and $\alpha = 2.33$.

When the firm is close to default, the negotiated bank debt price tends to be above the competitive market debt price, and the corresponding credit spread of the bank debt is lower than that of the market debt. However, when the firm is far from default, the negotiated bank debt price is instead below the competitive market debt price, and the corresponding credit spread of the bank debt is higher than that of the market debt. Hence, bank debt appears to carry higher credit spreads and trade at lower prices than market debt when the firm is not highly levered. This is despite the fact that the bank debt is senior to the market debt. However, this pattern reverses when the firm moves closer to default, as the credit spreads of the market debt spike up (and prices of the market debt decline) more sharply than those of the bank debt. In other words, bank debt tends to be a more stable form of financing in the sense that its cost of financing is less variable compared to that of the market debt.

E. Suboptimality of Pari-Passu Bank Debt

I now consider the case in which the bank debt is pari-passu with the market debt. Without commitment, the firm's equity holders are incentivized to dilute existing lenders by issuing new market debt just before default. Covenants and legal restrictions (e.g., fraudulent conveyance) help limit such behavior. To examine the case with pari-passu bank debt, I consider two extreme legal regimes. Under the first regime, the equity holders are able to fully dilute the existing

lenders by issuing additional pari-passu debt, and the bank lender receives no liquidation value upon default. Under the second regime, there is perfect legal protection and the liquidation value $\alpha Y^\star$ is split between the bank lender and the market debt investors according to their holdings in the event of default.

From (3.11), when the bank lender and the firm negotiate bank debt issuance, the optimal amount of bank debt is determined by bargaining and only depends on the firm's EBIT Y_t and its market debt outstanding M_t. The post-negotiation optimal bank debt ratio $\bar{b}_t \equiv b_t + \vec{q}_t^b$ is thus a function of the EBIT-to-market-debt ratio y_t. I plot $\bar{b}_t$ under the two regimes in Panel A and Panel C of Figure 8. I also plot the (normalized) joint equity and bank debt value against b_t for the two regimes in Panel B and Panel D of Figure 8. If the bank debt is pari-passu with the market debt, the joint equity and bank debt value is monotonically decreasing in the bank debt ratio b_t under both regimes and the firm never issues any bank debt. Intuitively, when the bank debt and the market debt are pari-passu, the dilution effect of bank debt issuance is no longer at work. An increase in the face amount of the bank debt leads to a higher total leverage and thus earlier default by the firm's equity holders. Through this leverage effect, the joint equity and bank debt value is negatively impacted by the issuance of bank debt. Hence, the optimal bank debt ratio $\bar{b}_t$, which maximizes the joint equity and bank debt value, is zero.

5 Bank Lending Shock

In this section, I consider the effect of a bank lending shock, modeled as a shock to the bank lender's discount rate. Specifically, I assume that the economy is either in a normal state or a shock state. I use $\xi \in \{n, s\}$ to indicate the state of the economy, with n denoting the normal state and s denoting the shock state. In the normal state, the bank lender discounts time at rate $r^n > 0$. In the shock state, the bank lender has a higher discount rate $r^s > r^n$. The economy transitions between states through a Markov switching process that is independent of the Poisson process governing bank-firm negotiations. The economy transitions from the shock state to the normal state with intensity $\zeta^s > 0$, and from the normal state to the shock state

Figure 8: Optimal pari-passu bank debt amount determined by bargaining
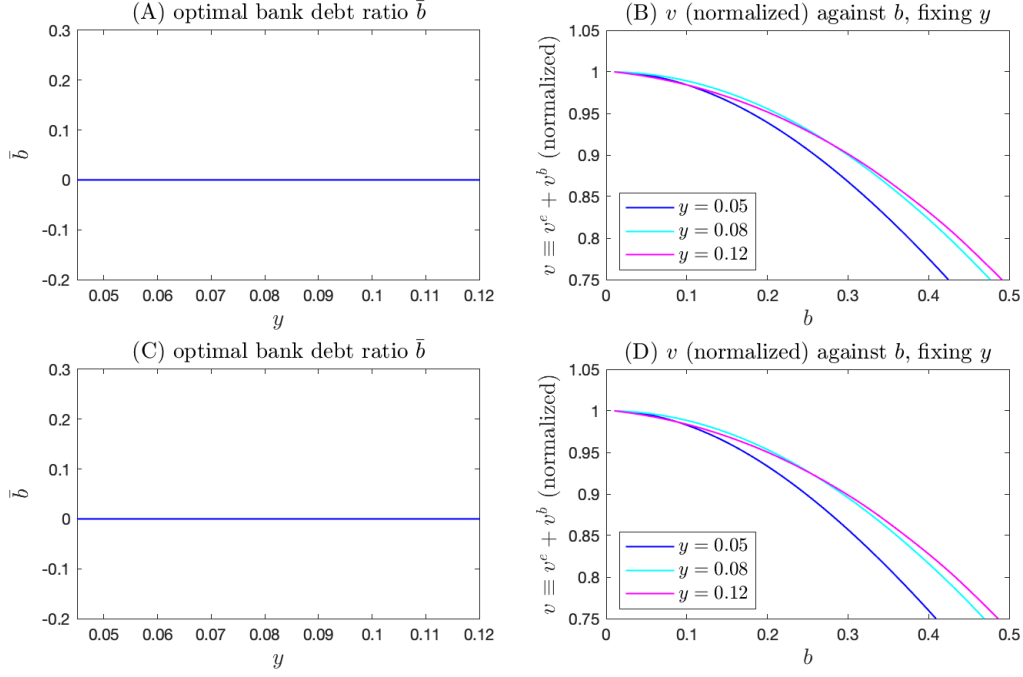
Note: The figure plots the bargaining-determined optimal bank debt ratio $\bar{b}_t$ and the normalized joint equity and bank debt value v under two legal regimes when the bank debt and the market debt are pari-passu. In the first regime, the bank lender receives no liquidation value upon default. In the second regime, the liquidation value is split between the bank lender and the market debt investors according to their holdings. Panel A plots $\bar{b}_t$ against the EBIT-to-market-debt ratio y_t under the first regime, and Panel C plots $\bar{b}_t$ against y_t under the second regime. Panel B plots the normalized joint equity and bank debt value against the bank debt ratio b_t for various y_t under the first regime. Panel D plots the same object under the second regime. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = (1-\pi)c = 0.05$, $\lambda = 1$, and $\alpha = 2.33$.

with intensity ζ^n which I normalize to zero. That is, the normal state is absorbing, and my analysis examines the impact of a transitory bank lending shock where the persistence of the shock is characterized by $1/\zeta^s$. For simplicity, I continue to focus on the case with a strong bank lender (i.e., $\theta = 0$).

Let $V^{e,\xi}(Y_t, M_t, B_t)$ and $V^{b,\xi}(Y_t, M_t, B_t)$ denote the equity holders' total value and the bank lender's value when the economy is in state ξ and the firm has EBIT Y_t, market debt outstanding M_t, and bank debt outstanding B_t. As shown in Appendix B.9, when the firm is

not in default, the equity holders' total value satisfies the HJB equation

$$
\begin{aligned}
rV^{e,\xi}(Y_t, M_t, B_t) =&(1 - \pi)\big[Y_t - c(B_t + M_t)\big] \\
&+ \mu Y_t V_Y^{e,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,\xi}(Y_t, M_t, B_t) \\
&+ \zeta^\xi\big[V^{e,-\xi}(Y_t, M_t, B_t) - V^{e,\xi}(Y_t, M_t, B_t)\big],
\end{aligned}
\tag{5.1}
$$

and the bank lender's value satisfies

$$
\begin{aligned}
r^\xi V^{b,\xi}(Y_t, M_t, B_t) =&cB_t + g_t^{m,\xi} M_t V_M^{b,\xi}(Y_t, M_t, B_t) \\
&+ \mu Y_t V_Y^{b,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{b,\xi}(Y_t, M_t, B_t) \\
&+ \lambda\big[V^\xi(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - V^\xi(Y_t, M_t, B_t)\big] \\
&+ \zeta^\xi\big[V^{b,-\xi}(Y_t, M_t, B_t) - V^{b,\xi}(Y_t, M_t, B_t)\big],
\end{aligned}
\tag{5.2}
$$

where $V^\xi(Y_t, M_t, B_t) \equiv V^{e,\xi}(Y_t, M_t, B_t) + V^{b,\xi}(Y_t, M_t, B_t)$ is the sum of the equity holders' total value and the bank lender's value. The market debt issuance rate $g_t^{m,\xi}$ is given by

$$
g_t^{m,\xi} = \frac{\pi c}{-M_t p_M^{m,\xi}(Y_t, M_t, B_t)} + \frac{\lambda[p^{m,\xi}(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - p^{m,\xi}(Y_t, M_t, B_t)]}{-M_t p_M^{m,\xi}(Y_t, M_t, B_t)}.
\tag{5.3}
$$

As before, I will continue to focus on $\tilde{g}_t^{m,\xi} \equiv g_t^{m,\xi} M_t/(B_t + M_t)$, the amount of market debt issuance scaled by the total debt outstanding, to capture the firm's market debt issuance policy.

A. Permanent Shock

For better intuition, I first consider a bank lending shock that is permanent. That is, $\zeta^s = 0$. For instance, the post-GFC banking regulations have increased banks' balance sheet constraints. As shown in the appendix, the firm's debt growth due to market debt issuance $\tilde{g}_t^\xi$ in state $\xi \in \{n, s\}$ is given by

$$
\tilde{g}_t^{m,\xi} = \frac{\pi c}{\gamma}\frac{r}{(1 - \pi)c}\Big[\frac{1 + \gamma}{\gamma}\frac{r}{(r - \mu)c}\frac{Y_t}{B_t + M_t}\Big]^\gamma + \frac{\lambda}{\gamma}\Big[1 - \Big(\frac{B_t + M_t + \bar{Q}_t^{b,\xi}}{B_t + M_t}\Big)^\gamma\Big].
\tag{5.4}
$$

Given the firm's EBIT Y_t, its market debt outstanding M_t, and its bank debt outstanding B_t, the firm's market debt issuance policy $\tilde{g}_t^\xi$ in state ξ depends on the negotiated bank debt issuance quantity $\bar{Q}_t^{b,\xi}$ (or equivalently, the post-issuance bank debt amount $\bar{B}_t^\xi \equiv B_t + \bar{Q}_t^{b,\xi}$). In Panel A of Figure 9, I compare the bank debt issuance quantity scaled by the amount of market debt in the shock state $\bar{q}_t^{b,s} \equiv \bar{Q}_t^{b,s}/M_t$ to that in the normal state $\bar{q}_t^{b,n} \equiv \bar{Q}_t^{b,n}/M_t$. To be more specific, I plot $\bar{q}_t^{b,s}$ and $\bar{q}_t^{b,n}$ against the firm's prevailing bank debt ratio b_t for given levels of the EBIT-to-market-debt ratio y_t. In Panel B of Figure 9, I plot the corresponding post-issuance bank debt ratio in the shock state $\bar{b}_t^s \equiv b_t + \bar{q}_t^{b,s}$ and the corresponding post-issuance bank debt ratio is the normal state $\bar{b}_t^n \equiv b_t + \bar{q}_t^{b,n}$ for given levels of y_t.

Figure 9: Negotiated bank debt issuance and post-issuance bank debt ratio
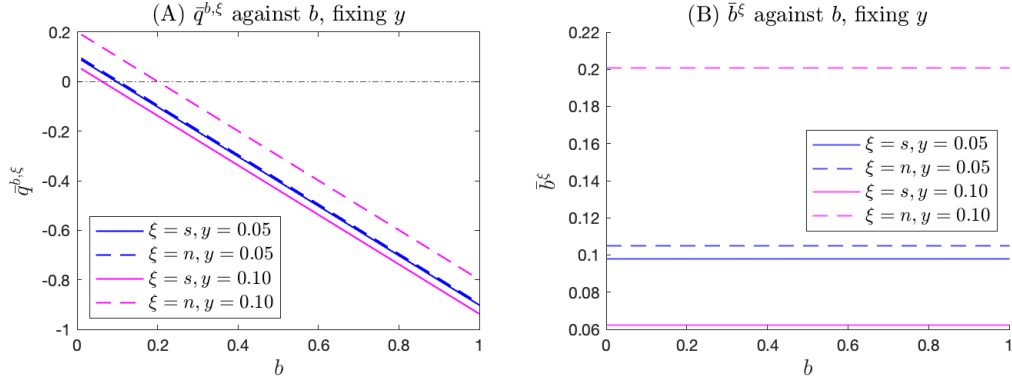

Note: Panel A of the figure plots the negotiated amount of bank debt issuance scaled by the amount of market debt $\bar{q}_t^{b,\xi} \equiv \bar{Q}_t^{b,\xi}/M_t$ against the bank debt ratio b_t for various levels of the EBIT-to-market-debt ratio y_t. Panel B plots the corresponding post-issuance bank debt ratio in the normal state and in the shock state, $\bar{b}_t^n$ and $\bar{b}_n^s$, for various levels of y_t. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r^n = r = c(1-\pi) = 0.05$, $r^s = 0.07$, $\lambda = 1$, and $\alpha = 2.33$.

The figure shows that given the prevailing states Y_t, M_t and B_t, the post-issuance optimal bank debt ratio is lower when the economy is in the shock state than when it is in the normal state. Intuitively, the shock to the bank lender's discount rate reduces the bank lender's willingness to lend. When the bank lender and the firm negotiate bank debt issuance, they find it optimal to carry a smaller balance of bank debt. Hence, the bank lending shock can be seen as a shock to the bank credit supply. From (5.4), it is immediate that $\tilde{g}^{m,s}(Y_t, M_t, B_t) > \tilde{g}^{m,n}(Y_t, M_t, B_t)$. Hence, a bank lending shock results in greater market debt issuance by the firm. In Figure 10, I plot $\tilde{g}_t^{m,s}$ and $\tilde{g}_t^{m,n}$ against the bank debt ratio b_t for given levels of the EBIT-to-market-debt

ratio y_t. The figure shows that the firm issues market debt at a higher rate upon the bank lending shock. The increase in the rate of market debt issuance is more pronounced when the firm is farther from default.

Figure 10: State-dependent market debt issuance $\tilde{g}_t^m$
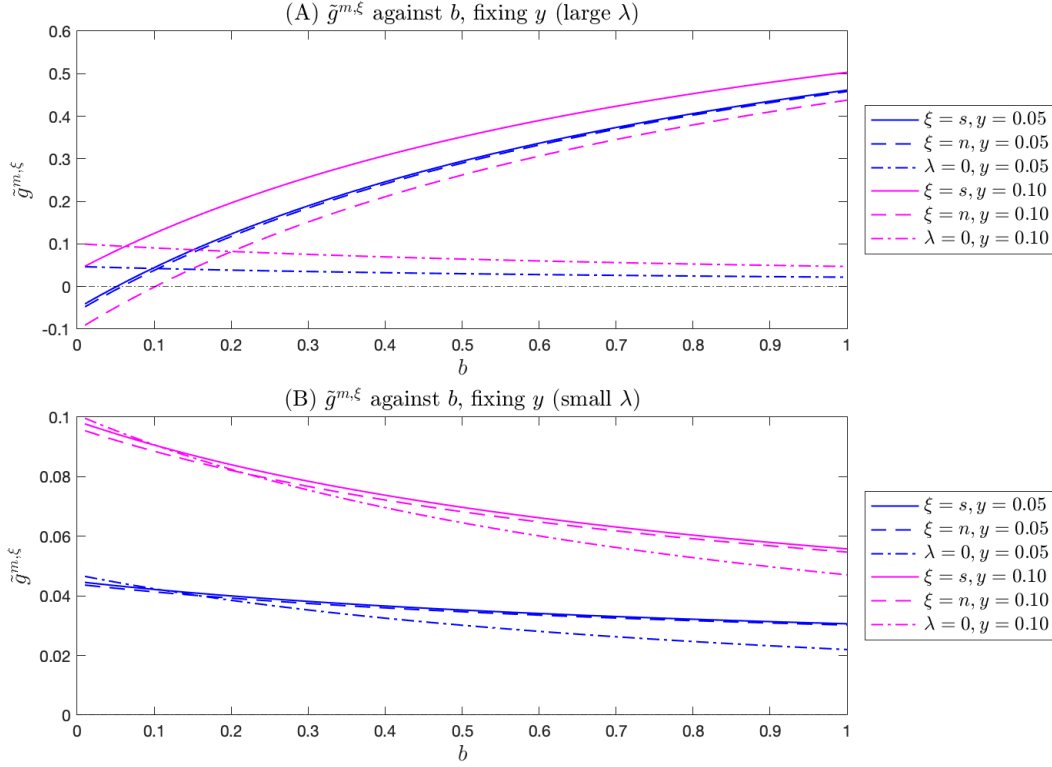
Note: This figure plots the rate of total debt growth due to the market debt issuance $\tilde{g}_t^{m,\xi}$ against the bank debt ratio b_t for various levels of the EBIT-to-market-debt ratio y_t, when the economy is in the normal state and when it is in the shock state. Panel A plots this relationship for $\lambda = 1$, while Panel B plots this relationship for $\lambda = 0.02$. The other parameters take the values: $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r^n = r = (1 - \pi)c = 0.05$, $r^s = 0.07$, and $\alpha = 2.33$.

The figure appears consistent with the empirical observation that firms increase their market debt issuance after a contraction in bank credit supply.[5] In the model, a bank lending shock reduces the bank lender's willingness to extend credit, leading to a lower share of bank debt in the firm's debt structure upon bank-firm negotiation over bank debt issuance or repurchase. As the bank debt ratio declines, the equity holders' value becomes less sensitive to market debt issuance, prompting the firm to issue more market debt.

---

[5]For example, see Becker and Ivashina (2014).

Importantly, the strategic complementarity between the bank debt and the market debt, as captured by the second term on the right-hand side of (5.4), is still present in both the normal state and the shock state. This can be seen from Figure 10 by comparing $\tilde{g}_t^{m,\xi}$ in both states of the economy with the counterfactual with $\lambda = 0$. When λ is sufficiently large, both $\tilde{g}_t^{m,n}$ and $\tilde{g}_t^{m,s}$ are increasing in b_t. That is, in both the normal state and the shock state, a higher proportion of bank debt in the debt structure is associated with greater market debt issuance, even though the firm issues more market debt after the bank lender experiences the discount rate shock. The strategic complementarity of bank and market debt tends to amplify the shock. This is because the bank lending shock reduces the proportion of bank debt in the firm's debt structure once the bank lender and the firm negotiate bank debt issuance or repurchase. The decrease in the proportion of bank debt constrains the market debt capacity compared to the counterfactual where the two types of debt are not strategically complementary.

B. Transitory Shock

A bank lending shock is often transitory. I now examine how the persistence of the shock affects the firm's market debt issuance behavior. In Figure 11, I plot $\tilde{g}_t^{m,s}$ and $\tilde{g}_t^{m,n}$ for various levels of shock persistence, ranging from highly persistent ($\zeta^s = 0.2$) to transitory ($\zeta^s = 2$). The pattern that the firm increases market debt issuance upon shock remains robust. However, the extent to which the firm ramps up its market debt issuance depends on the shock's persistence. As shown in the figure, the firm increases its market debt issuance more when the shock is more persistent.

6   Effects of Firm Bargaining Power

I now turn to the general case where the firm has positive bargaining power, that is $\theta > 0$. The general case can only be solved numerically. In particular, I am interested in knowing how the firm's bargaining power affects the relationship between the firm's market debt issuance decision and the proportion of bank debt in its debt structure. To this end, I plot the firm's

47

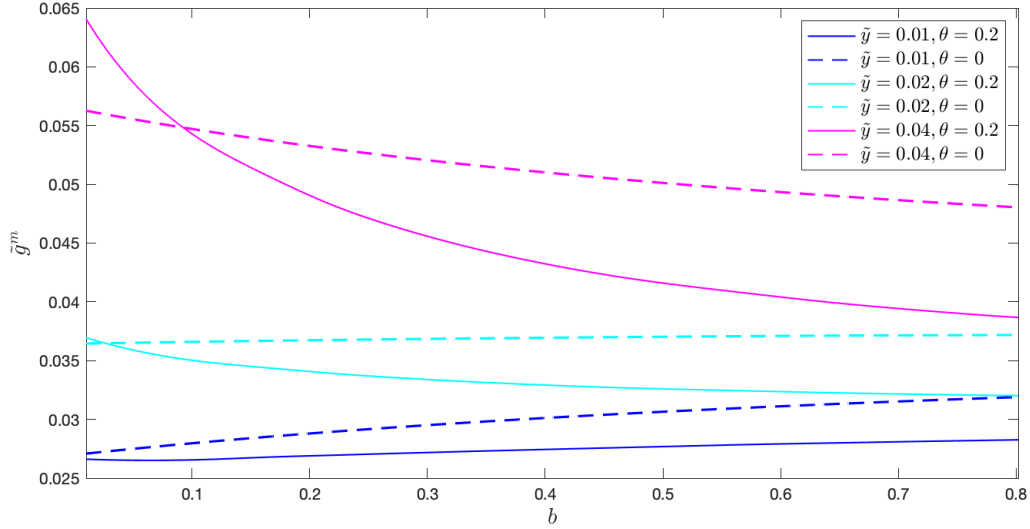**Figure 11:** Effect of shock persistence on market debt issuance $\tilde{g}_t^{m,\xi}$
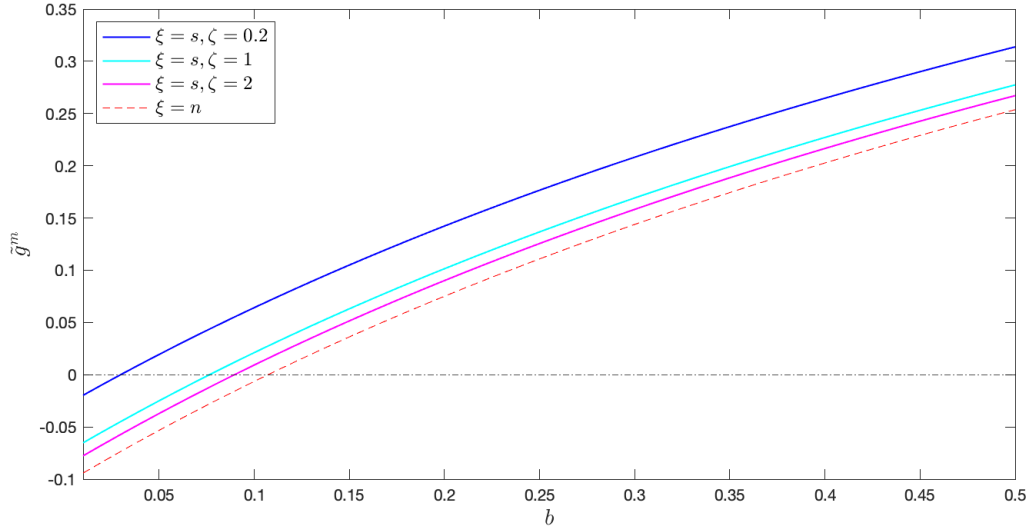
Note: This figure plots the rate of total debt growth due to the market debt issuance $\tilde{g}_t^{m,\xi}$ against the bank debt ratio b_t when the distance to default $\tilde{y}_t = 0.08$ for different levels of shock persistence. The parameters values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r^n = r = (1-\pi)c = 0.05$, $r^s = 0.07$, and $\alpha = 2.33$.

debt growth due to market debt issuance $\tilde{g}_t^m$ against the bank debt ratio b_t at various levels of distance to default, for both $\theta = 0.2$ and $\theta = 0$. The results are shown in Figure 12. The parameter values are as follows: $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = c(1-\pi) = 0.05$, $\lambda = 0.02$, and $\alpha = 2.33$.

In the figure, the solid lines correspond to $\theta = 0.2$ while the dashed lines correspond to $\theta = 0$. For each given distance to default (represented by different colors in the figure), the slope of the solid line is smaller than that of the dashed line. That is, for an increase in the firm's bank debt ratio b_t, the firm's debt growth due to market debt issuance $\tilde{g}_t^m$ rises less when the firm has positive bargaining power than when it has none. In other words, assigning bargaining power to the firm weakens the strategic complementarity between the bank debt and the market debt.

How does the allocation of bargaining power between the bank lender and the firm affect the total firm value? When the market debt investors are absent and there are only the bank lender and the firm, it is a well-known result that the allocation of bargaining power does not affect

48

**Figure 12:** Debt growth due to market debt issuance $\tilde{g}_t^m$

Note: This figure plots the rate of total debt growth due to the market debt issuance $\tilde{g}_t^m$ as a function of the bank debt ratio b_t at various levels of distance to default $\tilde{y}_t$ (including $\tilde{y}_t = 0.01, 0.02, 0.04$), for $\theta = 0.2$ and $\theta = 0$ respectively. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = (1 - \pi)c = 0.05$, $\lambda = 0.02$, and $\alpha = 2.33$.

the total value. However, in the presence of market debt investors as a silent non-negotiating party, this result no longer holds. This is because outcome of the bargaining game between the bank lender and the firm affects the value of the market debt investors. In Figure 13, I plot the total enterprise value against the firm's bank debt ratio for various levels of distance to default. The total enterprise value is defined as the firm value scaled by the amount of market debt outstanding. The figure shows that assigning bargaining power to the firm reduces the total enterprise value. Intuitively, the source of the bargaining surplus is the dilution effect due to the sandwich capital structure. If the equity holders have bargaining power, the bank lender's incentive to dilute decreases and the bargaining surplus also decreases.

7   Conclusion

This paper develops a dynamic model of firm debt structure that simultaneously incorporates both bank debt and market debt. Market debt investors are arm's-length, and market debt is

Figure 13: Total enterprise value $v + p^m$
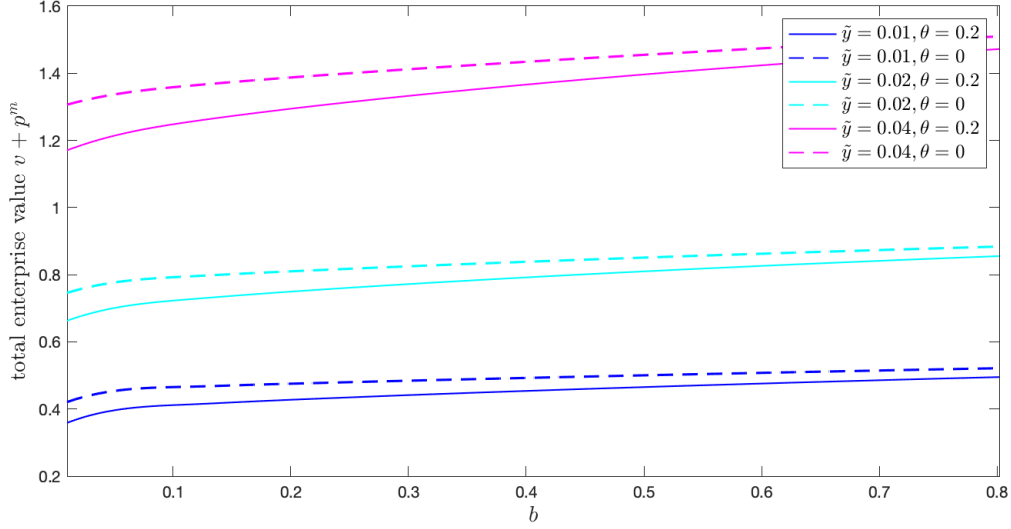
Note: This figure plots the total enterprise value $v + p^m$ as a function of the bank debt ratio b_t at various levels of distance to default $\tilde{y}_t$ (including $\tilde{y}_t = 0.01, 0.02, 0.04$), for $\theta = 0.2$ and $\theta = 0$ respectively. The parameter values are $\mu = 0.02$, $\sigma = 0.25$, $\pi = 0.3$, $r = (1 - \pi)c = 0.05$, $\lambda = 0.02$, and $\alpha = 2.33$.

issued or repurchased at its prevailing competitive prices. By contrast, the bank lender forms a bilateral relationship with the firm that is inherently strategic. Hence, bank debt issuances or repurchases are outcomes of negotiations between the bank lender and the firm that occur periodically. For example, the bank lender and the firm often renegotiate when a bank debt matures and is being rolled over, leading to changes in the firm's bank debt outstanding under negotiated terms. A key prediction of the model is the strategic complementarity between bank debt and market debt. When the firm has a high proportion of bank debt in its debt structure, negotiation between the bank lender and the firm would result in bank debt repurchase. Reduction in the face amount of bank debt boosts the market debt value, leading to greater market debt capacity today and more market debt issuance. Conversely, when the firm has a low proportion of bank debt in its debt structure, bank-firm negotiation would result in bank debt issuance, depressing market debt value and slowing market debt issuance. When the strategic complementarity is sufficiently strong, bank debt and market debt appear as complements. Empirically, bank debt and market debt are complementary in the sense that a greater proportion of bank debt predicts higher subsequent market debt issuance. The pattern

is more pronounced for smaller, less profitable and riskier firms as well as during periods of stress. Moreover the model is able to generate repurchases of market debt by the firm, which is consistent with empirical data. This is in contrast to the leverage ratchet effect which states that the firm never voluntarily repurchases its debt. The firm tends to buy back market debt when it experiences a large negative shock to its bank credit supply, when bank-firm negotiations are frequent (e.g., the bank debt is short in maturity), or when the firm is close to default. The strategic complementarity of bank and market debt is robust to the allocation of bargaining power between the bank lender and the firm. However, the allocation of bargaining power has efficiency implications.

References

Admati, Anat R., Peter M. DeMarzo, Martin F. Hellwig, and Paul Pfleiderer, "The Leverage Ratchet Effect," *Journal of Finance*, 2018, *73* (1), 145–198.

Becker, Bo and Victoria Ivashina, "Cyclicality of Credit Supply: Firm Level Evidence," *Journal of Monetary Economics*, 2014, *62*, 76–93.

Coase, Ronald H., "Durability and Monopoly," *Journal of Law and Economics*, 1972, *15* (1), 143–149.

Crouzet, Nicolas, "Aggregate Implications of Corporate Debt Choices," *Review of Economic Studies*, 2018, *85* (3), 1635–1682.

DeMarzo, Peter M. and Zhiguo He, "Leverage Dynamics without Commitment," *Journal of Finance*, 2021, *76* (3), 1195–1250.

Greenwood, Robin M., Samuel G. Hanson, and Lawrence J. Jin, "Reflexivity in Credit Markets," *Journal of Finance*, 2023, *forthcoming*.

Hackbarth, Dirk, Christopher A. Hennessy, and Hayne E. Leland, "Can the Trade-off Theory Explain Debt Structure?," *Review of Financial Studies*, 2007, *20* (5), 1389–1428.

Leland, Hayne E., "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance*, 1994, *49* (4), 1213–1252.

Miller, Merton H., "Debt and Taxes," *Journal of Finance*, 1977, *32* (2), 261–275.

Park, Cheol, "Monitoring and Structure of Debt Contracts," *Journal of Finance*, 2000, *55* (5), 2157–2195.

Rauh, Joshua D. and Amir Sufi, "Capital Structure and Debt Structure," *Review of Financial Studies*, 2010, *23* (12), 4242–4280.

Appendix

A Additional Empirical Results

A.1 Further Robustness Tests

In this appendix, I present a number of robustness tests to support the empirical results in Section 2. In the first set of robustness tests, I use alternative definitions for bank debt and market debt. In the second set of robustness tests, I resort to a matched-sample regression specification by including industry-year-size-leverage fixed effects as the main controls. Finally, I show the pattern of complementarity between bank and market debt using time-series aggregate balance sheet data from the Flow of Funds.

A. Alternative Definitions

In the baseline, I define bank debt as term loans, and define market debt as the sum of senior bonds and notes and subordinated bonds and notes. For robustness, I also account for short-term credit by expanding the definition of bank debt to include drawn credit lines and the definition of market debt to include commercial paper. Credit lines and commercial paper are both short-term financing instruments that differ in nature from either term loans or bonds and notes. However, excluding credit lines and commercial paper may impact my results if firms shift between credit lines and term loans or between commercial paper and bonds and notes. I re-estimate the specification (2.1) using the alternative definitions of bank and market debt, and report the results in Table A.1.

The coefficient of interest is the coefficient on the bank debt share variable. In all columns of Table A.1, the coefficient estimates on bank debt share are all positive and highly statistically significant. The magnitudes are also similar to the baseline results reported in Table 2. The results thus show that the baseline findings are robust to alternative definitions of bank debt and market debt.

Table A.1: Panel Regressions: Robustness Using Alternative Definitions

	Market Debt Net Issuance Rate					
	(1)	(2)	(3)	(4)	(5)	(6)
Bank Debt Share	0.857***	0.599***	0.931***	0.607***	0.374***	0.405***
	(0.214)	(0.100)	(0.235)	(0.101)	(0.090)	(0.102)
Log(Assets)	-0.199***	-0.155***	-0.202***	-0.163***	-0.252**	-0.285**
	(0.045)	(0.040)	(0.051)	(0.041)	(0.112)	(0.127)
Tangibility	0.268	0.025	0.203	-0.125	0.027	-0.002
	(0.280)	(0.361)	(0.249)	(0.280)	(0.325)	(0.358)
ROA	-0.040	-0.320*	-0.095	-0.351*	-0.167	-0.205
	(0.181)	(0.159)	(0.209)	(0.185)	(0.355)	(0.428)
Book Leverage	-1.367***	-1.285***	-1.445***	-1.368***	-1.064***	-1.184***
	(0.216)	(0.258)	(0.216)	(0.241)	(0.323)	(0.370)
Debt-to-EBITDA	-0.000	0.000	0.000	0.000	-0.000	-0.000
	(0.001)	(0.000)	(0.001)	(0.001)	(0.001)	(0.001)
Dividend Payout		-0.002		0.007	0.003	0.005*
		(0.005)		(0.006)	(0.002)	(0.002)
Market-to-Book		0.169***		0.190***	0.191***	0.212***
		(0.035)		(0.041)	(0.063)	(0.074)
Stock Return		-0.005		-0.004	-0.005	-0.007
		(0.007)		(0.008)	(0.007)	(0.007)
Sales Growth		-0.032		-0.048	0.044	0.056
		(0.057)		(0.065)	(0.078)	(0.091)
Sample	All	All	All	All	Rated	Rated
Conditional on Issuance	No	No	Yes	Yes	No	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes
Year FE	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.212	0.140	0.251	0.206	0.099	0.104
Observations	38,502	29,840	34,028	26,289	16,239	14,321

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The table presents results of the regression specification (2.1) using data from Compustat, S&P Credit Ratings, and Capital IQ for the period 2002-2024. The outcome variable is market debt net issuance rate defined as the net increase in market debt outstanding during the following year scaled by the total amount of bank and market debt. The coefficients of interest are the coefficients on bank debt share defined as the amount of bank debt divided by the sum of bank and market debt. Columns 1-4 use all firm-year observations from the estimation sample. Columns 5-6 restrict to observations with available S&P credit ratings. Columns 1, 2 ad 5 are unconditional on non-zero market debt changes while columns 3, 4 and 6 are conditional on non-zero market debt changes. In columns 1 and 3, the firm-year level controls include the log of book assets, tangibility, ROA, book leverage, and debt-to-EBITDA. Columns 2 and 4-6 also include dividend payout, market-to-book, stock return, and sales growth as additional control variables. Standard errors are double-clustered at the firm and year levels.

B. Matched-Sample Analysis

A potential concern is that firms' debt issuance decisions are affected by time-varying credit market conditions. For example, the observed increase in subsequent market debt net issuance following a higher bank debt share could be due to a decline in the relative credit spreads of bonds compared to bank loans. This could arise from increased investor demand for corporate bonds. Moreover, the magnitude of this credit spread differential between bonds and bank loans may vary across firms with different characteristics and across industries, further complicating identification. To address this concern, I tighten the regression specification by including the interacted industry-year-size-leverage fixed effects.

Table A.2: Panel Regressions: Robustness Using Alternative Definitions

	Market Debt Net Issuance Rate			
	(1)	(2)	(3)	(4)
Bank Debt Share	0.154**	0.159**	0.135**	0.133*
	(0.063)	(0.065)	(0.061)	(0.065)
Debt Definition	Baseline	Baseline	Robustness	Robustness
Conditional on Issuance	No	Yes	No	Yes
Industry-Year-Size-Leverage FEs	Yes	Yes	Yes	Yes
R^2	0.211	0.214	0.221	0.224
Observations	29,353	25,876	40,109	35,358

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: The table presents results from regressions of market debt net issuance rate (defined as the net increase in market debt outstanding during the following year scaled by the total amount of bank and market debt) on bank debt share (defined as the amount of bank debt divided by the sum of bank and market debt), while controlling for the interacted industry-year-size-leverage fixed effects. Specifically size corresponds to quintiles of firm size as measured by total assets, and leverage corresponds to quintiles of book leverage. Columns 1-2 follow baseline definitions of bank debt (term loans) and market debt (senior and subordinated bonds and notes), while columns 3-4 follow definitions of bank debt (term loans and drawn credit lines) and market debt (senior and subordinated bonds and notes and commercial paper) in the robustness tests. Columns 1 and 3 are unconditional on non-zero market debt changes while columns 2 and 4 are conditional on non-zero market debt changes. Data are from Compustat and Capital IQ for the period 2002-2024. Standard errors are double-clustered at the firm and year levels.

In particular, industry fixed effects are constructed using the first two digits of the NAICS code, capturing broad sectoral classifications. Size fixed effects are defined by quintiles of firm size as measured by book assets. Leverage fixed effects are defined by quintiles of book leverage computed as the total debt outstanding divided by book assets. The coefficient estimates on bank debt share are reported in Table A.2. In columns 1 and 2, I follow the baseline definitions of bank debt and market debt. That is, bank debt is defined as bank loans and market debt

includes senior and subordinated bonds and notes. In columns 3 and 4, I use the alternative definitions used in the robustness tests above. That is, bank debt is comprised of bank loans and drawn credit lines, while market debt includes commercial paper as well as senior and subordinated bonds and notes. Columns 1 and 3 are unconditional on market debt net issuance or repurchase, while columns 2 and 4 are conditional on non-zero market debt changes.

In all columns of Table A.2, the coefficient estimates on bank debt share are lower in magnitude compared to those reported in the baseline regressions. However, the coefficient estimates remain positive and statistically significant at conventional levels. The results suggest that the baseline findings are also robust to alternative specifications with tighter controls. These robustness tests thus provide further evidence supporting the complementarity between bank debt and market debt.

C. Aggregate-Level Evidence

I now document the complementarity of bank and market debt using aggregate time-series data. The aggregate data are obtained from the Flow of Funds, which contains quarterly balance sheet information of the U.S. non-financial corporate sector and is accessible via the FRED database of the St. Louis Fed. In Panel A of Figure A.1, I plot the relationship between bank debt share and net issuance rate of market debt in the following year, over the sample period from Q1 1953 to Q4 2024. The bank debt share is the ratio of depository institution loans to corporate bonds (ex. eREITs). The market debt net issuance rate is the net increase in corporate bonds scaled by the combined total of depository institution loans and corporate bonds. The figure shows a positive relationship between the subsequent market debt net issuance rate and the prevailing bank debt share. In other words, firms tend to issue more market debt following periods in which their debt structures contain a greater share of bank debt.

The positive relationship between bank debt share and the subsequent market debt net issuance rate may be a mechanical artifact of the secular decline in the bank-to-market-debt ratio over the sample period. To address this concern, I focus on the period between Q1 2012 and Q4 2024 when the bank debt share remained relatively stable. I exclude the first two

Figure A.1: Bank-to-Market-Debt Ratio and Subsequent Debt Issuances

(A) Subsequent Market Debt Net Issuance

(B) Subsequent Market Debt Net Issuance (Post-2012)
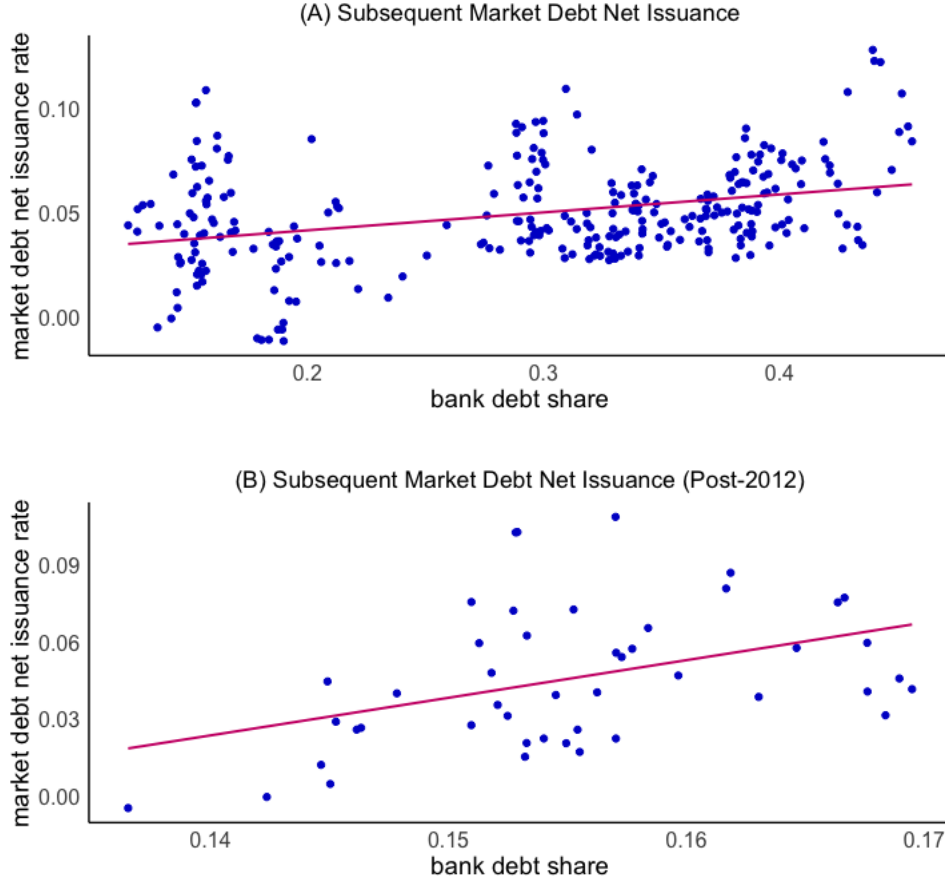
Note: The figure plots the relationship between bank debt share and market debt net issuance rate in the following year, for the full sample period (Panel A) and for the post-2012 subsample excluding the COVID-19 crisis period (Panel B). The full sample contains quarterly data from Q1 1965 to Q3 2024. The post-2012 subsample contains quarterly data from Q1 2012 to Q3 2024, while the COVID shock period is defined as the first two quarters of 2020. The bank debt share is defined as the ratio of depository institution loans to corporate bonds (ex. eREITs) obtained from the Flow of Funds. The market debt net issuance rate is the net increase in corporate bonds (ex. eREITs) scaled by the sum of depository institution loans and corporate bonds.

quarters of 2020 (COVID-19 shock period) from the sample to remove any impact due to unconventional Federal Reserve policies targeting bank lending and the corporate bond market. The positive relationship between bank debt share and market debt net issuance rate remains robust in this subsample period, as seen from Panel B of Figure A.1.

B Derivations and Proofs

B.1 Proof of Proposition 1

Suppose that the firm and the bank lender agree on the proposed terms $\{p_t^b, Q_t^b\}$ when negotiating the bank debt issuance. The equity holders receive a cash inflow of $p_t^b Q_t^b$ in issuance proceeds from the bank lender, and their value becomes $V^e(Y_t, M_t, B_t + Q_t^b)$. If the firm and the bank fail to agree on the proposed terms, the equity holders' total value remains at $V^e(Y_t, M_t, B_t)$. Hence, the equity holders' surplus from bargaining is given by

$$V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t). \tag{B.1}$$

Similarly, the bank lender's surplus from bargaining is

$$V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t). \tag{B.2}$$

The issuance price and the issuance quantity are both determined by Nash bargaining between the firm and the bank lender, where the firm has bargaining power θ. Let $\{\bar{p}_t^b, \bar{Q}_t^b\}$ denote the outcome of the Nash bargaining problem. Then

$$\{\bar{p}_t^b, \bar{Q}_t^b\} = \arg\max_{p_t^b, Q_t^b} \left[V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t) \right]^{\theta}$$
$$\left[V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t) \right]^{1-\theta}, \tag{B.3}$$

subject to $V^e(Y_t, M_t, B_t + Q_t^b) + p_t^b Q_t^b - V^e(Y_t, M_t, B_t) \geq 0$ and $V^b(Y_t, M_t, B_t + Q_t^b) - p_t^b Q_t^b - V^b(Y_t, M_t, B_t) \geq 0$. The solution to the Nash bargaining problem is characterized by

$$\bar{Q}_t^b = \arg\max_{Q_t^b} \left[V^e(Y_t, M_t, B_t + Q_t^b) + V^b(Y_t, M_t, B_t + Q_t^b) \right], \tag{B.4}$$

and given the issuance quantity $\bar{Q}_t^b$,

$$(1-\theta)\left[V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + \bar{p}_t^b \bar{Q}_t^b - V^e(Y_t, M_t, B_t)\right]$$
$$= \theta\left[V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - \bar{p}_t^b \bar{Q}_t^b - V^b(Y_t, M_t, B_t)\right]. \tag{B.5}$$

Rearranging (B.5) yields

$$\bar{p}_t^b = \frac{\theta[V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - V^b(Y_t, M_t, B_t)]}{\bar{Q}_t^b}$$
$$- \frac{(1-\theta)[V^e(Y_t, M_t, B_t + \bar{Q}_t^b) - V^e(Y_t, M_t, B_t)]}{\bar{Q}_t^b}, \tag{B.6}$$

where $\bar{Q}_t^b$ is given by (B.4). Furthermore, substituting (B.4) and (B.5) into the equity holders' surplus (B.1) and the bank lender's surplus (B.2), we have

$$V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + \bar{p}_t^b \bar{Q}_t^b - V^e(Y_t, M_t, B_t)$$
$$= \theta\left[V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - V^e(Y_t, M_t, B_t) - V^b(Y_t, M_t, B_t)\right], \tag{B.7}$$

and

$$V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - \bar{p}_t^b \bar{Q}_t^b - V^b(Y_t, M_t, B_t)$$
$$= (1-\theta)\left[V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - V^e(Y_t, M_t, B_t)\right. \tag{B.8}$$
$$\left. - V^b(Y_t, M_t, B_t)\right].$$

In words, the firm captures a fraction θ of the joint surplus and the bank lender captures a fraction $1 - \theta$ of the joint surplus.

I now verify that the solution $\{\bar{p}_t^b, \bar{Q}_t^b\}$ satisfy the conditions $V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + \bar{p}_t^b \bar{Q}_t^b - V^e(Y_t, M_t, B_t) \geq 0$ and $V^b(Y_t, M_t, B_t + \bar{Q}_t^b) - \bar{p}_t^b \bar{Q}_t^b - V^b(Y_t, M_t, B_t) \geq 0$ are satisfied. First, note that (B.4) implies that $V^e(Y_t, M_t, B_t + \bar{Q}_t^b) + V^b(Y_t, M_t, B_t + \bar{Q}_t^b) \geq V^e(Y_t, M_t, B_t) + V^b(Y_t, M_t, B_t)$. Substituting the inequality into (B.7) and (B.8), it is immediate that the equity holders' surplus and the bank lender's surplus are both non-negative.

B.2   Proof of Proposition 2

The market debt issuance rate can be obtained from the first-order condition (3.5). The equilibrium market debt price is denoted by $p^m(Y_t, M_t, B_t)$. Then the first-order condition can be written as

$$p_t^m \equiv p^m(Y_t, M_t, B_t) = -V_M^e(Y_t, M_t, B_t). \tag{B.9}$$

Differentiating both sides of (3.15) with respect to M_t yields

$$
\begin{aligned}
rV_M^e(Y_t, M_t, B_t) = & - (1 - \pi)c + \mu Y_t V_{MY}^e(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{MYY}^e(Y_t, M_t, B_t) \\
& + \theta\lambda \left[\frac{\partial V(Y_t, M_t, B_t + \bar{Q}_t^b)}{\partial M_t} - V_M(Y_t, M_t, B_t) \right].
\end{aligned}
\tag{B.10}
$$

Suppose that $V(Y_t, M_t, B_t)$ is concave, then $\bar{Q}_t^b$ must be such that the following first-order condition is satisfied

$$V_B(Y_t, M_t, B_t + \bar{Q}_t^b) = 0, \tag{B.11}$$

where V_B denotes the first-order derivative of $V(Y_t, M_t, B_t)$ with respect to B_t. Hence,

$$
\begin{aligned}
\frac{\partial V(Y_t, M_t, B_t + \bar{Q}_t^b)}{\partial M_t} &= V_M(Y_t, M_t, B_t + \bar{Q}_t^b) + V_B(Y_t, M_t, B_t + \bar{Q}_t^b)\frac{\partial \bar{Q}_t^b}{\partial M_t} \\
&= V_M(Y_t, M_t, B_t + \bar{Q}_t^b),
\end{aligned}
\tag{B.12}
$$

where the last equality follows from the first-order condition (B.11). Hence, the HJB equation (B.10) can be rewritten as

$$
\begin{aligned}
rV_M^e(Y_t, M_t, B_t) = & - (1 - \pi)c + \mu Y_t V_{MY}^e(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{MYY}^e(Y_t, M_t, B_t) \\
& + \theta\lambda \left[V_M(Y_t, M_t, B_t + \bar{Q}_t^b) - V_M(Y_t, M_t, B_t) \right].
\end{aligned}
\tag{B.13}
$$

Moreover, from (B.9),

$$p_Y^m(Y_t, M_t, B_t) = -V_{MY}^e(Y_t, M_t, B_t), \tag{B.14}$$

$$p_{YY}^m(Y_t, M_t, B_t) = -V_{MYY}^e(Y_t, M_t, B_t). \tag{B.15}$$

Hence, substituting (B.9), (B.14)-(B.15) into (B.13) yields

$$
\begin{aligned}
-rp^m(Y_t, M_t, B_t) = &- (1-\pi)c - \mu Y_t p_Y^m(Y_t, M_t, B_t) - \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^m(Y_t, M_t, B_t) \\
&+ \theta\lambda\big[V_M(Y_t, M_t, B_t + \bar{Q}_t^b) - V_M(Y_t, M_t, B_t)\big].
\end{aligned} \tag{B.16}
$$

Adding up (B.16) and (3.9) yields

$$
\begin{aligned}
0 = &\pi c + g_t^m M_t p_M^m(Y_t, M_t, B_t) + \lambda\big[p^m(Y_t, M_t, B_t + \bar{Q}_t^b) - p^m(Y_t, M_t, B_t)\big] \\
&+ \theta\lambda\big[V_M(Y_t, M_t, B_t + \bar{Q}_t^b) - V_M(Y_t, M_t, B_t)\big].
\end{aligned} \tag{B.17}
$$

Rearranging yields

$$
\begin{aligned}
g_t^m = &\frac{\pi c}{-M_t p_M^m(Y_t, M_t, B_t)} + \frac{\lambda[p^m(Y_t, M_t, B_t + \bar{Q}_t^b) - p^m(Y_t, M_t, B_t)]}{-M_t p_M^m(Y_t, M_t, B_t)} \\
&+ \frac{\theta\lambda[V_M(Y_t, M_t, B_t + \bar{Q}_t^b) - V_M(Y_t, M_t, B_t)]}{-M_t p_M^m(Y_t, M_t, B_t)}.
\end{aligned} \tag{B.18}
$$

B.3 Competitive Bank Debt

Denote by $V^{e,c}(Y_t, M_t, B_t)$ the equity holders' total value in the benchmark with competitive bank lenders. The HJB equation for the equity holders' total value is given by (3.18). When the firm issues bank debt upon Poisson arrivals, the issuance quantity is determined by

$$\bar{Q}_t^{b,c} = \underset{Q_t^{b,c}}{\arg\max} \ \big[V^{e,c}(Y_t, M_t, B_t + Q_t^{b,c}) + p_t^{b,c}Q_t^{b,c} - V^{e,c}(Y_t, M_t, B_t)\big]. \tag{B.19}$$

Taking the first-order condition with respect to $Q_t^{b,c}$, we see that the optimal issuance amount $\bar{Q}_t^{b,c}$ must satisfy

$$V_B^{e,c}(Y_t, M_t, B_t + \bar{Q}_t^{b,c}) + p_t^{b,c} = 0. \tag{B.20}$$

Note that $Q_t^{b,c}$ is a function of the state variables. Moreover, taking the first-order condition with respect to $g_t^{m,c}$ yields

$$p_t^{m,c} = -V_M^{e,c}(Y_t, M_t, B_t). \tag{B.21}$$

Substituting (B.20) and (B.21) into the HJB equation (3.18), we can then rewrite the HJB equation as

$$\begin{aligned} rV^{e,c}(Y_t, M_t, B_t) =& (1 - \pi)\big[Y_t - c(B_t + M_t)\big] \\ &+ \mu Y_t V_Y^{e,c}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,c}(Y_t, M_t, B_t) \\ &+ \lambda\big[V^{e,c}(Y_t, M_t, B_t + \bar{Q}_t^{b,c}) + p_t^{b,c}\bar{Q}_t^{b,c} - V^{e,c}(Y_t, M_t, B_t)\big]. \end{aligned} \tag{B.22}$$

Differentiating both sides with respect to M_t and using (B.20), we get

$$\begin{aligned} rV_M^{e,c}(Y_t, M_t, B_t) =& -(1 - \pi)c + \mu Y_t V_{MY}^{e,c}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{MYY}^{e,c}(Y_t, M_t, B_t) \\ &+ \lambda\big[V_M^{e,c}(Y_t, M_t, B_t + \bar{Q}_t^{b,c}) - V_M^{e,c}(Y_t, M_t, B_t)\big]. \end{aligned} \tag{B.23}$$

Let $p^{m,c}(Y_t, M_t, B_t)$ denote the equilibrium market debt price. From (B.21), we have

$$p^{m,c}(Y_t, M_t, B_t) = -V_M^{e,c}(Y_t, M_t, B_t). \tag{B.24}$$

Thus, (B.23) can be rewritten as

$$
\begin{aligned}
-r p^{m,c}(Y_t, M_t, B_t) = & - (1-\pi)c - \mu Y_t p_Y^{m,c}(Y_t, M_t, B_t) - \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^{m,c}(Y_t, M_t, B_t) \\
& + \lambda\big[- p^{m,c}(Y_t, M_t, B_t + \bar{Q}_t^{b,c}) + p^{m,c}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.25}
$$

On the other hand, as in (3.9), the equilibrium market debt price must satisfy the HJB equation

$$
\begin{aligned}
r p^{m,c}(Y_t, M_t, B_t) = & c + g_t^{m,c} M_t p_M^{m,c}(Y_t, M_t, B_t) \\
& + \mu Y_t p_Y^{m,c}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^{m,c}(Y_t, M_t, B_t) \\
& + \lambda\big[p^{m,c}(Y_t, M_t, B_t + \bar{Q}_t^{b,c}) - p^{m,c}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.26}
$$

Summing up (B.25) and (B.26) yields

$$
0 = \pi c + g_t^{m,c} M_t p_M^{m,c}(Y_t, M_t, B_t).
\tag{B.27}
$$

Thus,

$$
g_t^{m,c} = \frac{\pi c}{-M_t p_M^{m,c}(Y_t, M_t, B_t)}.
\tag{B.28}
$$

B.4 Proof of Proposition 3

The equity holders' no-issuance value $V^{e,0}(Y_t; M, B)$ satisfies the ODE given by (4.1). If the firm never defaults, then the cash flows can be evaluated as growing perpetuities. Let $\tilde{V}^{e,0}(Y_t; M, B)$ denote the equity holders' no-issuance value when there is never default, then

$$
\tilde{V}^{e,0}(Y_t; M, B) = \frac{1-\pi}{r-\mu}Y_t - \frac{(1-\pi)c(B+M)}{r}.
\tag{B.29}
$$

The first term on the right-hand side is the no-default present value of after-tax earnings, and the second term on the right-hand side is the no-default present value of after-tax coupon payments.

The equity holders' no-issuance value $V^{e,0}(Y_t; M, B)$ must thus equal the no-default value as given by (B.29) plus the value of a default option. That is,

$$V^{e,0}(Y_t; M, B) = \tilde{V}^{e,0}(Y_t; M, B) + \omega(Y_t; M, B)\big[0 - \tilde{V}^{e,0}(Y^\star(M, B); M, B)\big], \qquad \text{(B.30)}$$

where $Y^\star(M, B)$ is the default boundary at which the equity holders choose to default and $\omega(Y_t; M, B)$ is the discount factor applied to the default option and follows the homogeneous version of the ODE (4.1). That is,

$$r\omega(Y_t; M, B) = \mu Y_t \omega'(Y_t; M, B) + \frac{1}{2}\sigma^2 Y_t^2 \omega''(Y_t; M, B). \qquad \text{(B.31)}$$

The general solution to (B.31) is

$$\omega(Y_t; M, B) = K_\gamma Y_t^{-\gamma} + K_\eta Y_t^\eta, \qquad \text{(B.32)}$$

where

$$-\gamma \equiv -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}, \qquad \text{(B.33)}$$

$$\eta \equiv -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \qquad \text{(B.34)}$$

It is easy to verify that $-\gamma < 0$ and $\eta > 1$. The constants K_γ and K_η as well as the default boundary $Y^\star(M, B)$ are determined by the boundary conditions. First, the no-bubble condition as $Y_t \to \infty$ says that $\omega(\infty; B, M) = 0$, implying that $K_\eta = 0$. The value matching condition at the default boundary requires that $\omega(Y^\star(M, B); M, B) = 1$, thus $K_\gamma = Y^\star(M, B)^\gamma$ and

$$\omega(Y_t; M, B) = \left[\frac{Y_t}{Y^\star(M, B)}\right]^{-\gamma}. \qquad \text{(B.35)}$$

From (B.30), the equity holders' no-issuance value can thus be rewritten as

$$
\begin{aligned}
V^{e,0}(Y_t; M, B) =& \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c(B+M)}{r} \\
& \underbrace{- \left[\frac{1-\pi}{r-\mu} Y^{\star}(M, B) - \frac{(1-\pi)c(B+M)}{r} \right] \left[\frac{Y_t}{Y^{\star}(M, B)} \right]^{-\gamma}}_{\text{default option value}}.
\end{aligned}
\tag{B.36}
$$

The smooth pasting condition at the default boundary requires that $V^{e,0\prime}(Y_t; M, B) = 0$. From (B.36), the default boundary is thus given by

$$
Y^{\star}(M, B) = \frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} (B+M).
\tag{B.37}
$$

By inspecting (B.36), the default option as captured by the last term on the right-hand side is indeed maximized at the default boundary $Y^{\star}(M, B)$. Substituting the default boundary into (B.36) yields

$$
\begin{aligned}
V^{e,0}(Y_t; M, B) =& \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c}{r} (B+M) \\
& + \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^{\gamma} (B+M)^{1+\gamma} Y_t^{-\gamma}.
\end{aligned}
\tag{B.38}
$$

Notice that the default option value as captured by the last term on the right-hand side is decreasing in Y_t. The equity holders choose to default when Y_t reaches $Y^{\star}(M, B)$ from the above. In equilibrium, the equity holders' total value equals the equity holders' no-issuance value. That is, $V^e(Y_t, M_t, B_t) = V^{e,0}(Y_t; M_t, B_t)$. From (B.38),

$$
\begin{aligned}
V^e(Y_t, M_t, B_t) =& \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c}{r} (B_t + M_t) \\
& + \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^{\gamma} (B_t + M_t)^{1+\gamma} Y_t^{-\gamma}.
\end{aligned}
\tag{B.39}
$$

B.5   Proof of Proposition 4

The equilibrium market debt price satisfies the first-order condition as given by (3.5), that is $p^m(Y_t, M_t, B_t) = -V_M^e(Y_t, M_t, B_t)$. Differentiating (B.39) with respect to M_t yields

$$
\begin{aligned}
V_M^e(Y_t, M_t, B_t) &= -\frac{(1-\pi)c}{r} + \frac{(1-\pi)c}{r}\Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\Big]^\gamma (B_t + M_t)^\gamma Y_t^{-\gamma} \\
&= -\frac{(1-\pi)c}{r}\Big\{1 - \Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\Big]^\gamma \Big(\frac{B_t + M_t}{Y_t}\Big)^\gamma\Big\}.
\end{aligned}
\tag{B.40}
$$

Thus, the equilibrium market debt price is

$$
p^m(Y_t, M_t, B_t) = \frac{(1-\pi)c}{r}\Big\{1 - \Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\Big]^\gamma \Big(\frac{B_t + M_t}{Y_t}\Big)^\gamma\Big\}.
\tag{B.41}
$$

Differentiating (B.41) with respect to M_t yields

$$
p_M^m(Y_t, M_t, B_t) = -\gamma\frac{(1-\pi)c}{r}\Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\Big]^\gamma \frac{(B_t + M_t)^{\gamma-1}}{Y_t^\gamma}.
\tag{B.42}
$$

Thus,

$$
-M_t p_M^m(Y_t, M_t, B_t) = \gamma\frac{(1-\pi)c}{r}\Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\frac{B_t + M_t}{Y_t}\Big]^\gamma \frac{M_t}{B_t + M_t}.
\tag{B.43}
$$

Moreover,

$$
\begin{aligned}
&p^m(Y_t, M_t, B_t + \bar{Q}_t^b) - p^m(Y_t, M_t, B_t) \\
&= \frac{(1-\pi)c}{r}\Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\Big]^\gamma \Big[\Big(\frac{B_t + M_t}{Y_t}\Big)^\gamma - \Big(\frac{B_t + M_t + \bar{Q}_t^b}{Y_t}\Big)^\gamma\Big].
\end{aligned}
\tag{B.44}
$$

The firm's optimal market debt issuance rate g_t^m is obtained from (3.17). When $\theta = 0$, using (B.43) and (B.44), we get

$$
\begin{aligned}
g_t^m &\equiv g^m(Y_t, M_t, B_t) \\
&= \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{B_t + M_t} \right]^\gamma \frac{B_t + M_t}{M_t} \\
&\quad + \frac{\lambda}{\gamma} \left[1 - \left(\frac{B_t + M_t + \bar{Q}_t^b}{B_t + M_t} \right)^\gamma \right] \frac{B_t + M_t}{M_t}.
\end{aligned}
\tag{B.45}
$$

Let $\tilde{g}_t^m \equiv g_t^m M_t / (B_t + M_t)$ be the rate of increase in the firm's total debt outstanding due to its market debt issuance. From (B.45),

$$
\tilde{g}_t^m = \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{B_t + M_t} \right]^\gamma + \frac{\lambda}{\gamma} \left[1 - \left(\frac{B_t + M_t + \bar{Q}_t^b}{B_t + M_t} \right)^\gamma \right].
\tag{B.46}
$$

Since $\gamma > 0$, it is immediate that the first term on the right-hand side is decreasing in B_t. To see that the second term is increasing in B_t, note that

$$
\bar{Q}_t^b = \arg\max_{Q_t^b} \left[V^e(Y_t, M_t, B_t + Q_t^b) + V^b(Y_t, M_t, B_t + Q_t^b) \right].
\tag{B.47}
$$

Choosing the issuance quantity Q_t^b that maximizes the joint value is equivalent to choosing the post-issuance amount of bank debt $B_t' = B_t + Q_t^b$ that maximizes the joint value. Hence, (B.47) is the same as

$$
\bar{B}_t = \arg\max_{B_t'} \left[V^e(Y_t, M_t, B_t') + V^b(Y_t, M_t, B_t') \right].
\tag{B.48}
$$

It is immediate that $\bar{B}_t$ depends only on Y_t and M_t. We can thus write $\bar{B}_t = \bar{B}(Y_t, M_t)$. Then (B.46) can be rewritten as

$$
\tilde{g}_t^m = \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{B_t + M_t} \right]^\gamma + \frac{\lambda}{\gamma} \left[1 - \left(\frac{\bar{B}(Y_t, M_t) + M_t}{B_t + M_t} \right)^\gamma \right].
\tag{B.49}
$$

The second term on the right-hand side is increasing in B_t.

B.6 Market Debt Only

Consider the case where the firm can only issue market debt. This is the same setting as DeMarzo and He (2021). The only two state variables are the firm's EBIT Y_t and the market debt outstanding M_t. The equity holders' total value satisfies the following HJB equation

$$rV^e(Y_t, M_t) = \max_{g_t^m} \left\{ (1-\pi)(Y_t - cM_t) + p_t^m g_t^m M_t + g_t^m M_t V_M^e(Y_t, M_t) \right. \tag{B.50}$$
$$\left. + \mu Y_t V_Y^e(Y_t, M_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^e(Y_t, M_t) \right\}.$$

Taking the first-order condition with respect to g_t^m, we get $p_t^m = -V_M^e(Y_t, M_t)$. Substituting the first-order condition into the HJB equation yields

$$rV^e(Y_t, M_t) = (1-\pi)(Y_t - cM_t) + \mu Y_t V_Y^e(Y_t, M_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^e(Y_t, M_t). \tag{B.51}$$

Differentiating both sides with respect to M_t and using the first-order condition with respect to the market debt issuance rate, we get

$$-rp^m(Y_t, M_t) = -(1-\pi)c - \mu Y_t p_Y^m(Y_t, M_t) - \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^m(Y_t, M_t). \tag{B.52}$$

The HJB equation for the equilibrium market debt price $p^m(Y_t, M_t)$ is

$$rp^m(Y_t, M_t) = c + g_t^m M_t p_M^m(Y_t, M_t) + \mu Y_t p_Y^m(Y_t, M_t) + \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^m(Y_t, M_t). \tag{B.53}$$

Summing up (B.52) and (B.53) and rearranging

$$g_t^m = \frac{\pi c}{-M_t p_M^m(Y_t, M_t)}. \tag{B.54}$$

From (B.51), the equity holders' total value $V^e(Y_t, M_t)$ can be solved as if the firm never issues any debt. Let $V^{e,0}(Y_t; M)$ denote the equity holders' no-issuance value given the amount

of debt outstanding M. Then

$$rV^{e,0}(Y_t; M) = (1 - \pi)(Y_t - cM) + \mu Y_t V^{e,0\prime}(Y_t; M) + \frac{1}{2}\sigma^2 Y_t^2 V^{e,0\prime\prime}(Y_t; M). \tag{B.55}$$

If the firm never defaults, the equity holders' no-issuance value would be

$$\tilde{V}^{e,0}(Y_t; M) = \frac{1 - \pi}{r - \mu} Y_t - \frac{(1 - \pi)cM}{r}. \tag{B.56}$$

The equity holders' no-issuance value $V^{e,0}(Y_t; M)$ must equal the no-default value $\tilde{V}^{e,0}(Y_t; M)$ plus the value of a default option. That is,

$$V^{e,0}(Y_t; M) = \tilde{V}^{e,0}(Y_t; M) + \omega(Y_t; M)\big[0 - \tilde{V}^{e,0}(Y^\star(M); M)\big], \tag{B.57}$$

where $Y^\star(M)$ is the default boundary and $\omega(Y_t; M)$ is the discount factor applied to the default option and follows the homogeneous version of the ODE (B.55). That is,

$$r\omega(Y_t; M) = \mu Y_t \omega'(Y_t; M) + \frac{1}{2}\sigma^2 Y_t^2 \omega''(Y_t; M). \tag{B.58}$$

The general solution to the ODE is given by

$$\omega(Y_t; M) = H_\gamma Y_t^{-\gamma} + H_\eta Y_t^\eta, \tag{B.59}$$

where

$$-\gamma \equiv -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}, \tag{B.60}$$

$$\eta \equiv -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{B.61}$$

It is easy to verify that $-\gamma < 0$ and $\eta > 1$. The no-bubble condition as $Y_t \to \infty$ implies that $H_\eta = 0$. The value matching condition requires that $\omega(Y^\star(M); M) = 1$, thus $H_\gamma = Y^\star(M)^\gamma$.

Then from (B.57),

$$
\begin{aligned}
V^{e,0}(Y_t; M) = & \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c}{r} M \\
& - \left[\frac{1-\pi}{r-\mu} Y^\star(M) - \frac{(1-\pi)c}{r} M \right] \left[\frac{Y_t}{Y^\star(M)} \right]^{-\gamma}.
\end{aligned}
\tag{B.62}
$$

The smooth pasting condition at the default boundary requires that $V^{e,0\prime}(Y_t; M) = 0$. From (B.62), the default boundary is given by

$$
Y^\star(M) = \frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} M.
\tag{B.63}
$$

Hence, the equity holders' no-issuance value can be rewritten as

$$
\begin{aligned}
V^{e,0}(Y_t; M) = & \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c}{r} M \\
& + \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^\gamma M^{1+\gamma} Y_t^{-\gamma}.
\end{aligned}
\tag{B.64}
$$

In equilibrium, the equity holders' total value equals the equity holders' no-issuance value. That is, $V^e(Y_t; M) = V^{e,0}(Y_t; M)$. Then

$$
\begin{aligned}
V^e(Y_t, M_t) = & \frac{1-\pi}{r-\mu} Y_t - \frac{(1-\pi)c}{r} M_t \\
& + \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^\gamma M_t^{1+\gamma} Y_t^{-\gamma}.
\end{aligned}
\tag{B.65}
$$

Using the first-order condition with respect to the market debt issuance rate, the equilibrium market debt price is

$$
p^m(Y_t, M_t) = - V_M^e(Y_t, M_t) = \frac{(1-\pi)c}{r} \left\{ 1 - \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^\gamma \left(\frac{M_t}{Y_t} \right)^\gamma \right\}.
\tag{B.66}
$$

70

Differentiating with respect to M_t yields

$$p_M^m(Y_t, M_t) = -\gamma \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^\gamma \frac{M_t^{\gamma-1}}{Y_t^\gamma}. \tag{B.67}$$

Thus,

$$-M_t p_M^m(Y_t, M_t) = \gamma \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \frac{M_t}{Y_t} \right]^\gamma. \tag{B.68}$$

The firm's optimal market debt issuance rate g_t^m can be obtained from (B.54). From (B.68),

$$g_t^m = \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{Y_t}{M_t} \right]^\gamma. \tag{B.69}$$

When the firm can access market debt financing, Y_t/M_t is the total debt coverage ratio, and g_t^m is also the rate of total debt increase due to market debt issuance.

B.7 Bank Debt Value When $\theta = 0$

The firm is scale invariant in the setting. Thus, the bank lender's value $V^b(Y_t, M_t, B_t)$ is homogeneous of degree one. That is,

$$V^b(Y_t, M_t, B_t) = V^b\left(\frac{Y_t}{M_t}, 1, \frac{B_t}{M_t} \right) M_t \equiv v^b(y_t, b_t) M_t. \tag{B.70}$$

Hence, the joint value of the equity holders and the bank lender $V(Y_t, M_t, B_t) \equiv V^e(Y_t, M_t, B_t) + V^b(Y_t, M_t, B_t)$ is also homogeneous of degree one. That is,

$$V(Y_t, M_t, B_t) = V\left(\frac{Y_t}{M_t}, 1, \frac{B_t}{M_t} \right) M_t \equiv v(y_t, b_t) M_t, \tag{B.71}$$

where $v(y_t, b_t) \equiv v^e(y_t, b_t) + v^b(y_t, b_t)$ with $v^e(y_t, b_t) \equiv V^e(Y_t, M_t, B_t)/M_t$. From (3.11) of Proposition 1,

$$\bar{Q}_t^b = \arg\max_{Q_t^b} \ V(Y_t, M_t, B_t + Q_t^b). \tag{B.72}$$

Given the amount of market debt outstanding M_t, (B.72) is equivalent to

$$\bar{q}_t^b \equiv \frac{\bar{Q}_t^b}{M_t} = \arg\max_{q_t^b} \ v(y_t, b_t + q_t^b). \tag{B.73}$$

Here, $\bar{q}_t^b$ is the negotiated amount of bank debt issuance scaled by the amount of market debt outstanding. From (B.73), it is easy to see that $\bar{q}_t^b$ is a function of the firm's EBIT-to-market-debt ratio y_t as well as the bank debt ratio b_t, that is $\bar{q}_t^b \equiv \bar{q}^b(y_t, b_t)$. Note that

$$V(Y_t, M_t, B_t + \bar{Q}_t^b) = v(y_t, b_t + \bar{q}_t^b)M_t, \tag{B.74}$$

where $\bar{q}_t^b$ is given by (B.73). Thus, from (B.73) and (B.74),

$$V(Y_t, M_t, B_t + \bar{Q}_t^b) = \max_{q_t^b} \ v(y_t, b_t + q_t^b)M_t. \tag{B.75}$$

From (B.70),

$$V_M^b(Y_t, M_t, B_t) = v^b(y_t, b_t) - y_t v_y^b(y_t, b_t) - b_t v_b^b(y_t, b_t), \tag{B.76}$$

$$V_Y^b(Y_t, M_t, B_t) = v_y^b(y_t, b_t), \tag{B.77}$$

$$V_{YY}^b(Y_t, M_t, B_t) = \frac{1}{M_t} v_{yy}^b(y_t, b_t). \tag{B.78}$$

Substituting (B.70), (B.71), and (B.75)-(B.78) into the bank lender's HJB equation (3.16), we obtain the following HJB equation for the case with $\theta = 0$

$$
\begin{aligned}
rv^b(y_t, b_t)M_t =& cB_t + g_t^m M_t \big[v^b(y_t, b_t) - y_t v_y^b(y_t, b_t) - b_t v_b^b(y_t, b_t) \big] \\
& + \mu Y_t v_y^b(y_t, b_t) + \frac{1}{2}\sigma^2 \frac{Y_t^2}{M_t} v_{yy}^b(y_t, b_t) \\
& + \lambda \big[\max_{q_t^b} v(y_t, b_t + q_t^b) - v(y_t, b_t) \big] M_t.
\end{aligned}
\tag{B.79}
$$

Dividing both sides by M_t and rearranging, we get

$$
\begin{aligned}
(r - g_t^m)v^b(y_t, b_t) =& cb_t - g_t^m b_t v_b^b(y_t, b_t) + (\mu - g_t^m) y_t v_y^b(y_t, b_t) + \frac{1}{2}\sigma^2 y_t^2 v_{yy}^b(y_t, b_t) \\
& + \lambda \big[\max_{q_t^b} v(y_t, b_t + q_t^b) - v(y_t, b_t) \big],
\end{aligned}
\tag{B.80}
$$

where $v_b^b(y_t, b_t)$ denotes the first-order partial derivative with respect to b_t, while $v_y^b(y_t, b_t)$ and $v_{yy}^b(y_t, b_t)$ are the first-order and the second-order partial derivatives with respect to y_t.

The market debt issuance rate g_t^m is given by (4.6), which can be rewritten as

$$
\begin{aligned}
g_t^m \equiv & g^m(y_t, b_t) \\
= & \frac{\pi c}{\gamma}\frac{r}{(1-\pi)c}\Big[\frac{1+\gamma}{\gamma}\frac{r}{(r-\mu)c}\frac{y_t}{1+b_t} \Big]^\gamma (1+b_t) \\
& + \frac{\lambda}{\gamma}\Big\{ 1 - \Big[\frac{1+b_t + \bar{q}^b(y_t, b_t)}{1+b_t} \Big]^\gamma \Big\}(1+b_t),
\end{aligned}
\tag{B.81}
$$

where $\bar{q}^b(y_t, b_t)$ satisfies (B.73). Moreover, $v(y_t, b_t) \equiv v^e(y_t, b_t) + v^b(y_t, b_t)$. From (4.2), we have

$$
\begin{aligned}
v^e(y_t, b_t) \equiv & \frac{V^e(Y_t, M_t, B_t)}{M_t} \\
= & \frac{1-\pi}{r-\mu}y_t - \frac{(1-\pi)c}{r}(1+b_t) \\
& + \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}(1+b_t)\Big[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r} \Big]^\gamma \Big(\frac{y_t}{1+b_t} \Big)^{-\gamma}.
\end{aligned}
\tag{B.82}
$$

73

Note that the two state variables are $y_t \equiv Y_t/M_t$ and $b_t \equiv B_t/M_t$. One can transform the state space such that the model is solved in terms of $\tilde{y}_t \equiv [Y_t - Y^\star(M_t, B_t)]/M_t = y_t - y^\star(b_t)$ and b_t. In the transformed state space, the default boundary is $\tilde{y}^\star = 0$. That is, the equity holders choose to default when the distance to default approaches zero from above. From (B.81), the market debt issuance rate becomes

$$
\begin{aligned}
g^m(\tilde{y}_t, b_t) =& \frac{\pi c}{\gamma} \frac{r}{(1-\pi)c} \left[\frac{1+\gamma}{\gamma} \frac{r}{(r-\mu)c} \frac{\tilde{y}_t + y^\star(b_t)}{1+b_t} \right]^\gamma (1+b_t) \\
&+ \frac{\lambda}{\gamma} \left\{ 1 - \left[\frac{1+b_t+\bar{q}_t^b}{1+b_t} \right]^\gamma \right\} (1+b_t),
\end{aligned}
\tag{B.83}
$$

where $\bar{q}_t^b$ is solution to (B.73), and $y^\star(b_t)$ can be obtained from (4.4), that is

$$
y^\star(b_t) = \frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}(1+b_t).
\tag{B.84}
$$

Similarly, from (B.82), the equity value can be written as

$$
\begin{aligned}
v^e(\tilde{y}_t, b_t) =& \frac{1-\pi}{r-\mu}[\tilde{y}_t + y^\star(b_t)] - \frac{(1-\pi)c}{r}(1+b_t) \\
&+ \frac{1}{1+\gamma} \frac{(1-\pi)c}{r}(1+b_t) \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \right]^\gamma \left[\frac{\tilde{y}_t + y^\star(b_t)}{1+b_t} \right]^{-\gamma}.
\end{aligned}
\tag{B.85}
$$

Note that

$$
\frac{\partial v^b}{\partial b_t} = v_b^b(\tilde{y}_t, b_t) - v_{\tilde{y}}^b(\tilde{y}_t, b_t)y^{\star\prime}(b_t),
\tag{B.86}
$$

$$
\frac{\partial v^b}{\partial y_t} = v_{\tilde{y}}^b(\tilde{y}_t, b_t),
\tag{B.87}
$$

$$
\frac{\partial^2 v^b}{\partial y_t^2} = v_{\tilde{y}\tilde{y}}^b(\tilde{y}_t, b_t).
\tag{B.88}
$$

Substituting (B.86)-(B.88) into (B.80) yields

$$
\begin{aligned}
(r + \lambda - g_t^m) v^b(\tilde{y}_t, b_t) =& c b_t - g_t^m b_t \big[v_b^b(\tilde{y}_t, b_t) - v_{\tilde{y}}^b(\tilde{y}_t, b_t) y^{\star\prime}(b_t) \big] \\
& + (\mu - g_t^m) \big[\tilde{y}_t + y^\star(b_t) \big] v_{\tilde{y}}^b(\tilde{y}_t, b_t) + \frac{1}{2} \sigma^2 \big[\tilde{y}_t + y^\star(b_t) \big]^2 v_{\tilde{y}\tilde{y}}^b(\tilde{y}_t, b_t) \quad \text{(B.89)} \\
& + \lambda \big[\bar{v}(\tilde{y}_t, b_t) - v^e(\tilde{y}_t, b_t) \big],
\end{aligned}
$$

where

$$
\bar{v}(\tilde{y}_t, b_t) = \max_{q_t^b} \, v\big(\tilde{y}_t + y^\star(b_t) - y^\star(b_t + q_t^b), b_t + q_t^b \big). \tag{B.90}
$$

Rearranging (B.89) yields

$$
\begin{aligned}
(r + \lambda - g_t^m) v^b(\tilde{y}_t, b_t) =& c b_t - g_t^m b_t v_b^b(\tilde{y}_t, b_t) \\
& + \big\{ (\mu - g_t^m) [\tilde{y}_t + y^\star(b_t)] + g_t^m b_t y^{\star\prime}(b_t) \big\} v_{\tilde{y}}^b(\tilde{y}_t, b_t) \quad \text{(B.91)} \\
& + \frac{1}{2} \sigma^2 \big[\tilde{y}_t + y^\star(b_t) \big]^2 v_{\tilde{y}\tilde{y}}^b(\tilde{y}_t, b_t) + \lambda \big[\bar{v}(\tilde{y}_t, b_t) - v^e(\tilde{y}_t, b_t) \big].
\end{aligned}
$$

B.8 Issuance Price of Bank Debt

From (3.12), when $\theta = 0$ we have

$$
\bar{p}_t^b = \frac{V^e(Y_t, M_t, B_t) - V^e(Y_t, M_t, B_t + \bar{Q}_t^b)}{\bar{Q}_t^b}. \tag{B.92}
$$

From (4.2),

$$
\begin{aligned}
& V^e(Y_t, M_t, B_t) - V^e(Y_t, M_t, B_t + \bar{Q}_t^b) \\
& = \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \Big[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r} \Big]^\gamma Y_t^{-\gamma} \Big[(B_t + M_t)^{1+\gamma} - (B_t + M_t + \bar{Q}_t^b)^{1+\gamma} \Big] \quad \text{(B.93)} \\
& \quad + \frac{(1-\pi)c}{r} \bar{Q}_t^b.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\bar{p}_t^b ={}& \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^\gamma Y_t^{-\gamma}\frac{(B_t+M_t)^{1+\gamma}-(B_t+M_t+\bar{Q}_t^b)^{1+\gamma}}{\bar{Q}_t^b}\\
&+\frac{(1-\pi)c}{r}\\
={}& \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^\gamma y_t^{-\gamma}\frac{(1+b_t)^{1+\gamma}-(1+b_t+\bar{q}_t^b)^{1+\gamma}}{\bar{q}_t^b}+\frac{(1-\pi)c}{r}.
\end{aligned}
\tag{B.94}
$$

B.9 Bank Lending Shock

Outside default, the equity holders' total value $V^{e,\xi}(Y_t, M_t, B_t)$ when the economy is in state $\xi \in \{n,s\}$ satisfies the following HJB equation

$$
\begin{aligned}
rV^{e,\xi}(Y_t,M_t,B_t) = \max_{g_t^{m,\xi}} \Big\{ & (1-\pi)\big[Y_t - c(B_t+M_t)\big]\\
& + p_t^{m,\xi}g_t^{m,\xi}M_t + g_t^{m,\xi}M_t V_M^{e,\xi}(Y_t,M_t,B_t)\\
& + \mu Y_t V_Y^{e,\xi}(Y_t,M_t,B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,\xi}(Y_t,M_t,B_t)\\
& + \lambda\big[V^{e,\xi}(Y_t,M_t,B_t+Q_t^{b,\xi}) + p_t^{b,\xi}Q_t^{b,\xi} - V^{e,\xi}(Y_t,M_t,B_t)\big]\\
& + \zeta^\xi\big[V^{e,-\xi}(Y_t,M_t,B_t) - V^{e,\xi}(Y_t,M_t,B_t)\big]\Big\},
\end{aligned}
\tag{B.95}
$$

where $-\xi \neq \xi \in \{n,s\}$, $p_t^{m,\xi}$ is the competitive market debt price in state ξ, and $p_t^{b,\xi}$ is the bank debt issuance price in state ξ determined by bargaining. Taking the first-order condition with respect to the market debt issuance rate yields

$$
p_t^{m,\xi} = -V_M^{e,\xi}(Y_t,M_t,B_t).
\tag{B.96}
$$

Hence, the equity holders' HJB equation can be rewritten as

$$
\begin{aligned}
rV^{e,\xi}(Y_t, M_t, B_t) =& (1 - \pi)\big[Y_t - c(B_t + M_t)\big] \\
& + \mu Y_t V_Y^{e,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,\xi}(Y_t, M_t, B_t) \\
& + \lambda\big[V^{e,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) + p_t^{b,\xi}Q_t^{b,\xi} - V^{e,\xi}(Y_t, M_t, B_t)\big] \\
& + \zeta^\xi\big[V^{e,-\xi}(Y_t, M_t, B_t) - V^{e,\xi}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.97}
$$

Similarly, the bank lender's value $V^{b,\xi}(Y_t, M_t, B_t)$ when the economy is in state ξ satisfies the HJB equation

$$
\begin{aligned}
r^\xi V^{b,\xi}(Y_t, M_t, B_t) =& cB_t + g_t^{m,\xi}M_t V_M^{b,\xi}(Y_t, M_t, B_t) \\
& + \mu Y_t V_Y^{b,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{b,\xi}(Y_t, M_t, B_t) \\
& + \lambda\big[V^{b,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) - p_t^{b,\xi}Q_t^{b,\xi} - V^{b,\xi}(Y_t, M_t, B_t)\big] \\
& + \zeta^\xi\big[V^{b,-\xi}(Y_t, M_t, B_t) - V^{b,\xi}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.98}
$$

The bank debt issuance price and the bank debt issuance quantity are both determined by Nash bargaining between the bank lender and the firm. Let $\{\bar{p}_t^{b,\xi}, \bar{Q}_t^{b,\xi}\}$ be the outcome of the Nash bargaining problem, then

$$
\begin{aligned}
\{\bar{p}_t^{b,\xi}, \bar{Q}_t^{b,\xi}\} = \underset{p_t^{b,\xi}, Q_t^{b,\xi}}{\arg\max}\ & \big[V^{e,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) + p_t^{b,\xi}Q_t^{b,\xi} - V^{e,\xi}(Y_t, M_t, B_t)\big]^\theta \\
& \big[V^{b,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) - p_t^{b,\xi}Q_t^{b,\xi} - V^{b,\xi}(Y_t, M_t, B_t)\big]^{1-\theta},
\end{aligned}
\tag{B.99}
$$

subject to $V^{e,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) + p_t^{b,\xi}Q_t^{b,\xi} - V^{e,\xi}(Y_t, M_t, B_t) \geq 0$ and $V^{b,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) - p_t^{b,\xi}Q_t^{b,\xi} - V^{b,\xi}(Y_t, M_t, B_t) \geq 0$. Following derivations in Appendix B.1, when $\theta = 0$ we have

$$
\bar{Q}_t^{b,\xi} = \underset{Q_t^{b,\xi}}{\arg\max}\ \big[V^{e,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi}) + V^{b,\xi}(Y_t, M_t, B_t + Q_t^{b,\xi})\big],
\tag{B.100}
$$

and

$$\bar{p}_t^{b,\xi} = \frac{V^{e,\xi}(Y_t, M_t, B_t) - V^{e,\xi}(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi})}{\bar{Q}_t^{b,\xi}}. \tag{B.101}$$

Substituting (B.101) into (B.97) yields

$$
\begin{aligned}
rV^{e,\xi}(Y_t, M_t, B_t) =& (1 - \pi)\big[Y_t - c(B_t + M_t)\big] \\
&+ \mu Y_t V_Y^{e,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{e,\xi}(Y_t, M_t, B_t) \\
&+ \zeta^\xi \big[V^{e,-\xi}(Y_t, M_t, B_t) - V^{e,\xi}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.102}
$$

Differentiating (B.102) with respect to M_t and using the first-order condition (B.96),

$$
\begin{aligned}
-rp^{m,\xi}(Y_t, M_t, B_t) =& -(1 - \pi)c - \mu Y_t p_Y^{m,\xi}(Y_t, M_t, B_t) - \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^{m,\xi}(Y_t, M_t, B_t) \\
&+ \zeta^\xi \big[-p^{m,-\xi}(Y_t, M_t, B_t) + p^{m,\xi}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.103}
$$

Moreover, let $p^{m,\xi}(Y_t, M_t, B_t)$ denote the equilibrium market debt price in state ξ, then

$$
\begin{aligned}
rp^{m,\xi}(Y_t, M_t, B_t) =& c + g_t^{m,\xi} M_t p_M^{m,\xi}(Y_t, M_t, B_t) \\
&+ \mu Y_t p_Y^{m,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 p_{YY}^{m,\xi}(Y_t, M_t, B_t) \\
&+ \lambda \big[p^{m,\xi}(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - p^{m,\xi}(Y_t, M_t, B_t)\big] \\
&+ \zeta^\xi \big[p^{m,-\xi}(Y_t, M_t, B_t) - p^{m,\xi}(Y_t, M_t, B_t)\big].
\end{aligned}
\tag{B.104}
$$

Adding up (B.103) and (B.104) yields

$$0 = \pi c + g_t^{m,\xi} M_t p_M^{m,\xi}(Y_t, M_t, B_t) + \lambda\big[p^{m,\xi}(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - p^{m,\xi}(Y_t, M_t, B_t)\big]. \tag{B.105}$$

Rearranging yields

$$g_t^{m,\xi} = \frac{\pi c}{-M_t p_M^{m,\xi}(Y_t, M_t, B_t)} + \frac{\lambda[p^{m,\xi}(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - p^{m,\xi}(Y_t, M_t, B_t)]}{-M_t p_M^{m,\xi}(Y_t, M_t, B_t)}. \tag{B.106}$$

A. Permanent Shock

I now consider a permanent bank lending shock, that is $\zeta^s = 0$. The equity holders' total value can be solved as if the firm issues neither market debt nor bank debt. Let $V^{e,\xi,0}(Y_t; M, B)$ denote the equity holders' no-issuance value given the market debt outstanding $M_t = M$ and the bank debt outstanding $B_t = B$. Then

$$
\begin{aligned}
rV^{e,\xi,0}(Y_t; M, B) =&(1 - \pi)\big[Y_t - c(B + M)\big] \\
&+ \mu Y_t V^{e,\xi,0\prime}(Y_t; M, B) + \frac{1}{2}\sigma^2 Y_t^2 V^{e,\xi,0\prime\prime}(Y_t; M, B),
\end{aligned}
\tag{B.107}
$$

for $\xi \in \{n, s\}$. Following the derivations in Appendix B.4,

$$
\begin{aligned}
V^{e,\xi,0}(Y_t; M, B) =&\frac{1 - \pi}{r - \mu}Y_t - \frac{(1 - \pi)c}{r}(B + M) \\
&+ \frac{1}{1 + \gamma}\frac{(1 - \pi)c}{r}\Big[\frac{\gamma}{1 + \gamma}\frac{(r - \mu)c}{r}\Big]^{\gamma}(B + M)^{1+\gamma}Y_t^{-\gamma}.
\end{aligned}
\tag{B.108}
$$

Hence,

$$
\begin{aligned}
V^{e,\xi}(Y_t, M_t, B_t) =&\frac{1 - \pi}{r - \mu}Y_t - \frac{(1 - \pi)c}{r}(B_t + M_t) \\
&+ \frac{1}{1 + \gamma}\frac{(1 - \pi)c}{r}\Big[\frac{\gamma}{1 + \gamma}\frac{(r - \mu)c}{r}\Big]^{\gamma}(B_t + M_t)^{1+\gamma}Y_t^{-\gamma}.
\end{aligned}
\tag{B.109}
$$

Using the first-order condition with respect to the market debt issuance rate (B.96), the equilibrium market debt price in state ξ is then

$$
p^{m,\xi}(Y_t, M_t, B_t) = \frac{(1 - \pi)c}{r}\left\{1 - \Big[\frac{\gamma}{1 + \gamma}\frac{(r - \mu)c}{r}\Big]^{\gamma}\Big(\frac{B_t + M_t}{Y_t}\Big)^{\gamma}\right\}.
\tag{B.110}
$$

The market debt issuance rate in state ξ is given by

$$
\begin{aligned}
g_t^{m,\xi} =&\frac{\pi c}{\gamma}\frac{r}{(1 - \pi)c}\Big[\frac{1 + \gamma}{\gamma}\frac{r}{(r - \mu)c}\frac{Y_t}{B_t + M_t}\Big]^{\gamma}\frac{B_t + M_t}{M_t} \\
&+ \frac{\lambda}{\gamma}\Big[1 - \Big(\frac{B_t + M_t + \bar{Q}_t^{b,\xi}}{B_t + M_t}\Big)^{\gamma}\Big]\frac{B_t + M_t}{M_t}.
\end{aligned}
\tag{B.111}
$$

B. Transitory Shock

Let $V^{e,\xi,0}(Y_t; M, B)$ denote the equity holders' no-issuance value given the market debt outstanding $M_t = M$ and the bank debt outstanding $B_t = B$. It must then satisfy the following HJB equation

$$
\begin{aligned}
rV^{e,\xi,0}(Y_t; M, B) =& (1 - \pi)\big[Y_t - c(B + M)\big] \\
&+ \mu Y_t V^{e,\xi,0\prime}(Y_t; M, B) + \frac{1}{2}\sigma^2 Y_t^2 V^{e,\xi,0\prime\prime}(Y_t; M, B) \\
&+ \zeta^\xi \big[V^{e,-\xi,0}(Y_t; M, B) - V^{e,\xi,0}(Y_t; M, B)\big].
\end{aligned}
\tag{B.112}
$$

The equity holders' no-issuance value is homogeneous of degree one. Hence

$$
V^{e,\xi,0}(Y_t; M, B) = v^{e,\xi,0}(y_t; b)M,
\tag{B.113}
$$

where $y_t \equiv Y_t/M_t$, $b \equiv B/M$, and $v^{e,\xi,0}(y_t; b)$ is the equity holders' no-issuance value scaled by the amount of market debt outstanding. I refer to $v^{e,\xi,0}(y_t; b)$ as the no-issuance equity value. The HJB equation (B.112) can be rewritten as

$$
\begin{aligned}
rv^{e,\xi,0}(y_t; b) =& (1 - \pi)\big[y_t - c(1 + b)\big] + \mu y_t v^{e,\xi,0\prime}(y_t; b) + \frac{1}{2}\sigma^2 y_t^2 v^{e,\xi,0\prime\prime}(y_t; b) \\
&+ \zeta^\xi \big[v^{e,-\xi,0}(y_t; b) - v^{e,\xi,0}(y_t; b)\big].
\end{aligned}
\tag{B.114}
$$

That is, $v^{e,n,0}(y_t; b)$ and $v^{e,s,0}(y_t; b)$ respectively satisfy

$$
rv^{e,n,0}(y_t; b) = (1 - \pi)\big[y_t - c(1 + b)\big] + \mu y_t v^{e,n,0\prime}(y_t; b) + \frac{1}{2}\sigma^2 y_t^2 v^{e,n,0\prime\prime}(y_t; b),
\tag{B.115}
$$

and

$$
\begin{aligned}
rv^{e,s,0}(y_t; b) =& (1 - \pi)\big[y_t - c(1 + b)\big] + \mu y_t v^{e,s,0\prime}(y_t; b) + \frac{1}{2}\sigma^2 y_t^2 v^{e,s,0\prime\prime}(y_t; b) \\
&+ \zeta^s \big[v^{e,n,0}(y_t; b) - v^{e,s,0}(y_t; b)\big].
\end{aligned}
\tag{B.116}
$$

When the economy is in the normal state, at the default boundary $y^{\star,n}(b)$ the no-issuance equity value satisfies the value matching condition $v^{e,n,0}(y^{\star,n}(b); b) = 0$ and the smooth pasting condition $v^{e,n,0\prime}(y^{\star,n}(b); b) = 0$. Following the derivations in Appendix B.4, the default boundary is given by

$$y^{\star,n}(b) = \frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}(1+b),$$ (B.117)

where

$$-\gamma \equiv -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$ (B.118)

The no-issuance equity value in the normal state is then

$$
\begin{aligned}
v^{e,n,0}(y_t; b) =& \frac{1-\pi}{r-\mu}y_t - \frac{(1-\pi)c}{r}(1+b) \\
&+ \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^{\gamma}(1+b)^{1+\gamma}y_t^{-\gamma}.
\end{aligned}
$$ (B.119)

In equilibrium, the equity holders' total value equals the equity holders' no-issuance value, that is $V^{e,n}(Y_t, M_t, B_t) = V^{e,n,0}(Y_t; M_t, B_t)$. Let $v^{e,\xi}(y_t, b_t) \equiv V^{e,\xi}(Y_t, M_t, B_t)/M_t$ be the equity value in state ξ, where $b_t \equiv B_t/M_t$. Then $v^{e,n}(y_t, b_t) = v^{e,n,0}(y_t; b_t)$. Thus,

$$
\begin{aligned}
v^{e,n}(y_t, b_t) =& \frac{1-\pi}{r-\mu}y_t - \frac{(1-\pi)c}{r}(1+b_t) \\
&+ \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^{\gamma}(1+b_t)^{1+\gamma}y_t^{-\gamma}.
\end{aligned}
$$ (B.120)

Let

$$f(y_t; b) \equiv v^{e,n,0}(y_t; b) - v^{e,s,0}(y_t; b).$$ (B.121)

Subtracting (B.116) from (B.115) yields

$$(r + \zeta^s)f(y_t; b) = \mu y_t f'(y_t; b) + \frac{1}{2}\sigma^2 y_t^2 f''(y_t; b). \tag{B.122}$$

The general solution to (B.122) is

$$f(y_t; b) = H_\phi y_t^{-\phi} + H_\psi y_t^\psi. \tag{B.123}$$

where

$$-\phi \equiv -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + \zeta^s)}}{\sigma^2} < 0, \tag{B.124}$$

$$\psi \equiv -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + \zeta^s)}}{\sigma^2} > 1. \tag{B.125}$$

Hence, from (B.119), (B.121) and (B.123)

$$\begin{aligned}
v^{e,s,0}(y_t; b) =& -H_\phi y_t^{-\phi} - H_\psi y_t^\psi + \frac{1-\pi}{r-\mu}y_t - \frac{(1-\pi)c}{r}(1+b) \\
&+ \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^\gamma (1+b)^{1+\gamma}y_t^{-\gamma}.
\end{aligned} \tag{B.126}$$

The no-bubble condition at $y_t \to \infty$ implies that $H_\psi = 0$. At the default boundary $y^{\star,s}(b)$, the no-issuance equity value satisfies the value matching condition $v^{e,s,0}(y^{\star,s}(b); b) = 0$ and the smooth pasting condition $v^{e,s,0\prime}(y^{\star,s}(b); b) = 0$. That is

$$\begin{aligned}
0 =& -H_\phi y^{\star,s}(b)^{-\phi} + \frac{1-\pi}{r-\mu}y^{\star,s}(b) - \frac{(1-\pi)c}{r}(1+b) \\
&+ \frac{1}{1+\gamma}\frac{(1-\pi)c}{r}\left[\frac{\gamma}{1+\gamma}\frac{(r-\mu)c}{r}\right]^\gamma (1+b)^{1+\gamma}y^{\star,s}(b)^{-\gamma},
\end{aligned} \tag{B.127}$$

and

$$0 = \phi H_\phi y^{\star,s}(b)^{-\phi-1} + \frac{1-\pi}{r-\mu}$$
$$- \frac{\gamma}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}\right]^\gamma (1+b)^{1+\gamma} y^{\star,s}(b)^{-\gamma-1}. \tag{B.128}$$

The value matching condition and the smooth pasting condition together pin down H_ϕ and $y^{\star,s}(b)$. Then

$$v^{e,s}(y_t, b_t) = \frac{1-\pi}{r-\mu} y_t - \frac{(1-\pi)c}{r}(1+b_t)$$
$$+ \frac{1}{1+\gamma} \frac{(1-\pi)c}{r} \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}\right]^\gamma (1+b_t)^{1+\gamma} y_t^{-\gamma} - H_\phi y_t^{-\phi}. \tag{B.129}$$

From (B.96),

$$p_t^{m,n} = y_t v_y^{e,n}(y_t, b_t) + b_t v_b^{e,n}(y_t, b_t) - v^{e,n}(y_t, b_t)$$
$$= \frac{(1-\pi)c}{r} \left\{1 - \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}\right]^\gamma \left(\frac{1+b_t}{y_t}\right)^\gamma\right\}. \tag{B.130}$$

Similarly,

$$p_t^{m,s} = y_t v_y^{e,s}(y_t, b_t) + b_t v_b^{e,s}(y_t, b_t) - v^{e,s}(y_t, b_t)$$
$$= \frac{(1-\pi)c}{r} \left\{1 - \left[\frac{\gamma}{1+\gamma} \frac{(r-\mu)c}{r}\right]^\gamma \left(\frac{1+b_t}{y_t}\right)^\gamma\right\} + \left[(1+\phi)H_\phi - b_t \frac{dH_\phi}{db_t}\right] y_t^{-\phi}. \tag{B.131}$$

Substituting (B.101) into (B.98) yields

$$r^\xi V^{b,\xi}(Y_t, M_t, B_t) = cB_t + g_t^{m,\xi} M_t V_M^{b,\xi}(Y_t, M_t, B_t)$$
$$+ \mu Y_t V_Y^{b,\xi}(Y_t, M_t, B_t) + \frac{1}{2}\sigma^2 Y_t^2 V_{YY}^{b,\xi}(Y_t, M_t, B_t)$$
$$+ \lambda\left[V^\xi(Y_t, M_t, B_t + \bar{Q}_t^{b,\xi}) - V^\xi(Y_t, M_t, B_t)\right]$$
$$+ \zeta^\xi\left[V^{b,-\xi}(Y_t, M_t, B_t) - V^{b,\xi}(Y_t, M_t, B_t)\right], \tag{B.132}$$

where $V^\xi(Y_t, M_t, B_t) \equiv V^{e,\xi}(Y_t, M_t, B_t) + V^{b,\xi}(Y_t, M_t, B_t)$, and the market debt issuance rate $g_t^{m,\xi}$ is given by (B.106).

Given the scale invariance in the setting, the bank lender's value $V^{b,\xi}(Y_t, M_t, B_t)$ is homogeneous of degree one. That is

$$V^{b,\xi}(Y_t, M_t, B_t) = V^{b,\xi}\Big(\frac{Y_t}{M_t}, 1, \frac{B_t}{M_t}\Big) M_t \equiv v^{b,\xi}(y_t, b_t) M_t. \tag{B.133}$$

Similarly,

$$V^\xi(Y_t, M_t, B_t) = V^\xi\Big(\frac{Y_t}{M_t}, 1, \frac{B_t}{M_t}\Big) M_t \equiv v^\xi(y_t, b_t) M_t, \tag{B.134}$$

where $v^\xi(y_t, b_t) \equiv v^{e,\xi}(y_t, b_t) + v^{b,\xi}(y_t, b_t)$ with $v^{e,\xi}(y_t, b_t) \equiv V^{e,\xi}(Y_t, M_t, B_t)/M_t$. From (B.100),

$$\bar{q}_t^{b,\xi} \equiv \frac{\bar{Q}_t^{b,\xi}}{M_t} = \arg\max_{q_t^{b,\xi}} \ v^\xi(y_t, b_t + q_t^{b,\xi}). \tag{B.135}$$

Hence, one can rewrite (B.98) as

$$
\begin{aligned}
r^\xi v^{b,\xi}(y_t, b_t) M_t =& cB_t + g_t^{m,\xi} M_t \big[v^{b,\xi}(y_t, b_t) - y_t v_y^{b,\xi}(y_t, b_t) - b_t v_b^{b,\xi}(y_t, b_t)\big] \\
& + \mu Y_t v_y^{b,\xi}(y_t, b_t) + \frac{1}{2}\sigma^2 \frac{Y_t^2}{M_t} v_{yy}^{b,\xi}(y_t, b_t) \\
& + \lambda \big[v^\xi(y_t, b_t + \bar{q}_t^{b,\xi}) - v^\xi(y_t, b_t)\big] M_t \\
& + \zeta^\xi \big[v^{b,-\xi}(y_t, b_t) - v^{b,\xi}(y_t, b_t)\big] M_t.
\end{aligned}
\tag{B.136}
$$

Rearranging yields

$$
\begin{aligned}
(r^\xi + \zeta^\xi - g_t^{m,\xi}) v^{b,\xi}(y_t, b_t) =& cb_t - g_t^{m,\xi} b_t v_b^{b,\xi}(y_t, b_t) + \zeta^\xi v^{b,-\xi}(y_t, b_t) \\
& + (\mu - g_t^{m,\xi}) y_t v_y^{b,\xi}(y_t, b_t) + \frac{1}{2}\sigma^2 y_t^2 v_{yy}^{b,\xi}(y_t, b_t) \\
& + \lambda \big[\max_{q_t^{b,\xi}} \ v^\xi(y_t, b_t + q_t^{b,\xi}) - v^\xi(y_t, b_t)\big].
\end{aligned}
\tag{B.137}
$$

84