Equity Risk Premium with Intertemporal Hedging *

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March 2025

Abstract

The equity risk premium at a given horizon T_1 depends on the risks of future intertemporal shifts in the economic environment. These risks matter beyond T_1 and up to the representative agent's investment horizon $T_N \geq T_1$. We derive novel bounds on the equity risk premium which account for these risks. Our bounds embed information on the term structure of market return moments up to T_N. We estimate them using options and find that they improve the out-of-sample R^2 of market return prediction by up to 40%. In particular, intertemporal hedging shifts the equity risk premium up in times of market calm, leading to a term structure of equity risk premium that is essentially flat during market calm and downward-sloping in stress times.

JEL classification: G11, G12, G13, G17.

Keywords: equity risk premium, intertemporal hedging, term structure.

*We are grateful to Patrick Augustin, Martin Boyer, Jean-Édouard Colliard, Kevin Crotty (discussant), Mathieu Fournier, Johan Hombert, Piotr Orlowski, Alexei Ovtchinnikov, Olivier Scaillet, Grigory Vilkov, Irina Zviadadze and conference and seminar participants at the University of California Riverside, University of Cincinnati, HEC Paris, CICF 2022, and HEC Montreal for helpful comments and suggestions. This research has received support from a grant of the French National Research Agency (ANR), "Investissements d'Avenir" (LabEx Ecodec/ANR-11-LABX-0047). This paper subsumes an earlier draft titled "Conditional leverage and the term structure of option-implied risk premia."

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1 Introduction

The equity risk premium—the expected return on the equity market over the risk-free rate—is a crucial input for corporate valuation and portfolio allocation. Unfortunately, it is also notoriously hard to estimate *ex ante*. Martin (2017) shows how the risk-neutral market variance discounted at the risk-free rate provides a lower bound for the equity risk premium, in a one-period economy that ignores the higher-order moments of market returns. A major benefit of his approach is that the risk-neutral variance can be easily computed from observed option prices. Chabi-Yo and Loudis (2020) and Tetlock (2023) extend the approach of Martin (2017) and provide bounds for the equity risk premium that account for higher-order risks, still in a one-period model.

Restricting the economy to a one-period economy allows simplifying the analysis, but at the expense of strong assumptions. In particular, it ignores the risks of future intertemporal shifts in the economic environment, e.g., changes in the expected returns or return volatility. Consider, for example, forecast horizon $T_1 > t$. A one-period model assumes that investors choose their portfolio allocation at time t ignoring the risks beyond time T_1. These risks, however, impact future consumption. Merton (1973) shows that investors optimally seek to hedge these risks by tilting their portfolio allocation towards assets that deliver higher returns when consumption is negatively affected. Intertemporal hedging after time T_1 therefore affects demand, and thus equilibrium prices and returns at horizon T_1.

We derive novel bounds for the equity risk premium, which take into account both higher order risks and intertemporal hedging. Our model features a multi-period economy, in which the representative investor chooses the optimal allocation to the market, to maximize the expected utility of the wealth accumulated between time t and time $T_N \geq T_1$. T_N represents the investment horizon of the investor. In this economy, we derive bounds for the equity risk premium with horizon T_1, using a Taylor expansion of the inverse marginal utility. The resulting equity risk premium depends on the conditional moments of the horizon T_1-market returns, but also on time-t expected conditional moments of returns over $[T_1, T_N]$. All return moments can be readily estimated using available option prices. Whereas the bounds of Martin (2017) and Chabi-Yo and Loudis (2020) only need options expiring at T_1 to forecast the equity risk premium at horizon T_1, our method uses options at horizons T_1 and T_N.

We estimate bounds for the equity risk premium on the S&P 500 from 1996 to 2023, over horizons ranging from 10 days to 18 months. We show that accounting for intertemporal hedging leads to an increase of the equity risk premium, in particular during times of market calm. Intertemporal hedging accounts for up to 80% of the total equity risk premium during these periods, and around 30% during NBER recessions. Furthermore, our risk premium allows us to improve the out-of-sample R^2 of return prediction, compared to the bounds of Martin (2017) and Chabi-Yo and Loudis (2020). For all forecast horizons T_1 from 10 days to 18 months, the out-of-sample R^2 increases with the investors' horizon T_N, up to a given T_N. For example, for T_1 at 10 days, the maximum out-of-sample R^2 is achieved at 6 months. For T_1 larger than two months, the maximum R^2 is obtained for the longest horizon for which we have available option maturities, namely $T_N = 2$ years. We also construct market-timing strategies and compute realized mean-variance certainty equivalents. These certainty equivalents indicate that our risk premium reaches better forecasts of both the first and second return moments, and that the improvements upon the forecasts of Chabi-Yo and Loudis (2020) are statistically significant.

We define the implied investors' horizon $T^*_{N,t}$, as the investment horizon which at each time t maximizes the fit of our equity risk premium estimate to the data. Specifically, $T^*_{N,t}$ is chosen so that it maximizes the in-sample R^2 of returns over a window of three months $[t-3m, t]$. We find that the implied investors' horizon switches between the longest available horizon T_N, e.g., two years, and the shortest horizon $T_N > T_1$. When the probability of a crash is high (above 10%), the implied investors' horizon is short, and it is equal to two years otherwise. This result provides empirical evidence to the theory of Hirshleifer and Subrahmanyam (1993), which predicts that investors' time horizons shorten during periods of uncertainty due to increased risk aversion and limited attention. It is also in line with

Campbell and Vuolteenaho (2004), who find that in volatile markets, investors become more sensitive to "bad beta" – short-term cash flow shocks–, than to "good beta" – long-term discount rate changes–.

Whenever the probability of a crash is low, the representative agent thus behaves as a long-term investor, and intertemporal hedging shifts the equity risk premium upward.

Given these switches in the implied investors' horizon, we further optimize our equity risk premium by setting it, at each time t, equal to the risk premium at investment horizon $T^*_{N,t}$ –the implied investors' horizon at time t–. We thus obtain an equity risk premium estimate which matches the estimate at $T_N = 2$ years during most of the time series, and switches to the estimate at the shortest available $T_N > T_1$ when the probability of a crash is high. The resulting equity risk premium is higher than the one of Chabi-Yo and Loudis (2020) under normal market conditions, and roughly at the same level during market stress.

Intertemporal hedging increases the equity risk premium at short horizons more than it does at longer horizons. Therefore, it also impacts the term structure of equity risk premium, which we define as the hold-to-maturity yield on the S&P 500 implied by our estimates at various horizons. Where as the term structure of equity risk premium of Chabi-Yo and Loudis (2020) is upward sloping under normal market conditions, we obtain a term structure of equity risk premium which is essentially flat. During market stress, it is strongly downward sloping.

These results are robust to changes in our main assumptions. Our main results are based on preference parameters that are fixed. We estimate these parameters over the period 1996-2023, as linear functions of past returns. We show that the resulting preference parameters vary with market conditions, and generate larger out-of-sample R^2. However, estimating them over the full time period yields a look-ahead bias. We overcome this issue by estimating these parameters over a telescopic window of data, initially ranging from 1996 to 2006, and expanding with time. We show, however, that the resulting equity risk premium estimates do not improve upon our main estimates in terms of out-of-sample R^2, over the period 2006-2023. We also study an extension of our setup that allows the representative investor to rebalance her portfolio between times T_1 and T_N. Our conclusions survive this change.

We contribute to different strands of literature. The first strand uses options prices to infer information about the return distribution under the physical probability measure. The risk-neutral leverage effect used in this paper is closely related to the asymmetric volatility implied correlation studied by Jackwerth and Vilkov (2019). They use short- and long-term options on the S&P 500 Index and options on VIX futures to calibrate the risk-neutral correlation between returns and future volatility. As options on VIX futures are available only starting in 2006, data availability prevents us from using their methodology.

Our work is also related to the vast literature on the importance of the variance risk premium—the difference between the physical and risk-neutral variance—for predicting the equity risk premium (see, Bollerslev, Tauchen, and Zhou, 2009). Hu, Jacobs, and Seo (2021) show that the leverage effect, measured under the physical probability measure, has a strong positive relation with the variance risk premium. We derive an expression that relates the equity risk premium to the variance and leverage effect under the risk-neutral measure.

We contribute to the growing literature that constructs *bounds* on physical return moments. Building on Martin (2017), Martin and Wagner (2019), Kadan and Tang (2020), and Chabi-Yo, Dim, and Vilkov (2021) build bounds for the expected return on individual stocks and Kremens and Martin (2019) provide a bound for currency expected exchange rate appreciation using Quanto index options. See Back, Crotty, and Kazempour (2022) for a discussion and empirical tests of bounds for individual stocks and the stock market. Our novel bound for the equity risk premium involves intertemporal terms implied from options prices.

The Recovery Theorem of Ross (2015) shows how to disentangle the physical probability distribution from the pricing kernel and risk-neutral probabilities, but has been challenged on theoretical and empirical grounds.[1] Instead of making assumptions about the pricing kernel process, Schneider and Trojani (2019) impose sign restrictions on the risk premia of return moments and find predictive power for future returns. Our approach differs in that we express the equity risk premium as a function of risk-neutral moments of returns at different horizons and preference parameters estimated from the data.

Finally, our paper is related to the literature on the equity term structure. van Binsbergen, Brandt, and Koijen (2012) show that the expected one-period return on claims on dividends decreases in the maturity of the dividend. Gormsen (2020) shows that this slope is countercyclical (see also, van Binsbergen, Hueskes, Koijen, and Vrugt, 2013; van Binsbergen and Koijen, 2017; Bansal, Miller, Song, and Yaron, 2021; Ulrich, Florig, and Seehuber, 2022; Giglio, Kelly, and Kozak, 2024). While the main object in this literature is the expected one-period return on claims on dividends several years in the future, we focus on the term structure of expected total market return with maturity of up to one year.

Our paper proceeds as follows. Section 2 presents our theoretical results based on a second-order approximation, Section 3 discusses our empirical framework to build equity risk premium forecasts. Section 4 presents our main empirical results. In Section 5 we show the results when estimating the preference parameters of our model. Sections 6 and 7 study the robustness of our results to two extensions. Finally, Section 8 concludes.

2 Theoretical framework

In this section, we provide our main theoretical results. We derive a lower bound on the equity risk premium in a multi-period economy, accounting for the risks of future intertemporal shifts in the economic environment. We further use our methodology to derive the probability of a crash under the physical measure. We highlight the new components of the

[1] Borovička, Hansen, and Scheinkman (2016) show that Ross' assumptions rule out realistic models. Bakshi, Chabi-Yo, and Gao (2018) do not find support for the implications of the Recovery Theorem using U.S. Treasury bond futures. While Audrino, Huitema, and Ludwig (2019) find some forecasting power, Jensen, Lando, and Pedersen (2019) generalize the assumptions of Ross' (2015) model and find weak predictive power for future realized returns.

equity risk premium and crash probabilities, compared to estimates that do not account for intertemporal hedging. These components capture conditional moments of market returns beyond the forecast horizon. All proofs are provided in Appendix A.

2.1 Equity risk premium in a multi-period economy

We consider a three-date (two-period) economy with dates t, T_1, and T_N.[2] T_1 is the forecast horizon at which we aim to build a lower bound for the equity risk premium. T_N is the representative investor's horizon. We assume that this economy is arbitrage-free, which guarantees the existence of a risk-neutral measure. Consider a representative investor, who can invest, at time t, in an asset delivering the risk-free gross return $R_{f,t\rightarrow T_1}$, and in a set of risky assets. The gross return vector is denoted by $R_{t\rightarrow T_1}$. The intermediate wealth at forecast horizon T_1 is $W_{t\rightarrow T_1} = W_t\left(\omega_t^{\intercal} R_{t\rightarrow T_1}\right)$, where ω_t is the vector of portfolio weights. At forecast horizon T_1, the investor can rebalance her portfolio so that her terminal wealth at T_N is $W_{t\rightarrow T_N} = W_{t\rightarrow T_1}\left(\omega_{T_1}^{\intercal} R_{T_1\rightarrow T_N}\right)$, where ω_{T_1} is the vector of portfolio weights at time T_1.

The investor maximizes her expected utility of terminal wealth[3] over the period $[t, T_N] = [t, T_1] \cup [T_1, T_N]$:[4]

$$\max_{\omega_t, \omega_{T_1}} \mathbb{E}_t u\left[W_{t\rightarrow T_N}\right]. \tag{1}$$

The main innovation of our approach is that the investor considers what happens over $[T_1, T_N]$ when solving the portfolio allocation problem. In contrast, the bounds of Martin (2017); Chabi-Yo and Loudis (2020) and Tetlock (2023) are derived in an economy in which the investor maximizes the expected utility of wealth over $[t, T_1]$.

For simplicity, we assume no interest rate risk. Provided that no-arbitrage conditions hold in this economy, we show in Appendix A.1 that we can express the one-period stochastic

[2] We use the notation $T_0 = t$ for simplicity.

[3] The utility function $u[.]$ is well-defined, its derivatives up to order four exist, and their signs obey the following economic theory restriction: $\text{sign}\left(u^{(i)}\left[\cdot\right]\right) = \text{sign}(-1)^{i+1}$ (Eeckhoudt and Schlesinger, 2006; Deck and Schlesinger, 2014).

[4] We exclude consumption in (1) for simplicity. In the Internet Appendix D, we show that under minimal assumptions regarding the sign of the correlation between the consumption wealth ratio and the market return, the expected return derived in this section still holds.

discount factor (SDF) from t to T_1, $m_{t\to T_1}$, in terms of marginal utility and expectations under the risk-neutral measure as,

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\upsilon_{T_1}}{\mathbb{E}_t^*\left(\upsilon_{T_1}\right)} \text{ with } \upsilon_{T_1} = \mathbb{E}_{T_1}^*\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right), \tag{2}$$

where $\mathbb{E}_{T_i}^*\left(\cdot\right)$ denotes the expected value at time T_i under the risk-neutral measure. Since there is no interest rate risk, $R_{f,t\to T_N} = R_{f,t\to T_1} R_{f,T_1\to T_N}$.

Under no-arbitrage conditions, the expected excess return on an individual risky asset can be expressed as the risk-neutral covariance between the asset return and the inverse SDF:[5]

$$\mathbb{E}_t\left(R_{k,t\to T_1} - R_{f,t\to T_1}\right) = \mathbb{COV}_t^*\left(R_{k,t\to T_1}, \frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right). \tag{3}$$

The inverse marginal utility function is a function of $R_{M,T_{i-1}\to T_N} = \frac{W_{T_{i-1}\to T_N}}{W_{T_{i-1}}}$ for $i = 1, 2$. Its functional form is unknown. A second-order Taylor expansion series of the inverse marginal utility (see Equation (2)) around $(R_{M,t\to T_1}, R_{M,T_1\to T_N}) = (R_{f,t\to T_1}, R_{f,T_1\to T_N})$ produces a one-period SDF of the form[6]

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \approx \frac{\left(1 + z_{T_1}\right)}{\mathbb{E}_t^*\left(1 + z_{T_1}\right)}, \tag{4}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) + \frac{a_{2,t}}{R_{f,t\to T_1}^2}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{M}_{T_1\to T_N}^{*(2)} \tag{5}$$

and $\mathbb{M}_{T_1\to T_N}^{*(2)} = \mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2$ is the risk-neutral variance at time T_1. The coefficients $a_{1,t}$, $a_{2,t}$ and $a_{3,t}$ in the Taylor expansion series are functions of the investor's

[5]This identity is reminiscent of the well-known asset pricing equation in which the expected excess return is negatively related to the covariance between the return and the SDF under the physical measure. The proof of this identity follows from no-arbitrage conditions and no interest rate risk assumption and is given in Appendix A.2.

[6]While the representative agent re-balances her portfolio at any time t with $T_1 \leq t < T_N$, the Taylor expansion-series implicitly exploits the information about re-balancing at T_1 only. Later, we will provide expressions of the expected return that uses information about re-balancing at all discrete times t such as $T_1 \leq t < T_N$.

risk, skewness and kurtosis tolerance parameters τ_t, ρ_t and κ_t:

$$a_{1,t} = \tfrac{1}{\tau_t}, \qquad a_{2,t} = \tfrac{(1-\rho_t)}{\tau_t^2}, \qquad a_{3,t} = \tfrac{(\kappa_t+1-2\rho_t)}{\tau_t^3}, \tag{6}$$

where

$$\begin{aligned}
\tau_t &= -\frac{u^{(1)}\left[W_t R_{f,t\to T_N}\right]}{W_t R_{f,t\to T_N} u^{(2)}\left[W_t R_{f,t\to T_N}\right]}, \\
\rho_t &= \frac{1}{2!}\frac{u^{(3)}\left[W_t R_{f,t\to T_N}\right] u^{(1)}\left[W_t R_{f,t\to T_N}\right]}{\left(u^{(2)}\left[W_t R_{f,t\to T_N}\right]\right)^2}, \\
\kappa_t &= \frac{1}{3!}\frac{u^{(4)}\left[W_t R_{f,t\to T_N}\right]\left(u^{(1)}\left[W_t R_{f,t\to T_N}\right]\right)^2}{\left(u^{(2)}\left[W_t R_{f,t\to T_N}\right]\right)^3}.
\end{aligned} \tag{7}$$

The proof of Equation (4) is in Appendix A.3.[7]

Equations (4) and (5) show that the inverse of the SDF is a function of three terms: the excess market return, the squared excess market return, and the market risk-neutral variance $\mathbb{M}^{*(2)}_{T_1\to T_N}$ at time T_1. This risk-neutral variance term is new and only arises in a two-period economy. We know from Merton's ICAPM that shocks to risk can generate hedging demand and so can be priced. But Merton's ICAPM shows that market physical volatility is determinant in explaining the expected excess return on a stock. Merton's model argument is not about risk neutral market volatility. Strong evidence of time-varying volatility risk premium suggests that the risk neutral market variance and the physical market variance are distinct and carry different sets of information. Thus, our theoretical results are distinct from implications from Merton's ICAPM model. Further, Merton's ICAPM was not intended to derive closed-form expression of the risk premium on the market as a function of risk neutral correlation between market return and market risk neutral volatility.

We present our main theoretical result in Proposition 1 below. In this proposition, we combine the risk premium expression in Equation (3) with the SDF expression (4) to

[7] Our baseline results do not involve kurtosis preference, but we define the kurtosis preference parameter together with the risk aversion and skewness preference parameters for completeness. We will use the kurtosis preference parameter in Section 7, where we apply third-order Taylor expansion series.

provide a closed-form solution to the conditional expected excess market return in terms of risk-neutral moments.

Proposition 1 *Up to a second-order expansion-series, consistent with (4), under no-arbitrage conditions, the one-period expected excess market return is a function of risk neutral return moments:*

$$RP_{t\to T_1,T_N} \equiv \mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \quad (8)$$

where

$$\mathbb{LEV}^*_t = \mathbb{COV}^*_t\left(r_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_1\to T_N}\right), \quad (9)$$

and

$$\mathbb{M}^{*(n)}_{T_i\to T_j} = \mathbb{E}^*_{T_i}\left(R_{M,T_i\to T_j} - R_{f,T_i\to T_j}\right)^n, \text{ with } i<j,\ i=0,1,\ T_0=t, \text{ and } n>1. \quad (10)$$

Proof. See Appendix A.4. ■

Two new terms contribute to the equity risk premium in a two-period economy, compared to a one-period economy: the risk-neutral leverage effect $\mathbb{LEV}^*_t$ and the expected future variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$. Our conjecture is that the risk-neutral leverage effect, $\mathbb{LEV}^*_t$, is negative and as a result increases the equity risk premium due to the compensation required by investors for exposure to the future risk-neutral variance. There is a vast literature on leverage under the physical measure. Still, to our knowledge, our paper is the first to show how relevant leverage under the risk-neutral measure is for computing the one-period conditional expected excess market return in a two-period economy. Provided that $a_{2,t}$ is negative, a negative risk-neutral leverage contributes positively to the conditional equity risk premium.

We further show in the Internet Appendix D.3, that Eq. (8) remains a lower bound to the expected excess market return provided that odd market risk neutral moments are negative and conditions $1/\tau_t \geq 1$and $\rho_t - 1 \geq 1$ hold.

2.2 Comparison to existing bounds

The computation of the risk-neutral leverage effect $\mathbb{LEV}_t^*$ and of the expected future variance $\mathbb{E}_t^* \mathbb{M}_{T_1 \to T_N}^{*(2)}$ relies on information from options of maturities T_1 and T_N. In contrast, the existing bounds of Martin (2017) and Chabi-Yo and Loudis (2020) and the equity risk premium estimate of Tetlock (2023) only rely on options with maturity T_1. The bound in Martin (2017) corresponds to the expected excess return when the representative agent is endowed with a myopic log utility. The log utility assumption corresponds to $\tau_t = 1$ $(a_{1,t} = 1)$ and $\rho_t = 1$ $(a_{2,t} = 0)$, making higher-order moments and the leverage under the risk-neutral measure irrelevant in a two-period economy. In case of a CRRA utility with relative risk aversion α, an equivalent expression of (8) can be obtained by recognizing that (7) reduces to $\frac{1}{\tau_t} = \alpha$, $\rho_t = \frac{1}{2}\frac{(\alpha+1)}{\alpha}$, and $\kappa_t = \frac{1}{6}\frac{(\alpha+1)(\alpha+2)}{\alpha^2}$. In case of a CARA utility with absolute risk aversion $\widetilde{\alpha}$, an equivalent expression of (8) can be obtained by recognizing that (7) reduce to $\frac{1}{\tau_t} = \alpha_t$, $\rho_t = \frac{1}{2}$, and $\kappa_t = \frac{1}{6}$ with $\alpha_t = \widetilde{\alpha} W_t R_{f,t\to T_N}$.

To compare our measure to the one of Chabi-Yo and Loudis (2020), we first introduce Corollary 2, which expresses the conditional expected excess market return as a weighted average of two risk premia.

Corollary 2 *Up to a second-order expansion-series, consistent with (4), the expected excess market return is a weighted average of two premia:*

$$\mathbb{E}_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \pi_t^* RP_{t\to T_1} + \left(1 - \pi_t^*\right) \mathbb{RP}_t^{\upsilon}, \tag{11}$$

where

$$RP_{t\to T_1} = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}, \tag{12}$$

and

$$\mathbb{RP}^{\nu}_t = \frac{\mathbb{LEV}^*_t}{\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{13}$$

with

$$\pi^*_t = \frac{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}. \tag{14}$$

Proof. See Appendix A.5. ∎

The first risk premium $RP_{t\to T_1}$ in Equation (12), which corresponds to the measure obtained by Chabi-Yo and Loudis (2020) in a one-period economy, involves the risk-neutral variance and skewness of market returns.[8] The novelty of decomposition is the contribution of the risk-neutral leverage effect $\mathbb{LEV}^*_t$ and expected future variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$ to the conditional risk premium. Provided that odd market risk neutral moments are negative and conditions $1/\tau_t \geq 1$ and $\rho_t - 1 \geq 1$ hold, Equation (8) can be bounded as follows:

$$RP_{t\to T_1,T_N} \geq \frac{\frac{1}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} - \frac{1}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}{1 - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \frac{1}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}. \tag{15}$$

2.3 Conditional term premium

We build on Corollary 2 and derive the conditional term premium, which we define as the difference between a one-period expected excess market return in a two-period (three-date) economy and a one-period expected excess market return in a one-period (two-date) economy. Note that our definition of conditional term premium is similar but distinct from the equity term premium definition widely used in the literature.[9]

[8] Chabi-Yo and Loudis (2020) derive their expression using a third-order expansion-series of the inverse marginal utility. The expression provided in Equation (12) is the counterpart of the one given by Chabi-Yo and Loudis (2020) when using a second-order expansion-series of the inverse marginal utility.

[9] van Binsbergen, Brandt, and Koijen (2012) analyze returns on assets with claims to short-term dividends (i.e., up to three years ahead) on the S&P 500 Index. They show that claims to short-term dividends earn

Corollary 3 *Up to a second-order expansion-series, the conditional market term premium is*

$$\mathbb{CMTP}_{t\to T_1} = \underbrace{\pi_t^* RP_{t\to T_1} + (1-\pi_t^*)\,\mathbb{RP}_t^{\upsilon}}_{\substack{\text{One-period expected excess} \\ \text{return in a two-period economy}}} - \underbrace{RP_{t\to T_1},}_{\substack{\text{One-period expected excess} \\ \text{return in a one-period economy}}} \tag{16}$$

and can be alternatively written as

$$\mathbb{CMTP}_{t\to T_1} = (\pi_t^* - 1)\left(RP_{t\to T_1} - \mathbb{RP}_t^{\upsilon}\right), \tag{17}$$

where $RP_{t\to T_1}$, $\mathbb{RP}_t^{\upsilon}$, π_t^* *are defined in* (12), (13) *and* (14), *respectively.*

positive. A positive value indicates that our risk premia, $RP_{t\to T_1,T_N}$, will be higher than $RP_{t\to T_1}$. The difference in the shape of the term structure of risk premia depends on how $\mathbb{CMTP}_{t\to T_1}$ varies across T_1.

2.4 Probability of a crash

We further use our methodology to obtain the probability of a crash under the physical measure. We define the probability of a crash as $\mathbb{P}_t\left(R_{M,t\to T_1} < \alpha\right)$ where α is given. For example, $\alpha = 0.8$ for a 20% crash. We then exploit the no-arbitrage assumption that allows us to move from the physical measure to the risk-neutral measure. While the coefficient α could be time-varying or constant, we remove the time subscript on α to ease notations.

Proposition 4 *Up to a second-order expansion-series of the inverse marginal utilities, the conditional crash probability defined as* $\Pi_{t\to T_1,T_N}[\alpha] \equiv P_t\left(R_{M,t\to T} < \alpha\right)$ *can be expressed in*

higher average returns than claims on longer-term dividends: the unconditional dividend term structure is downward sloping. This finding contrasts with predictions from leading asset pricing models. Few asset pricing models with exogenous stochastic discount factors (e.g., Lettau and Wachter, 2007; Croce, Lettau, and Ludvigson, 2015) are able to generate returns consistent with their findings.

terms of risk neutral quantities

$$\Pi_{t\to T_1,T_N}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,\upsilon}[\alpha]}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \quad (18)$$

where $\mathbb{M}^{*(n)}_{t\to T_1}[\alpha] = \mathbb{E}^*_t\left((R_{M,t\to T_1} - R_{f,t\to T_1})^n \mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)$ *and* $\mathbb{M}^*_{t,\upsilon}[\alpha] = \mathbb{E}^*_t\left(\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)$.

Proof. See Appendix A.6. ■

Proposition 4 shows that truncated market moments matter for extracting the probability of the market crash. But more importantly, it shows that when the SDF is a function of future risk-neutral volatility as in (5), the tail of the distribution of risk-neutral volatility, captured by $\mathbb{M}^*_{t,\upsilon}[\alpha]$, has an impact on the probability of a crash. When the expected future volatility is not present in the SDF (4), the probability of a market crash reduces to

$$\Pi_{t\to T_1}[\alpha] \equiv P_t\left(R_{M,t\to T} < \alpha\right) = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha]}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}. \quad (19)$$

3 Empirical framework

We show in this section how the theoretical expressions derived in Section 2 can be brought to the data.

3.1 Leverage and future risk-neutral variance

The equity risk premium and crash probabilities are functions of risk-neutral moments, including $\mathbb{LEV}^*_t$ and $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$ which involve T_1- and T_N-horizon quantities. While closed-form expressions of risk-neutral moments for a given maturity in terms of option prices are directly available using the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000), closed-form expressions of the risk-neutral leverage effect and expected future moments are not directly available.

We propose a method to compute $\mathbb{LEV}_t^*$ and $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}$ using options with maturity T_1 and T_N. As the future variance is a function of the information set at T_1, we assume that it can be written as a nonlinear function f of $R_{M,t\to T_1} - R_{f,t\to T_1}$:

$$\mathbb{M}_{T_1\to T_N}^{*(2)} = \theta_t f(R_{M,t\to T_1} - R_{f,t\to T_1}) + \epsilon_t, \tag{20}$$

with $\mathbb{E}_t^*\left(\epsilon_t|R_{M,t\to T_1}\right) = \mathbb{E}_t^*\left(\epsilon_t\right) = 0$. Multiplying both sides of Equation (20) by $R_{M,t\to T_1}^2$ and taking the time-t risk-neutral expectation, we obtain

$$\theta_t = \frac{\mathbb{M}_{t\to T_N}^{*(2)} - R_{f,T_1\to T_N}^2\mathbb{M}_{t\to T_1}^{*(2)}}{\mathbb{E}_t^*\left(R_{M,t\to T_1}^2 f(R_{M,t\to T_1} - R_{f,t\to T_1})\right)}, \tag{21}$$

and

$$\mathbb{M}_{T_1\to T_N}^{*(2)} = \frac{\mathbb{M}_{t\to T_N}^{*(2)} - R_{f,T_1\to T_N}^2\mathbb{M}_{t\to T_1}^{*(2)}}{\mathbb{E}_t^*\left(R_{M,t\to T_1}^2 f(R_{M,t\to T_1} - R_{f,t\to T_1})\right)} f(R_{M,t\to T_1} - R_{f,t\to T_1}) + \epsilon_t. \tag{22}$$

Note that (20) is distinct from the assumption that the risk neutral volatility follows a GARCH process. The returns of interest in the left- and right-handsides of equation (20) are different. The risk neutral quantity in the left-handside of (20) is obtained from the return from time T_1 to T_N while the quantity in the right-handside of Equation (20) is a function of the realized return from t to T_1. We further show in the Internet Appendix D.1, that the key risk-neutral volatility dynamics implied by (20) is distinct from that of a GARCH process. Hence, a direct comparison cannot be made with a GARCH process. To obtain the expected future variance, $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}$, and leverage, $\mathbb{LEV}_t^*$, we compute the time-t risk-neutral expected values of Equation (22) and the product of $R_{M,t\to T_1} - R_{f,t\to T_1}$ and Equation (22), respectively.

The final step consists to choose the function $f(\cdot)$. We use $\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2$ for two reasons. First, note that the numerator of θ_t is always positive in the data. Therefore, our choice of function $f(\cdot)$ ensures that the expected future variance is a positive number. Second, as $\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2$ is a proxy for the first period conditional variance, this function captures the well-documented fact that conditional variances are highly positively correlated over time.

With this choice for the function $f(\cdot)$, we have,

$$\mathbb{E}_t^* \mathbb{M}_{T_1 \to T_N}^{*(2)} = \frac{\mathbb{M}_{t \to T_N}^{*(2)} - R_{f,T_1 \to T_N}^2 \mathbb{M}_{t \to T_1}^{*(2)}}{\mathbb{M}_{t \to T_1}^{*(4)} + 2R_{f,t \to T_1} \mathbb{M}_{t \to T_1}^{*(3)} + R_{f,t \to T_1}^2 \mathbb{M}_{t \to T_1}^{*(2)}} \mathbb{M}_{t \to T_1}^{*(2)}, \tag{23}$$

and,

$$\mathbb{LEV}_t^* = \frac{\mathbb{M}_{t \to T_N}^{*(2)} - R_{f,T_1 \to T_N}^2 \mathbb{M}_{t \to T_1}^{*(2)}}{\mathbb{M}_{t \to T_1}^{*(4)} + 2R_{f,t \to T_1} \mathbb{M}_{t \to T_1}^{*(3)} + R_{f,t \to T_1}^2 \mathbb{M}_{t \to T_1}^{*(2)}} \mathbb{M}_{t \to T_1}^{*(3)}. \tag{24}$$

Substituting Equations (23) and (24) in Equation (8) highlights that our expression for the equity risk premium is a non-linear function of T_1-return moments and the T_N-return variance.

3.2 Data

We use the S&P 500 index as the market portfolio. We obtain volatility surfaces, index levels, and forward term structures for the S&P 500 Index and the zero-coupon rate term structures from Ivy DB OptionMetrics. The data cover the period January 1996 to February 2023. When computing the excess returns on the S&P 500 index before January 1996, we use its level and the Fama term structures on U.S. Treasuries from the Center for Research in Security Prices (CRSP).

Implementing our risk premia requires the evaluation of different functions of risk-neutral expected values. We estimate these expected values at the end of each month and for each maturity provided in OptionMetrics' Volatility Surface File (10, 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 days). We refer to these maturities as one week, one month, two months, one quarter, four, five, six, and nine months, one year, 18 months, and two years.

We import annualized continuously-compounded zero-coupon yields from Jing Cynthia Wu's website, Liu and Wu (2021). We interpolate the term structure of zero-coupon rates using Nelson and Siegel (1987) model to find each maturity's risk-free rate.

Following Chabi-Yo, Dim, and Vilkov (2021), we define a moneyness grid of 1,000 equally spaced points from 1/3 to 3. We use a piecewise cubic Hermite polynomial to interpolate the implied volatility surface to the moneyness grid. We extrapolate the implied volatility using the closest value for moneyness points outside the implied volatility surface. Finally, we use the Black-Scholes formula to convert implied volatilities to call and put prices for each moneyness level.

3.3 Risk-neutral moments

We compute the risk-neutral moments of market returns and excess returns using the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000), as described in Appendix B.1. We report in Figure 1 excess return moments over time for horizons of one week to two years. To compare values across horizons, we report the annualized volatility in the top graph $\left(\sqrt{(365/T_1)\,\mathbb{M}^{*(2)}_{t\to T_1}}\right)$, skewness in the middle graph $\left(\mathbb{M}^{*(3)}_{t\to T_1}/\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^{\frac{3}{2}}\right)$, and kurtosis in the bottom graph $\left(\mathbb{M}^{*(4)}_{t\to T_1}/\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^{2}\right)$. We also report the expected future second moments and leverage in Figure 2, using Equations (23) and (24).

Risk-neutral volatilities and expected future volatilies vary over time, reaching a peak during the financial crisis of 2008. Risk-neutral skewness values are almost always negative and decrease over the sample period. Risk-neutral kurtosis values range between three and eight and trend upward over the sample period. The risk-neutral leverage effect is always negative and exhibits large time variations.

3.4 Preference parameters

The expressions for the one-period equity risk premium and crash probabilities provided in Section 2 are all functions of the investor's preference parameters τ_t and ρ_t.

We first set these parameters to $\tau_t = 1$ and $\rho_t = 2$ for all t, which is equivalent to $a_{1,t} = 1$ and $a_{2,t} = -1$. We derive our main results in Section 4 based on these values. In

Section 5, we attempt to estimate the preference parameters but find little improvement in out-of-sample results. We further show that our main findings do not change.

4 Results

In this section, we discuss the ability of $RP_{t\to T_1,T_N}$ to capture accurately equity risk premia. Following Chabi-Yo and Loudis (2020), we first set these parameters to $\tau_t = 1$ and $\rho_t = 2$ for all t. We show that $RP_{t\to T_1,T_N}$ outperforms the existing premia for most horizons T_N, and underline the existence of an implied investors' horizon, which corresponds to the optimal T_N. This horizon is long in quiet times, when the probability of crash is low, and short during market turmoil, when the probability of crash is high.

4.1 Estimated equity risk premium

We report in Figure 3 the time series of risk premia for a horizon of $T_1 =$ one month using $T_N =$ one year as the investment horizon. We compare in the top graph $RP_{t\to T_1,T_N}$ to $RP_{t\to T_1}$. $RP_{t\to T_1,T_N}$ is larger than $RP_{t\to T_1}$ over the entire sample period. This ordering suggests that the risks of future shifts in the economic environment lead to an increase of the equity risk premium. In the bottom graph, we further investigate how the equity risk premium changes with the investment horizon T_N. The two lines depict the fraction of the total risk premium that intertemporal hedging is responsible for. This fraction is computed as the conditional term premium defined in Section 2.3 (i.e., the difference of the risk premium with intertemporal hedging, $RP_{t\to T_1,T_N}$ and of the risk premium without intertemporal hedging, $RP_{t\to T_1}$), as a percentage of the former. Intertemporal hedging represents up to 80% of the total risk premium. This fraction increases with T_N: the blue line corresponds to an investment horizon of T_N equal to one year, and the red line of two years. The red line remains consistently above the blue line. Note that in times of market turmoil however, e.g., during the NBER recessions indicated in grey, intertemporal hedging accounts for a smaller percentage of the total premium, around 30%. The equity risk premium thus generally increases with the investment horizon T_N. This increase is large during quiet market times, and smaller during market stress.

In Appendix D.3, we show that our risk premium is a lower bound to the expected excess return on the market. Therefore, irrespective of the forecast horizon used, increasing the investment horizon makes the estimate tighter. We further show in Appendix D.2 that, when consumption is introduced in the representative agent problem, under minimal realistic assumptions, our measure of risk premium remains a lower bound to the expected market return.

4.2 Conditional variance and variance risk premium

Higher order moments under the physical measure are computed similarly to risk-neutral moments, as described in Appendix B.2. Figure 4, Panel A, compares the physical variance obtained when ignoring intertemporal hedging, to the physical variance obtained with intertemporal hedging. The forecast horizon is set at 1 month, and the investment horizon at 1 year. The physical variance is lower with intertemporal hedging throughout the time period. We observe large differences in times of market turmoil.

Panel B displays the associated variance risk premium, computed as the difference between the conditional variance under the physical measure, and under the risk-neutral measure. As the risk-neutral variance is computed from options, it does not depend on the investment horizon. Therefore, the lower physical variance with intertemporal hedging translates directly into a variance risk premium that is larger in magnitude, more negative, than without intertemporal hedging.

4.3 Out-of-sample performance

We study whether accounting for intertemporal hedging improves the out-of-sample performance of the equity risk premium. To assess the change in performance, we use two different metrics.

First, we follow Welch and Goyal (2008) and Campbell and Thompson (2008) in computing the out-of-sample R^2 measure as,

$$R^2_{OOS} = 1 - \frac{\sum_t \left(r_{M,t\to T_1} - \tilde{r}_{M,t\to T_1}\right)^2}{\sum_t \left(r_{M,t\to T_1} - \bar{r}_{M,t\to T_1}\right)^2}, \tag{25}$$

where $\bar{r}_{M,t\to T_1}$ is the sample average of returns at horizon T_1 prior to week t and $\tilde{r}_{M,t\to T_1}$ is a risk premium forecast. A positive R^2_{OOS} indicates that the prediction $\tilde{r}_{M,t\to T_1}$ is more accurate than the past average realized returns, while a negative R^2_{OOS} would favour the past average realized returns.

We report in Panel A of Table 3 the R^2_{OOS}, in percent, for $\tilde{r}_{M,t\to T_1} = RP^{Log}_{t\to T_1}$, $RP_{t\to T_1}$, and $RP_{t\to T_1,T_N}$ over full period from 1996 to 2023. Forecast horizons T_1 range from one month to 18 months and all available investment horizons $T_N > T_1$ are considered. We do not report on a forecast horizon of two years because there are no options available with maturity T_N larger than two years.

For all forecast horizons T_1, $RP_{t\to T_1}$outperforms $RP^{Log}_{t\to T_1}$, and $RP_{t\to T_1,T_N}$ outperforms $RP_{t\to T_1}$for almost all investment horizons T_N. In particular, for the 10-day forecast horizon, $RP_{t\to T_1}$and $RP^{Log}_{t\to T_1}$both perform worse, out-of-sample, than a forecast based on the past average realized returns, as they have negative R^2_{OOS}. In contrast, $RP_{t\to T_1,T_N}$ exhibits positive R^2_{OOS} for T_N between three months and one year. We test whether the differences in performance between $RP_{t\to T_1}$and $RP_{t\to T_1,T_N}$ are statistically significant, using the Diebold and Mariano (1995) test. The outperformance of $RP_{t\to T_1,T_N}$ is significant for forecast horizons T_1 between three and nine months, and for most T_N. Therefore, our results indicate that accounting for intertemporal hedging in the equity risk premium leads to a large and significant increase in out-of-sample forecast performance.

Inspection of the R^2_{OOS} achieved by $RP_{t\to T_1,T_N}$ in Table 3 reveals the importance of T_N on the performance of our risk premium. For all forecast horizons T_1, the R^2_{OOS} increases with T_N, up to a given T_N. For $T_1 = 10$ days, it reaches its maximum at $T_N = 6$ months, for $T_1 = 1$ month at $T_N = 9$ months, and for $T_N = 2$ months at $T_N = 18$ months. For all T_1 equal to 10 days, 1 month and 2 months, the R^2_{OOS} drops after reaching it maximum value, when increasing T_N. For T_1 larger than two months, the R^2_{OOS} increases up to $T_N = 24$ months. The pattern of R^2_{OOS} that we observe for $T_1 \leq 2$ months suggests that for $T_1 > 2$ months, there exists an optimal T_N beyond 24 months. Since options with maturities beyond 24 months are not available, we assume that the maximum R^2_{OOS} is 24 months. Overall, the R^2_{OOS} suggests the existence of an optimal $T_N > T_1$. The past column indicates the performance of a prediction based on the average prediction across investment horizons T_N. Such prediction achieves R^2_{OOS} that are all larger than those of $RP_{t\to T_1}$.

Second, we construct market-timing strategies and compute realized mean-variance certainty equivalents. While the R^2_{OOS}s reported in Panel A of Table 3 convey how our methodology captures the expected excess market return, results in Panel B combine both first and second moment predictions. For each forecasting method, we compute the weight of the market portfolio in the optimal portfolio at time t as,

$$\omega_{t\to T_1} = \frac{\tilde{r}_{M,t\to T_1}}{\gamma \tilde{\sigma}^2_{t\to T_1}} \tag{26}$$

where γ is a risk aversion parameter and $\tilde{\sigma}^2_{t\to T_1}$ is the physical variance of returns computed for each method, as described in Section 4.2. Then, we compute the realized mean-variance certainty equivalent as,

$$CE = E(r_{p,t\to T_1}) - \frac{\gamma}{2}\text{Var}(r_{p,t\to T_1}), \tag{27}$$

where $r_{p,t} = r_{f,t\to T_1} + \omega_{t\to T_1} r_{M,t\to T_1}$ are portfolio returns. The certainty equivalent is estimated using the sample return average and variance using non-overlapping returns over horizon T_1.

We report realized certainty equivalents annualized in percent for $\gamma = 3$. We find better performance of $RP_{t\to T_1,T_N}$, compared to $RP_{t\to T_1}$ and $RP^{Log}_{t\to T_1}$, for investment horizons T_N up to one year. In line with the results reported in Panel A, the certainty equivalents increase with T_N, reaching a maximum for T_N between 9 months and 24 months. Negative values are not displayed. They are obtained for $T_N = 18$ and 24 months due to estimates of the physical variance that are close to zero. We block-bootstrap the time-series of realized portfolio returns to compute the significance of the certainty equivalent differences for each strategy, compared to the one based on $RP_{t\to T_1}$ (see Politis and Romano, 1994).[10] We find that almost all differences between $RP_{t\to T_1,T_N}$-based and $RP_{t\to T_1}$-based strategies are statistically significant at the 5% level, when T_N is less or equal to a year.

Both out-of-sample performance metrics –out-of-sample R^2 and realized certainty equivalents– thus indicate that accounting for intertemporal hedging in the construction of the equity risk premium allows reaching better forecasts of the first and second return moments. The differences are statistically significant.

These equity risk premium measures are lower bounds for the equity risk premium. As a last analysis, we follow the methodology of Back, Crotty, and Kazempour (2022) and test for the validity and tightness of these bounds. In Online Appendix 2, we find results exactly in line with theirs. For all measures and horizons T_1, we do not reject that they are valid lower bounds and reject that they are tight. However, as expected the magnitude of the error from our bound is lower than either $RP_{t\to T_1}$ and $RP^{Log}_{t\to T_1}$.

4.4 Implied investors' horizon

We have shown that the out-of-sample performance of the equity risk premium depends on the choice of the investment horizon T_N, for all forecast horizons T_1. Increasing T_N, up

[10] We use 10,000 bootstrap samples and a mean block length equivalent to three years.

to a threshold, improves the out-of-sample performance of our risk premium. The forecast however deteriorates when increasing T_N beyond that threshold. We study whether the optimal threshold is time-dependent, by optimizing the investment horizon T_N used to make the prediction at each time t.

We select the optimal T_N at each time t in sample, by maximizing the R^2 of the forecast over a window of 90 days. This window covers the interval $t - T_1 - 90$ days, up to $t - T_1$, ensuring that there is no look-ahead bias. We denote this optimal time-varying horizon by $T^*_{N,t}$.

Table 4 reports the out-of-sample R^2_{OOS} achieved with $T^*_{N,t}$, and compares them to the R^2_{OOS} achieved with T_N at one and two years, and with the one obtained with the prediction averaged across T_N. The study period starts in 2000[11]. Comparing the first two columns ($RP^{Log}_{t\to T_1}$ and $RP_{t\to T_1}$) to the next two columns ($T_N = 1$ year and $T_N = 2$ years) confirms that $RP_{t\to T_1}$ outperforms $RP^{Log}_{t\to T_1}$ for most T_1, but that none of the two outperforms the other systematically. The $T_N = 1$ year estimate tends to perform better for shorter maturities, whereas the $T_N = 2$ years tends to outperform for longer maturities. The average prediction in column 5 yields a more stable outperformance across forecast horizons. The largest gain, for all T_1 except 10 days, is achieved when optimizing upon T_N (last column). The R^2 increases, compared to Chabi-Yo and Loudis (2020), by 40% to nearly 200% for forecast horizons T_1 between 2 and 9 months. This increase is statistically significant. Similarly, the largest realized certainty equivalents are obtained when optimizing T_N, for all forecast horizons except 10 days and one month.

Figure 5 displays in Panel A the estimated risk premium obtained with $T^*_{N,t}$, for T_1 at four months. Panel B depicts the time series of $T^*_{N,t}$. It oscillates between the smallest possible value of T_N (five months) and its largest value (two years). In particular, it is at five months during the two NBER recession periods, and tends to be at two years at most other times. This result is robust to varying the forecast horizon T_1. We thus conclude that

[11] As the optimal T_N is computed, at each time t, over a window that ends at $t - T_1$, we cannot study the R^2_{OOS} over the full window of data as in Table 3.

in quiet times, the implied investors' horizon is long (here, at its maximum of two years). In contrast, in turbulent times, the implied investors' horizon is short. This conclusion provides empirical evidence in line with the asset pricing model of **??**, in which investors' time horizon decreases in periods of high uncertainty, due to heightened risk aversion and liquidity needs. It also echoes the results of **??**, who use a VAR approach to show that investors' horizons shorten in volatile or declining markets because they become more sensitive to "bad beta", i.e., short-term negative cash flow news.

In turbulent times, the short-term horizon implies that intertemporal hedging has a negligible impact. As a result, the equity risk premium remains close to the one of $RP_{t\to T_1}$. In contrast, it is important in calm times, and pushes the equity risk premium up, since $RP_{t\to T_1,T_N}$increases with T_N. To better understand these punctual switches between long and short implied investors' horizon, we investigate the crash probabilities implied by our methodology.

4.5 Crash probabilities

Figure 6 displays the conditional probabilities of a $1-\alpha = 10\%$ crash over a horizon of four months. We present the probabilities from Martin (2017) ($\Pi^{Log}_{t\to T_1}[\alpha]$), those obtained by excluding the future risk neutral volatility from the SDF specification ($\Pi_{t\to T_1}[\alpha]$), and those obtained with our methodology ($\Pi_{t\to T_1,T_N}[\alpha]$) using the implied investors' horizon as T_N. The green areas display the conditional probability term premium which corresponds to $\Pi_{t\to T_1,T_N}[\alpha] - \Pi_{t\to T_1}[\alpha]$ (see Equation (18)).

Crash probabilities obtained with our method are lower than either $\Pi^{Log}_{t\to T_1}[\alpha]$ or $\Pi_{t\to T_1}[\alpha]$at all times. The green areas are thus always in the range of negative values. Most of the differences between $\Pi_{t\to T_1,T_N}[\alpha]$and $\Pi_{t\to T_1}[\alpha]$come from quiet times. During NBER recessions, the crash probabilities almost overlap.

To determine whether these lower probabilities are more accurate, we assess in Table 5 out-of-sample prediction performances. For each horizon, we compute the loss function of our prediction as the negative of the log-likelihood function as,

$$l_{t\to T_1,T_N} = -\left(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\log\left(\Pi_{t\to T_1,T_N}[\alpha]\right) + (1-\mathbb{1}_{R_{M,t\to T_1}<\alpha})(1-\log\left(\Pi_{t\to T_1,T_N}[\alpha]\right))\right).$$

Similarly, we compute the loss function for $\Pi_{t\to T_1}[\alpha]$ and $\Pi^{Log}_{t\to T_1}[\alpha]$, which we respectively denote $l_{t\to T_1}$ and $l^{Log}_{t\to T_1}$. Next, we test the significance of the average difference in loss functions using the Diebold and Mariano (1995) test. We find that our probabilities for a 10% crash, reported in the third column, lead to significantly lower losses (i.e., higher realized log-likelihoods) than other benchmark probabilities for most horizons. Finally, we similarly find significantly superior predictions for a crash size of 20% for all horizons except one week.

Now that we have confidence in the crash probabilities implied by our method, we can assess whether they have a link with the investors' implied horizon. Figure 7 plots in Panel A, the four-month 10% crash probability, with in grey the times at which this probability exceeds its 80% quantile of 12.4%, calculated over the full 1996 to 2023 time period. These grey areas thus indicate the times of market stress, during which the probability of a crash is high. Panel B displays the implied investors' horizon, together with these stress periods. Stress times are almost systematically associated with a short implied investors' horizon. With T_1 at four months, the shortest possible investment horizon $T_N > T_1$ is five months. In stress times, the implied investors' horizon is thus equal to five months. The implied investors' horizon jumps back to two years as soon as the probability of a crash decreases.

This analysis confirms that the implied investors' horizons switches with market conditions.

4.6 Term structure of equity risk premium

As in Chabi-Yo and Loudis (2020), we define the term structure of equity risk premium to be the hold-to-maturity yield on the S&P 500 implied by our equity risk premium estimates at various horizons.[12] Figure 8 compares the term structure of equity risk premium without ($RP_{t\to T_1}$, Panel A) and with ($RP_{t\to T_1,T_N}$, Panel B) intertemporal hedging. Without intertemporal hedging, the equity risk premium tends to slightly increase in T_1 in quiet times, and to strongly decrease in T_1 during turbulent times, as documented by Chabi-Yo and Loudis (2020).[13]

With intertemporal hedging, the investors' implied horizon T_N is long in quiet times, pulling the equity risk premium up, and short in turbulent times, leaving it almost unchanged. As a result, the term structure of equity risk premium is most of the time decreasing in T_1. In times of market calm, it is nearly flat, and it is strongly decreasing in times of market stress.

5 Estimating preference parameters

In this section, we attempt to improve the results achieved in Section 4 by calibrating the preference parameters ρ_t and τ_t to past data. We show the challenge of estimating these coefficients, and underline the importance of having enough data, covering both market turmoil and calm, to do so.

5.1 Methodology

We estimate the preference parameters ρ_t and τ_t using a two-stage non-linear least squares approach, similar to Chabi-Yo and Loudis (2020). Specifically, we estimate the coefficients τ_t, ρ_t, $\beta_0^{(1)}$, and $\beta_0^{(2)}$ by minimizing the weighted sum of squared errors $w_1\epsilon_{t\to T_1}^{(1)\intercal}\epsilon_{t\to T_1}^{(1)}$ +

[12] This definition differs from the literature studying the term structure of equity yields, which are defined in analogy to bond yields and extracted from dividend strips data. See van Binsbergen, Brandt, and Koijen (2012); van Binsbergen, Hueskes, Koijen, and Vrugt (2013) and van Binsbergen and Koijen (2017). Bansal, Miller, Song, and Yaron (2021) raise the potential criticism that traded dividend strips may be illiquid, and that their results on the term structure of equity yields may be artefacts of this illiquidity. Giglio, Kelly, and Kozak (2024) do not use dividend strips and instead use equity returns to estimate an affine model and make inference on the term structure of equity yields.

[13] Ait-Sahalia, Karaman, and Mancini (2020) found similar dynamics of the term structure by estimating an affine model on variance swaps with maturities ranging from 2 to 24 months.

$w_2 \epsilon^{(2)\intercal}_{t\to T_1} \epsilon^{(2)}_{t\to T_1}$ in the following equations,

$$\begin{aligned} R_{M,t\to T_1} - R_{f,t\to T_1} &= \beta_0^{(1)} + RP_{t\to T_1,T_N} + \epsilon^{(1)}_{t\to T_1}, \\ (R_{M,t\to T_1} - R_{f,t\to T_1})^2 &= \beta_0^{(2)} + \mathbb{E}_t\,(R_{M,t\to T_1} - R_{f,t\to T_1})^2 + \epsilon^{(2)}_{t\to T_1}. \end{aligned} \tag{28}$$

In the first stage, we set $w_1 = w_2 = 1$. In the second stage, we weigh each sum of squared errors by the inverse of the standard deviations of first-stage errors. Note that parameters τ_t and ρ_t enter the above equations through $RP_{t\to T_1,T_N}$ and $\mathbb{E}_t\,(R_{M,t\to T_1} - R_{f,t\to T_1})^2$.[14] We estimate parameters separately for each horizon T_1 and T_N. We restrict the parameter space such that the resulting risk premiums be positive.

5.2 Performance with in-sample estimation

Similar to Chabi-Yo and Loudis (2020), we use all observations in our time sample from 1996 to 2023 to estimate the preference parameters.[15]

We first estimate constant preference parameters over the period. We find estimates of τ that are between 0.86 and 0.88 for all forecast horizons T_1 and investment horizons T_N. In contrast, the estimates of ρ vary much more. Specifically, the estimated ρ for the bound $RP_{t\to T_1}$decreases sharply with T_1, from 5.06 to 1.20. The estimate of ρ also decreases with T_N. The estimate for $T_N = 2$ years is quite stable, between 1.20 and 1.60 for all T_1.

Table 6 compares the out-of-sample R^2 achieved when setting $\tau = 1$ and $\rho = 2$, as in Section 4, to those obtained when estimating these parameters. Columns (4) and (5) contain the R^2 for the bound of $RP_{t\to T_1}$and our new bound, with T_N optimized, using constant preference parameters. Estimating these parameters yields an increase in the R^2 and realized certainty equivalents of $RP_{t\to T_1}$. But for our bounds, the results are less clear. The R^2 obtained are still larger than those of $RP_{t\to T_1}$for all forecast horizons larger than one

[14]The physical second moments are provided in Appendix B.2.

[15]This estimation introduces a look-ahead bias when computing the out-of-sample performance measures. We will eliminate this bias in Section 5.3.

month, but they are smaller than those obtained when setting $\tau = 1$ and $\rho = 2$, for forecast horizons up to five months. The lack of performance of our estimate when estimating the preference parameters is due to the positivity constraint we impose on the risk premium estimates. Our estimation is performed over a time series ranging from 2000 to 2023, thus covering various economic conditions. The positivity constraint restricts the set of admissible preference parameters considerably, thereby limiting the gain of letting them free.

Second, we model τ and ρ as linear functions of past three-month returns, and estimate the loadings over the whole data period. The estimated time series of τ_t are displayed in Panel A of Figure 9, for a forecast horizon T_1 of 1 month. τ_t increases and gets closer to 1 when the investor horizon T_N increases. It exhibits variation both in the time series and term structure dimension. In particular, in times of market stress, τ_t decreases, in line with investors' risk aversion being higher. In quiet times τ_t is closer to 1, indicating that investors are less risk averse. The estimated time series of ρ_t are displayed in Panel B. ρ_t exhibits large time series variation. It is close to 2 in calms markets but increases sharply during the financial crisis. In particular, inspection of the results reveals that the estimated values of ρ and τ obtained when estimating them constant (τ =0.88 and ρ = 5.06) are now only reached during the Financial Crisis. The values of ρ_t are, as in the first estimation, larger when $T_N = T_1$ ($RP_{t \to T_1}$), than the values with intertemporal hedging. For longer investment horizons T_2, ρ_t oscillates around 2 and exhibits less volatility.

Columns (6) and (7) of Table 6 report the out-of-sample prediction results obtained when modelling τ and ρ as linear functions of past three-month returns. This additional degree of flexibility improves the performance of the two bounds $RP_{t \to T_1}$and $RP_{t \to T_1, T_N}$. Our bound provides R^2 that are consistently larger than those of $RP_{t \to T_1}$, with an improvement up to 45%.

These results show that a more precise estimation of the preference parameters, using a time series as large as possible, improves the performance of our bound.

5.3 Telescopic and rolling window estimations

In order to avoid a look-ahead bias, we now estimate a set of parameters using a telescopic window of past observations. We start in 2006 and use the past ten years of data to ensure we have enough stability in our estimated parameters.

Figure 10 displays the estimated time series of preference parameters, when assuming them constant over the estimation period. These time series make it clear that the values achieved in Section 5.2 result from realized returns during the Financial Crisis. From 2010, the preference parameter estimates stabilize, to only change slightly during the Covid period.

Table 7 reports the results when the parameters τ and ρ are assumed constant and estimated on window that at each time t does not include any data further to t. The first striking results is that for forecast horizons that are shorter than five months, estimating the preference parameters without look-ahead bias produces poor results for both $RP_{t\to T_1}$ and our bound. The values that are left blank in the table are negative and smaller than -1, indicating that the prediction is far worse than the long-term mean. For forecast horizons of 6 months and more, the best results are obtained with our bound, and a telescopic estimation of the preference parameters. Inspection of the certainty equivalents however shows that the estimation of the second moment is poor for all estimations except the one which sets $\tau = 1$ and $\rho = 2$.

These results illustrate the challenge of achieving good out-of-sample performance when estimating the preference parameters. The time series of estimated τ_t and ρ_t suggest that the instabilities in the telescopic estimation may be linked to the high values achieved during the 2006-2009 period. We now re-assess the out-of-sample performance of the different risk premia, excluding this time period from the evaluation. Table 8 provides the results.

Excluding the 2006-2009 period, the R^2_{OOS} achieved by $RP_{t\to T_1}$ with both telescopic and rolling window estimations of τ_t and ρ_t are higher than those with ρ fixed, for all forecast horizons, except for T_1 at ten days in the rolling window estimation. Furthermore, the rolling window estimation fails at delivering high R^2_{OOS}, but the telescopic estimation achieves R^2_{OOS}

for $RP_{t\to T_1,T_N}$ that further improve upon $RP_{t\to T_1}$. Our results therefore illustrate the need for an estimation window that includes large negative returns (as in 2008).

6 Portfolio rebalancing

The results derived so far were under the assumption that the representative agent could only rebalance her portfolio at time T_1. In this section, we relax this assumption and let the representative agent rebalance her portfolio at any time t such that $T_1 < t < T_N$. We assess whether this extension changes our main results.

6.1 Theoretical setup

As before, we use a second-order Taylor expansion-series of the inverse marginal utility (term inside the conditional expectation in (2). The novelty is that the Taylor-expansion uses the information that the agent re-balances her portfolio at any time t such that $T_1 < t < T_N$.

We denote

$$R_{M,t\to T_N} = \prod_{j=1}^{J} R_{M,T_{Q_{j-1}}\to T_{Q_j}} \text{ and } R_{f,t\to T_N} = \prod_{j=1}^{J} R_{f,T_{Q_{j-1}}\to T_{Q_j}}$$

with $T_0 = t$ and

$$x_j = R_{M,T_{Q_{j-1}}\to T_{Q_j}} \text{ and } x_{0,j} = R_{f,T_{Q_{j-1}}\to T_{Q_j}}$$

where $Q_{j-1} \in \{0, 1, ..., N-1\}$. In the previous section, we set $J = 2$. A second-order Taylor expansion-series of the inverse marginal utility (term inside the conditional expectation in (2)) around $(x_1, ..., x_N) = (x_{0.1,}, ..., x_{0.N})$ and taking the expectation under the risk neutral measure at time T_1 allows us to write (2) as

$$\upsilon_{T_1} = 1 + \frac{1}{\tau_t x_{0,1}}(x_1 - x_{0.1}) + \frac{1}{x_{0,1}^2}\frac{(1-\rho_t)}{\tau_t^2}(x_1 - x_{0,1})^2 + \frac{(1-\rho_t)}{\tau_t^2}\sum_{j>1}^{J}\frac{1}{x_{0,j}^2}\mathbb{E}_{T_1}^*(x_j - x_{0.j})^2 .$$

We replace this expression in (3) and derive the expected excess return on the market:

$$RP_{t\to T_1,T_N} = \frac{\frac{1}{\tau_t R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{1}{R^2_{f,t\to T_1}}\frac{(1-\rho_t)}{\tau_t^2}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{(1-\rho_t)}{\tau_t^2}\mathcal{LEV}^*_t}{1 + \frac{1}{R^2_{f,t\to T_1}}\frac{(1-\rho_t)}{\tau_t^2}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}^*_t\mathcal{M}^{*(2)}_{t,T_N}}. \tag{29}$$

where

$$\begin{aligned}\mathcal{LEV}^*_t &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathcal{M}^{*(2)}_{t,T_N}\right),\\ \mathcal{M}^{*(2)}_{t,T_N} &= \sum_{j>1}^{J}\frac{1}{R^2_{f,T_{Q_{j-1}}\to T_{Q_j}}}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}.\end{aligned}$$

Provided that preference parameters are estimated, expression (29) enables us to extract the risk premium from option prices if the risk neutral quantities $\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}$ can be recovered from option prices with various maturities.

6.2 Implementation

To compute the risk neutral quantities, we use an approach similar to (20) by considering the decomposition:

$$\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}}\right)^2 + \eta_{T_{Q_{j-1}}} \tag{30}$$

with $\mathbb{E}^*\left(\eta_{T_{Q_{j-1}}}|R_{M,t\to T_{Q_{-1}}}\right) = 0$. We then show:

$$\theta_{T_{Q_{j-1}}\to T_{Q_j}} = \frac{\mathbb{M}^{*(2)}_{t\to T_{Q_j}} - R^2_{f,T_{Q_{j-1}}\to T_{Q_j}}\mathbb{M}^{*(2)}_{t\to T_{Q_{j-1}}}}{\mathbb{E}^*_t\left(R^2_{M,t\to T_{Q_{j-1}}}\left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}}\right)^2\right)}.$$

and

$$\mathcal{LEV}^*_t = \sum_{j>1}^{J}\frac{1}{R^2_{f,T_{Q_{j-1}}\to T_{Q_j}}}\mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right). \tag{31}$$

with

$$\mathbb{COV}_t^*\left(R_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) = \mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) - R_{f,t\to T_1}\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} \quad (32)$$

In Appendix ??, we show that the second term in the right-hand-side of the covariance expression (32) simplifies to

$$\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\mathbb{M}^{*(2)}_{t\to T_{Q_{j-1}}}$$

If $T_{Q_{j-1}} = T_1$, the first term in the right-hand-side of (32) simplifies to

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_{Q_j}}\right) = \theta_{T_1\to T_{Q_j}}\left(\mathbb{M}^{*(3)}_{t\to T_1} + R_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}\right)$$

Now, assume that $T_{Q_{j-1}} > T_1$, the first term of the covariance expression simplifies to

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\theta_{T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right)\\ +R^2_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right) - 2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{M}^{*(2)}_{t\to T_1}\\ -2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}R^2_{f,t\to T_1} + R^2_{f,t\to T_{Q_{j-1}}}R_{f,t\to T_1}\end{array}\right\}$$

We leave the proof details of these expressions in Appendix ??. Provided that odd market risk neutral moments and the risk neutral leverage $\mathcal{LEV}_t^*$ are negative and conditions $1/\tau_t \geq 1$and $\rho_t - 1 \geq 1$ hold, we can further bound (29) as follows:

$$RP_{t\to T_1,T_N} \geq \frac{\frac{1}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} - \mathcal{LEV}_t^*}{1 - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \mathbb{E}_t^*\mathcal{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}}.$$

We then use option prices to recover the expected excess market for a fixed value J. For each fixed value J, there is a finite set of maturities that can be considered. To avoid presenting the results for all sets of maturities, we search for each J, the set of maturities that produces the best out-of-sample R-squared.

6.3 Empirical results

Table 10 summarizes the results when portfolio rebalancing is allowed. The new bound is very close to the bound obtained without rebalancing, for all forecast horizons T_1. Therefore, it still outperforms the bound $RP_{t\to T_1}$ and our results do not change.

7 Higher-order approximation of the equity risk premium

When using a second-order Taylor series-expansion, our theoretical results in the previous section show that $\mathbb{LEV}_t^*$ is a key contributor to the conditional expected excess market return. In this section, we investigate how higher-order leverage measures theoretically contribute to the conditional equity risk premium. We show that increasing the order of the approximation, therefore allowing for kurtosis preference, generates additional terms that contribute to the equity risk premium.

We show in Appendix C.3 that, under no-arbitrage assumptions, a third-order Taylor expansion-series produces a one-period SDF in a three-date (two-period) economy of the form

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \approx \frac{1 + z_{T_1} + z_{T_1}^v}{\mathbb{E}_t^* \left(1 + z_{T_1} + z_{T_1}^v\right)}, \tag{33}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t\to T_1}}(R_{M,t\to T_1} - R_{f,t\to T_1}) + \frac{a_{2,t}}{R_{f,t\to T_1}^2}(R_{M,t\to T_1} - R_{f,t\to T_1})^2 + \frac{a_{3,t}}{R_{f,t\to T_1}^3}(R_{M,t\to T_1} - R_{f,t\to T_1})^3,$$

$$z_{T_1}^v = \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{M}_{T_1\to T_N}^{*(2)} + \frac{a_{3,t}}{R_{f,T_1\to T_N}^3}\mathbb{M}_{T_1\to T_N}^{*(3)} + \frac{a_{2,3,t}}{R_{f,t\to T_1} R_{f,T_1\to T_N}^2}(R_{M,t\to T_1} - R_{f,t\to T_1})\mathbb{M}_{T_1\to T_N}^{*(2)}, \tag{34}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$. Using this third-order expansion, we next derive the conditional expected excess market return and the probability of a crash.

7.1 Equity risk premium

With the third-order Taylor expansion-series approach, Equation (33) depends on, in addition to risk-neutral variance, new terms such as risk-neutral skewness and cross-term between risk-neutral volatility and market excess return. These additional terms, as shown below, introduce additional high-order leverage effects in the expected excess return decomposition. To find a closed-form expression for the equity risk premium in terms of risk-neutral moments and high-order leverages, we first define high-order leverage effects under the risk-neutral measure as:

$$\mathbb{LES}_t^* = \mathbb{COV}_t^* \left(r_{M,t \to T_1}, \mathbb{M}^{*(3)}_{T_1 \to T_N} \right), \tag{35}$$

$$\mathbb{LEK}_t^* = \mathbb{COV}_t^* \left(r^2_{M,t \to T_1}, \mathbb{M}^{*(2)}_{T_1 \to T_N} \right). \tag{36}$$

We then show how the equity risk premium depends on these terms in the following Proposition.

Proposition 5 *Up to a third-order expansion-series, the one-period expected excess market return is*

$$RP^{3rd}_{t \to T_1, T_N} = \frac{\mathcal{D}_{1,t} + \mathcal{D}_{2,t}}{\mathcal{D}_{3,t} + \mathcal{D}_{4,t}} \tag{37}$$

with

$$\begin{aligned}
\mathcal{D}_{1,t} &= \sum_{k=1}^{3} \frac{a_{k,t}}{R^k_{f,t \to T_1}} \mathbb{M}^{*(k+1)}_{t \to T_1} \\
\mathcal{D}_{2,t} &= \frac{a_{2,t}}{R^2_{f,T_1 \to T_N}} \mathbb{LEV}_t^* + \frac{a_{3,t}}{R^3_{f,T_1 \to T_N}} \mathbb{LES}_t^* + \frac{a_{2,3,t}}{R_{f,t \to T_1} R^2_{f,T_1 \to T_N}} \left(\mathbb{LEK}_t^* + \mathbb{M}^{*(2)}_{t \to T_1} \mathbb{E}_t^* \mathbb{M}^{*(2)}_{T_1 \to T_N} \right) \\
\mathcal{D}_{3,t} &= 1 + \sum_{k=2}^{3} \frac{a_{k,t}}{R^k_{f,t \to T_1}} \mathbb{M}^{*(k)}_{t \to T_1} \\
\mathcal{D}_{4,t} &= \sum_{k=2}^{3} \frac{a_{k,t}}{R^k_{f,T_1 \to T_N}} \mathbb{E}_t^* \mathbb{M}^{*(k)}_{T_1 \to T_N} + \frac{a_{2,3,t}}{R_{f,t \to T_1} R^2_{f,T_1 \to T_N}} \mathbb{LEV}_t^*
\end{aligned}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ *and the risk-neutral quantities* $\mathbb{LEV}_t^*$, $\mathbb{M}_{T_i \to T_j}^{*(k)}$, $\mathbb{LES}_t^*$ *and* $\mathbb{LEK}_t^*$ *are defined in Equations* (9), (10), (35), *and* (36), *respectively.*

The proof of Proposition 5 is given in Appendix C.1.

We refer to our risk premium measure in Equation (39) as $RP_{t\to T_1,T_N}^{3rd}$. When (34) is removed from the SDF specification (33), which corresponds to a static SDF in a one-period economy, the equity risk premium reduces to

$$RP_{t\to T_1}^{3rd} = \frac{\mathcal{D}_{1,t}}{\mathcal{D}_{3,t}}, \tag{38}$$

Notice that the one-period expected excess market return obeys the following decomposition

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \pi_t^o RP_{t\to T_1}^{3rd} + (1-\pi_t^o)\,\mathbb{RP}_t^{\upsilon,s}, \tag{39}$$

with

$$RP_{t\to T_1}^{3rd} = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{a_{2,t}}{R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(3)} + \frac{a_{3,t}}{R_{f,t\to T_1}^3}\mathbb{M}_{t\to T_1}^{*(4)}}{1 + \frac{a_{2,t}}{R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{a_3}{R_{f,t\to T_1}^3}\mathbb{M}_{t\to T_1}^{*(3)}}, \tag{40}$$

$$\mathbb{RP}_t^{\upsilon,s} = \frac{\frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{LEV}_t^* + \frac{a_{3,t}}{R_{f,T_1\to T_N}^3}\mathbb{LES}_t^* + \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\left(\mathbb{LEK}_t^* + \mathbb{M}_{t\to T_1}^{*(2)}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}\right)}{\frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)} + \frac{a_3}{R_{f,T_1\to T_N}^3}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\mathbb{LEV}_t^*}, \tag{41}$$

and

$$\pi_t^o = \frac{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,t\to T_1}^k}\mathbb{M}_{t\to T_1}^{*(k)}}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,t\to T_1}^k}\mathbb{M}_{t\to T_1}^{*(k)} + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,T_1\to T_N}^k}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(k)} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\mathbb{LEV}_t^*}. \tag{42}$$

$RP_{t\to T_1}^{3rd}$ corresponds to the expected excess return in Chabi-Yo and Loudis.

7.2 Conditional crash probability

We next express the conditional probability of a crash using a third-order Taylor expansion-series for the SDF. To derive this probability, we define additional truncated moments as

$$\mathbb{M}^{*}_{t,s}\left[\alpha\right] = \mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(3)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right), \tag{43}$$

$$\mathbb{M}^{*}_{t,sv}\left[\alpha\right] = \mathbb{E}^{*}_{t}\left(r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right). \tag{44}$$

We then show:

Proposition 6 *Up to a third-order approximation, the conditional probability of a crash,* $\Pi^{3rd}_{t\to T_1}[\alpha] = P_t\left(R_{M,t\to T}<\alpha\right)$*, is*

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\left\{\begin{array}{c}\mathbb{M}^{*(0)}_{t\to T_1}\left[\alpha\right] + \sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}\left[\alpha\right] \\ +\frac{a_{2,t}}{R^{2}_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,v}\left[\alpha\right] + \frac{a_{3,t}}{R^{3}_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,s}\left[\alpha\right] + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^{2}_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,sv}\left[\alpha\right]\end{array}\right\}}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,T_1\to T_N}}\mathbb{E}^{*}_{t}\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^{2}_{f,T_1\to T_N}}\mathbb{E}^{*}_{t}r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}} \tag{45}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$.

The proof of Proposition 6 is given in Appendix C.2.

When $z^{v}_{T_1}$ is absent in the SDF expression (33), the SDF corresponds to the SDF in a one-period static economy. Under this scenario, the probability of crash reduces to

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}\left[\alpha\right] + \sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}\left[\alpha\right]}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^{k}_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}}$$

We refer to our crash probability in Equation (46) as $\Pi^{3rd}_{t\to T_1,T_N}[\alpha]$. This conditional crash probability in a two-period (three-date) economy is a weighted average of two probabilities:

$$\mathbb{P}_t\left(R_{M,t\to T_1}<\alpha\right) = \pi^{o}_{t}\Pi^{3rd}_{t\to T_1}[\alpha] + \left(1-\pi^{o}_{t}\right)\Pi^{v,s}_{t\to T_1}\left[\alpha\right], \tag{46}$$

with

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha] + \frac{a_{3,t}}{R^3_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}[\alpha]}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}}, \tag{47}$$

$$\Pi^{v,s}_{t\to T_1}[\alpha] = \frac{\frac{a_{2,t}}{\overline{R}^2_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,v}[\alpha] + \frac{a_{3,t}}{\overline{R}^3_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,s}[\alpha] + \frac{a_{2,3,t}}{R_{f,t\to T_1}\overline{R}^2_{f,T_1\to T_N}}\mathbb{M}^{*}_{t,sv}[\alpha]}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{\overline{R}^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}\overline{R}^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}, \tag{48}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ and π^o_t is defined in Equation (42)

7.3 Empirical results with fixed preference parameters

Table A3 reports the out-of-sample performance of our bound using the third-order Taylor expansion-series for the inverse SDF. We find that the predictions are overall not better than those of the second-order case. They are slightly worse for long investment horizons T_N, illustrating the challenge of accurately estimating higher order moments for long maturities, and slightly better for short maturities. While these results are in favour of our simpler second-order bounds, they are likely to improve should the liquidity of longer-maturity options improve with time, yielding better estimations of risk-neutral moments.

8 Conclusion

Given its importance in financial applications, there is considerable interest in improving our measurement of the conditional expected return on the market portfolio. Several methods using forward-looking information embedded in option prices have been proposed in recent years. Martin (2017), Chabi-Yo and Loudis (2020) and Tetlock (2023) measure a one-period expected excess return in a one-period, two-date economy. We contribute to the literature by deriving an expression accounting for intertemporal hedging.

We, theoretically and empirically, show a significant difference between a static and a dynamic estimation. In a dynamic economy, the SDF is a nonlinear function of the market return as in a one-period economy. But it also depends on novel risk-neutral quantities such as the expected future variance and skewness and the covariances between market returns and future variance and skewness, namely the leverage effects. We show how these quantities significantly impact the one-period conditional expected excess return on the market from the perspective of an investor who holds the market portfolio in a multi-period economy. We also derive expressions for the one-period conditional probability of a crash, in a multi-period economy, in terms of risk-neutral quantities.

Our methodology provides significantly better risk premium and crash predictions and market-timing allocations in empirical tests. We further use our measure to shed light on the shape and time variations of the term structure of equity risk premia, which we define as the expected excess market return as a function of the investment horizon. In a one-period economy, Chabi-Yo and Loudis (2020) find that the term structure is upward sloping on average and downward sloping during recessions. Our term structure slope is essentially flat during normal market conditions and downward sloping during recessions.

While we have used the S&P 500 index to proxy for the market portfolio, our methodology can be extended to individual assets and international markets. We leave these endeavors for future research.

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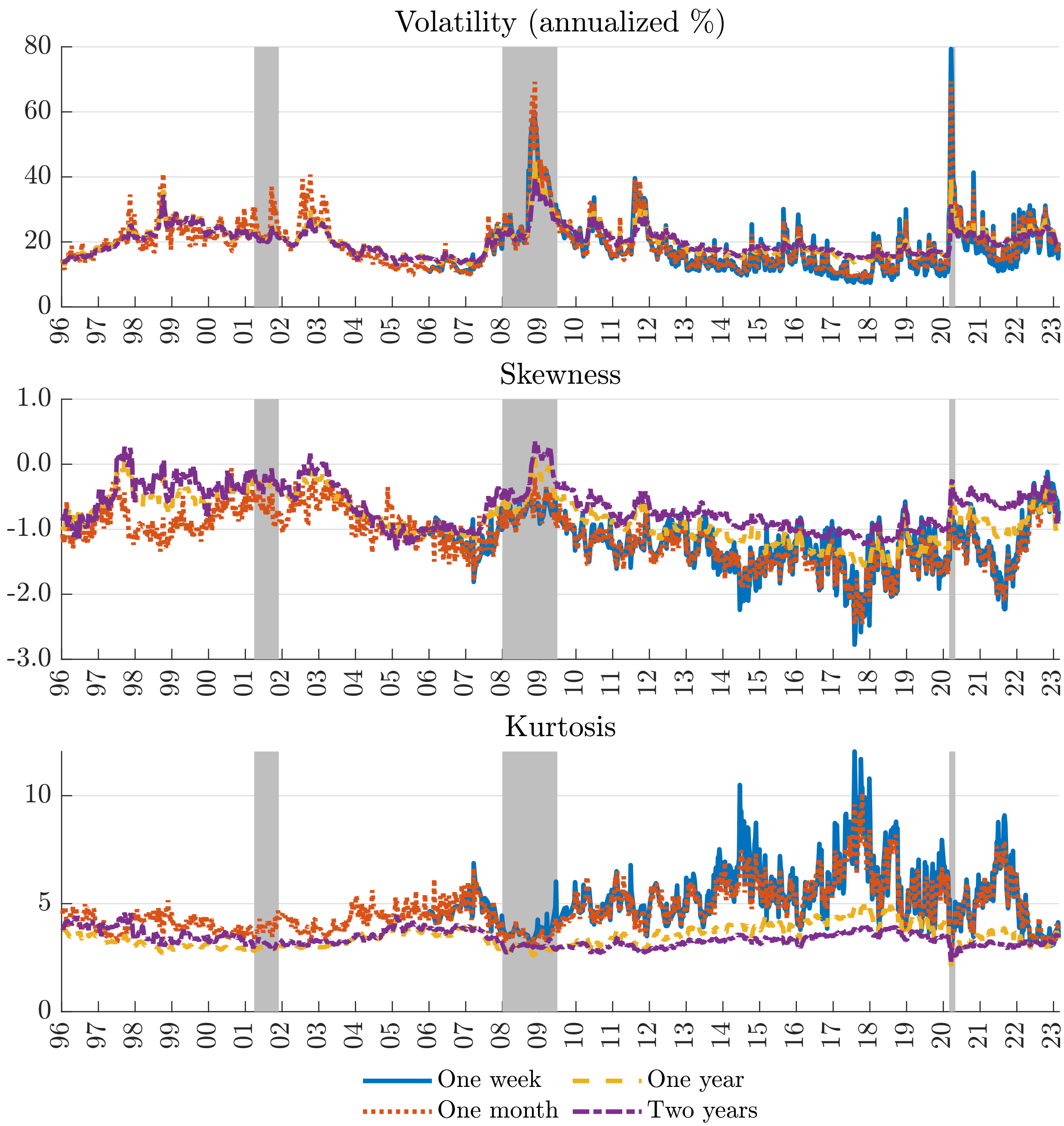

Figure 1: **Risk-neutral moments.**
We report option-implied risk-neutral volatility, skewness, and kurtosis for the S&P 500 index at a horizon of one week, one month, one year, and two years. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

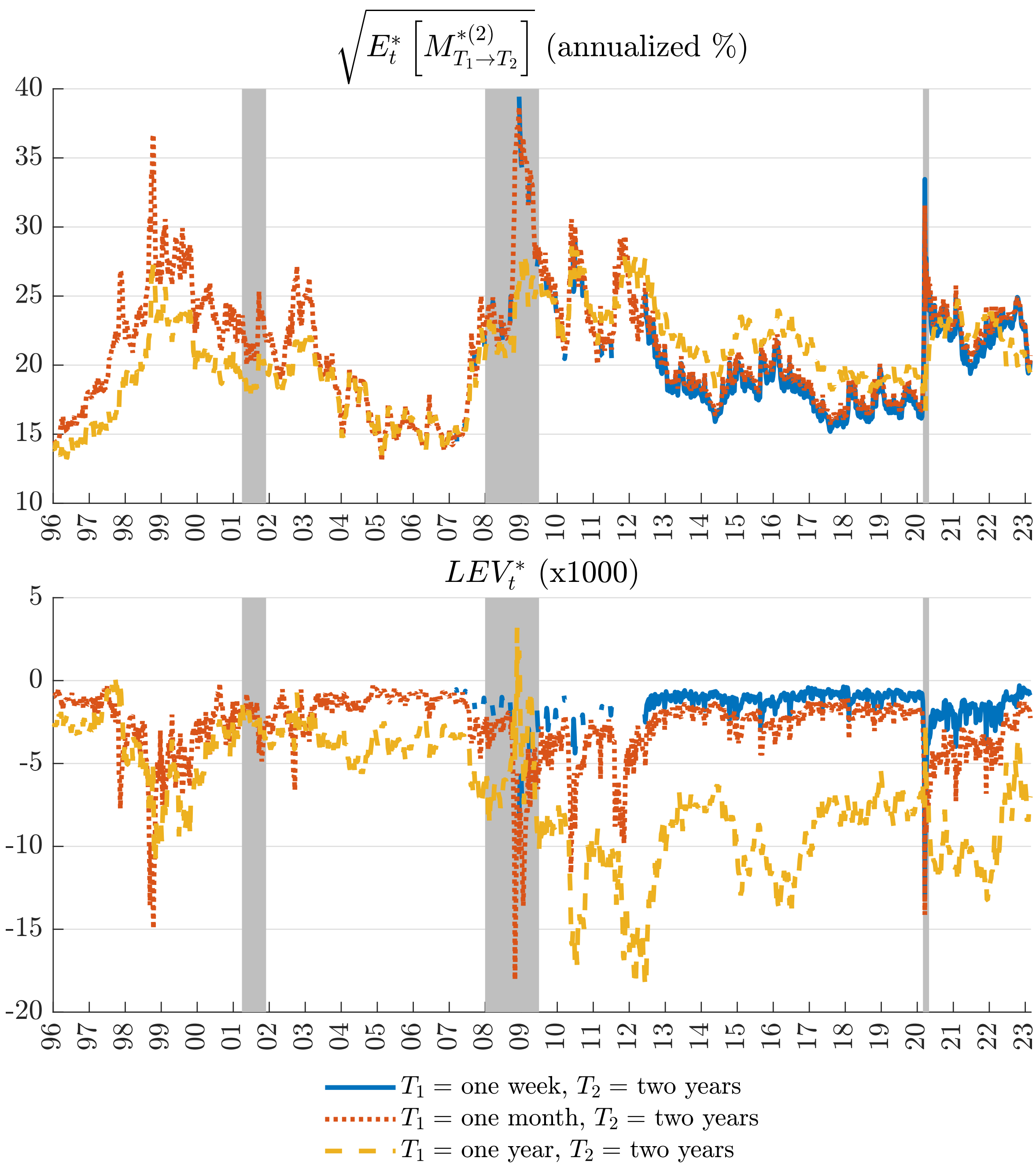

Figure 2: **Risk-neutral expected future variance and leverage.**
We report in the top graph the risk-neutral expected future volatility for the S&P 500 index. We report in the bottom graph the risk-neutral covariance between market returns and future variances in Equation (9). We use horizons T_1 of one week, one month, one quarter, and one year, and T_N = two years. We annualize each measure by multiplying by $\frac{365}{T_N - T_1}$. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

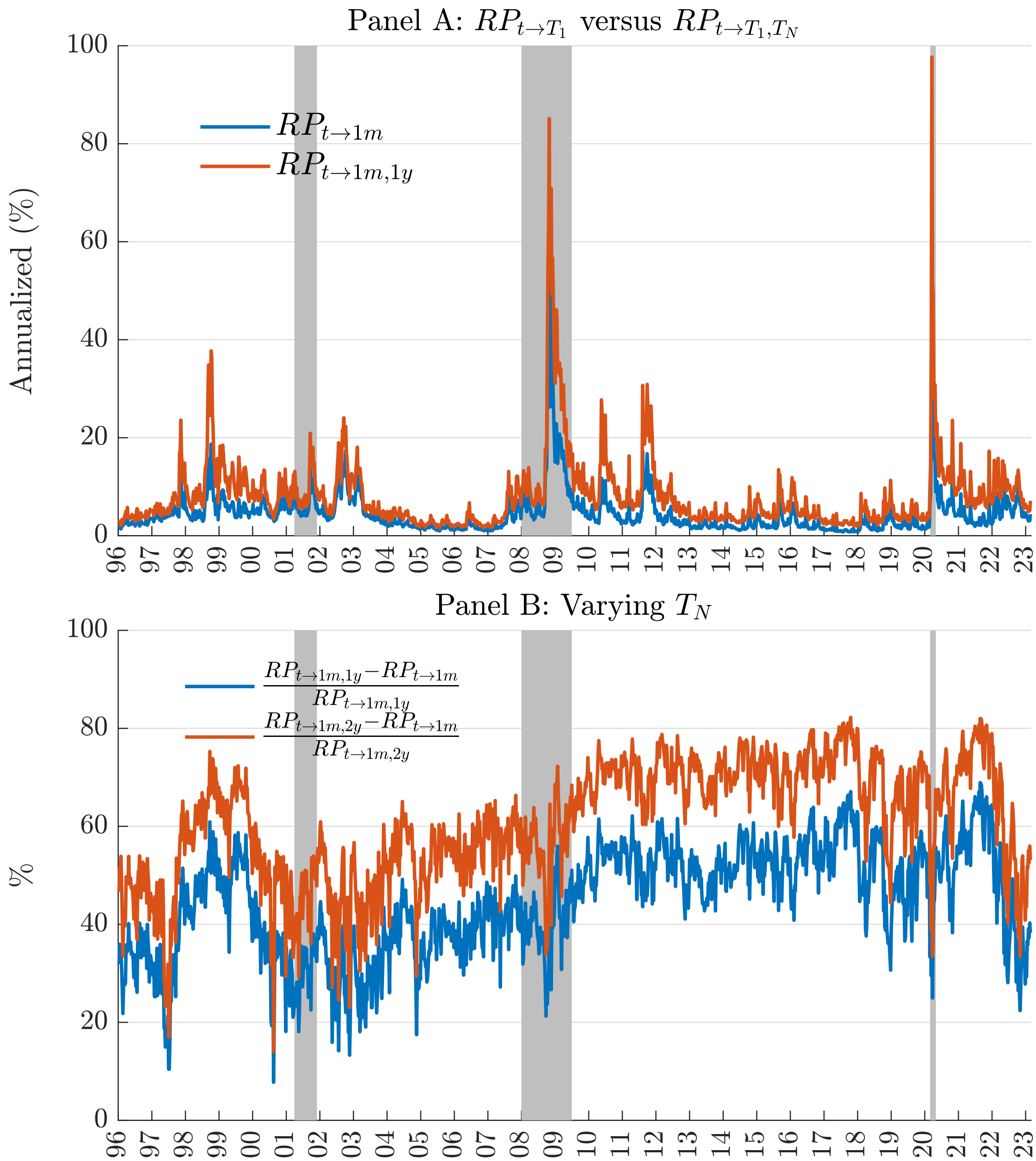

Figure 3: **Equity risk premium.**
This graph represents the different equity risk premium bounds. Panel A compares the bounds of Chabi-Yo and Loudis (2020), $RP_{t\to T_1}$, to our bound $RP_{t\to T_1,T_N}$ in Equation (8), for T_1- one month and $T_N =$ one year. Panel B displays the percentage of the total premium $RP_{t\to T_1,T_N}$ that is not part of $RP_{t\to T_1}$, i.e., the fraction of the total premium that comes from intertemporal hedging. The forecast horizon T_1 is of one month, and $T_N = 1$ year (blue) and 2 years (red).

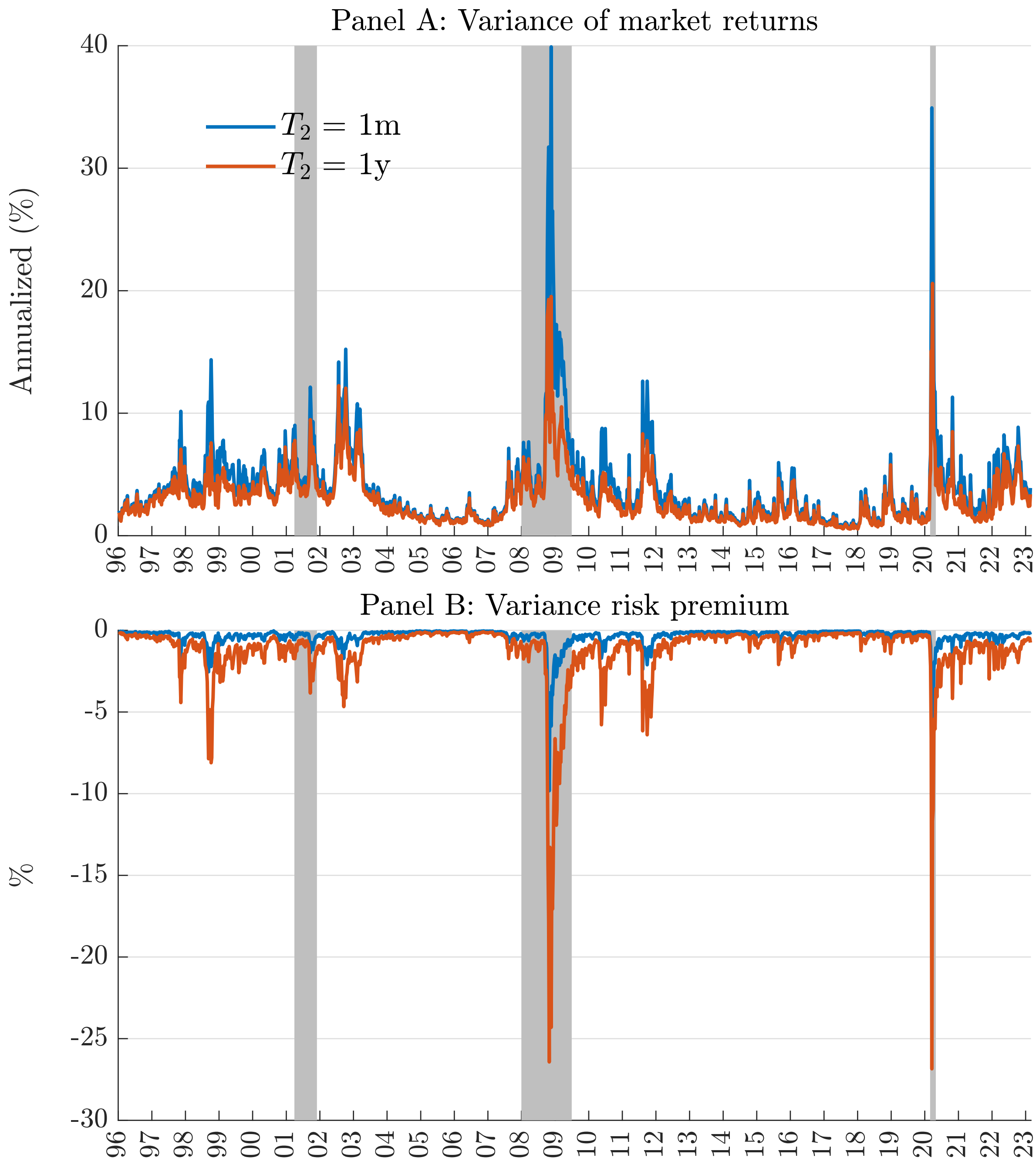

Figure 4: **Variance risk premium.**
This graph represents the expectation of the physical variance (Panel A) and the associated variance risk premium (Panel B) with and without intertemporal hedging. The forecast horizon T_1 is of one month, and $T_N = 1$ year with intertemporal hedging.

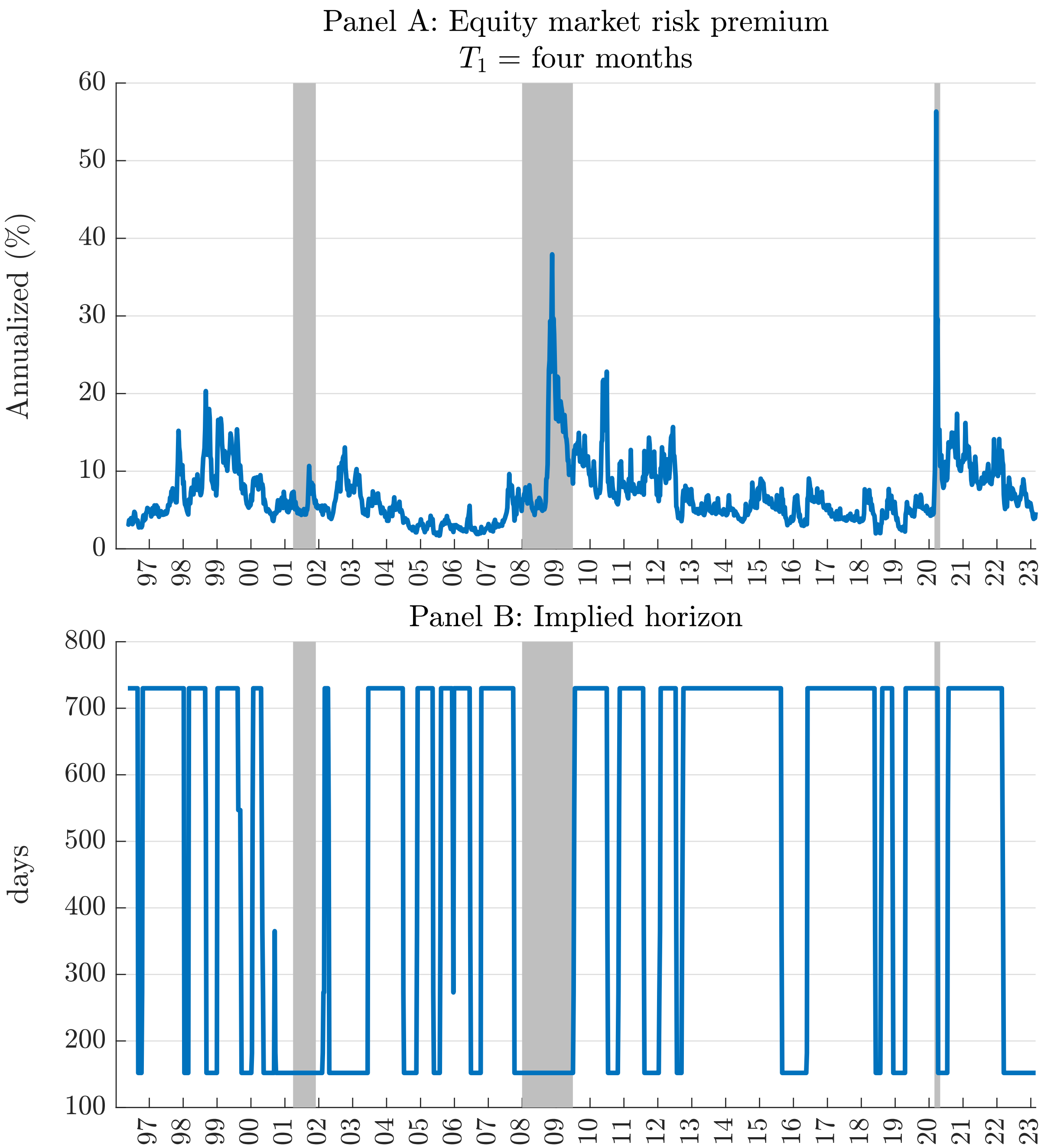

Figure 5: **Implied investors' horizon for $T_1 = 4$ months.**
This graph represents, in Panel A, the 4-month ERP obtained with an optimized investors' horizon. Panel B displays the implied investors' horizon $T^*_{N,t}$, which maximizes the in-sample fit of our bound to the realized returns, as measured by the R^2 over a window of 90 days.

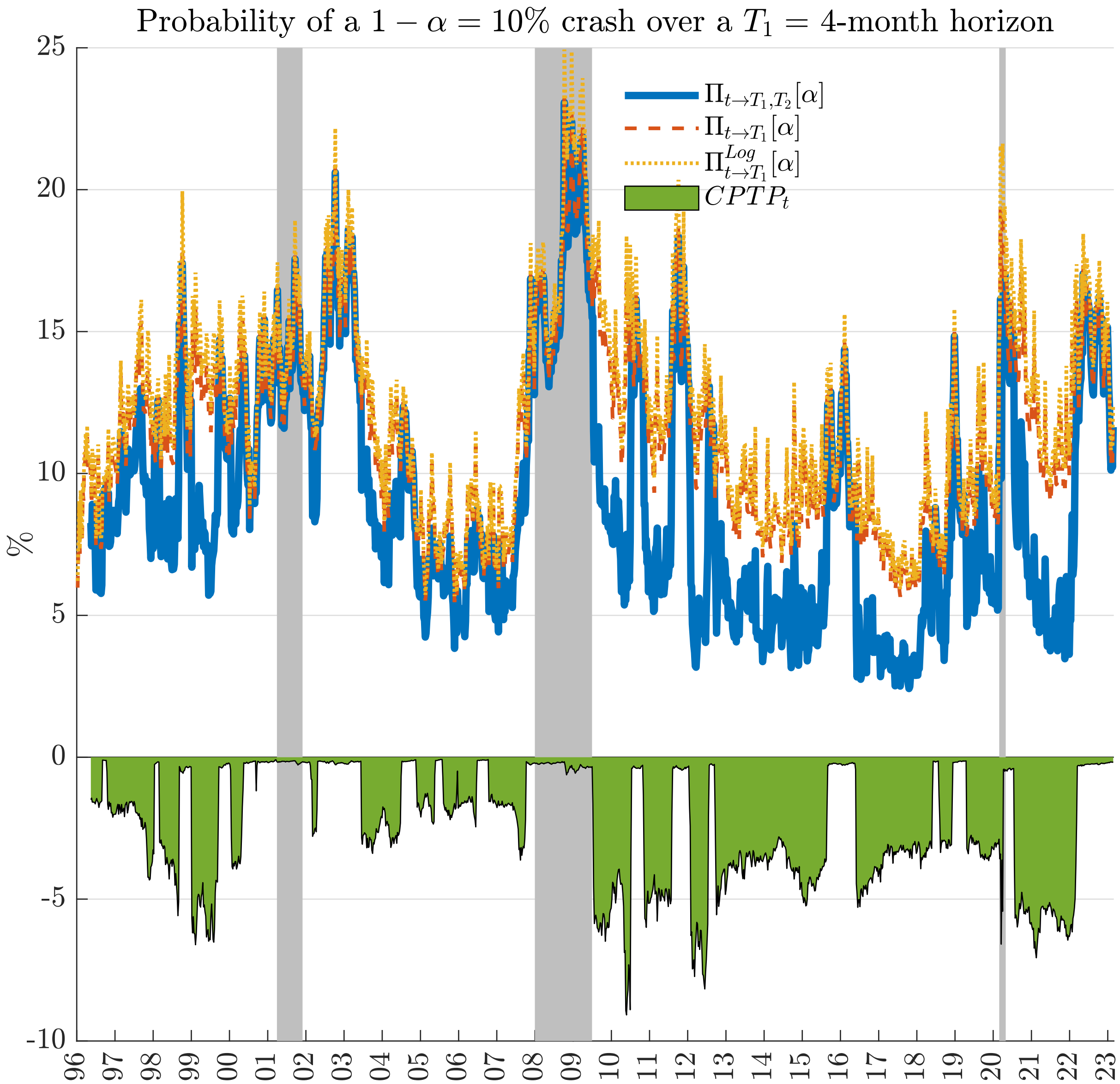

Figure 6: **Probability of a 10% market crash**
We report the time-varying probability of a 10% stock market crash from Proposition 4 and the conditional probability term premium ($CPTP_t$) from Corollary 16. The top graph reports on a horizon of T_1 = one month and the bottom graph report on a horizon of T_1 = one quarter. We use T_N = optimal. Gray areas are NBER recessions.

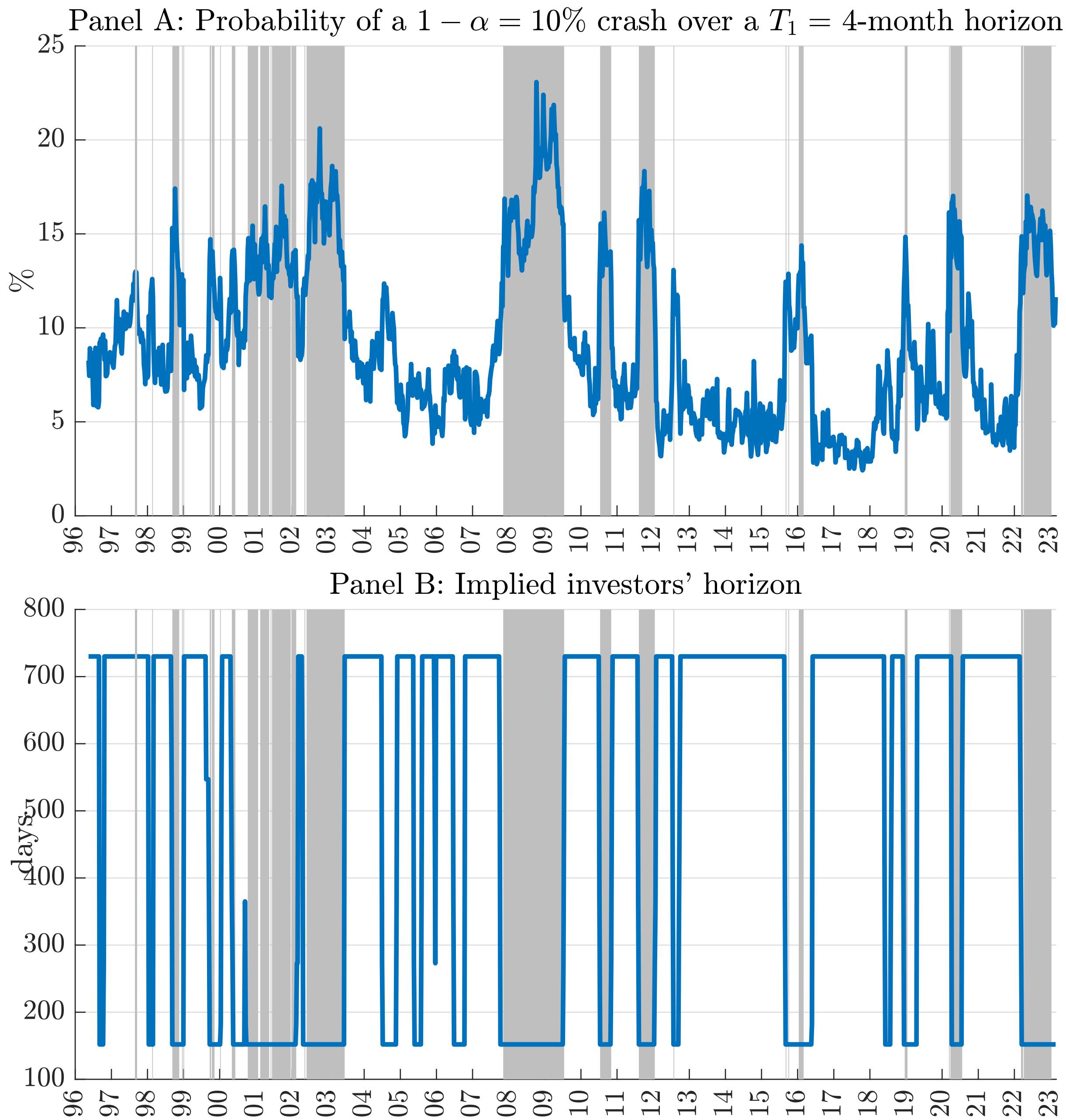

Figure 7: **Probability of a 10% market crash and implied investors' horizon.** Panel A displays the time-varying probability of a 10% stock market crash from Proposition 4. The grey areas are times when the probability of a 10% market crash is above its 80% quantile, calculated over the 1996-2023 time period. Panel B displays the implied investors' horizon $T^*_{N,t}$, which maximizes the in-sample fit of our bound to the realized returns, as measured by the R^2 over a window of 90 days.

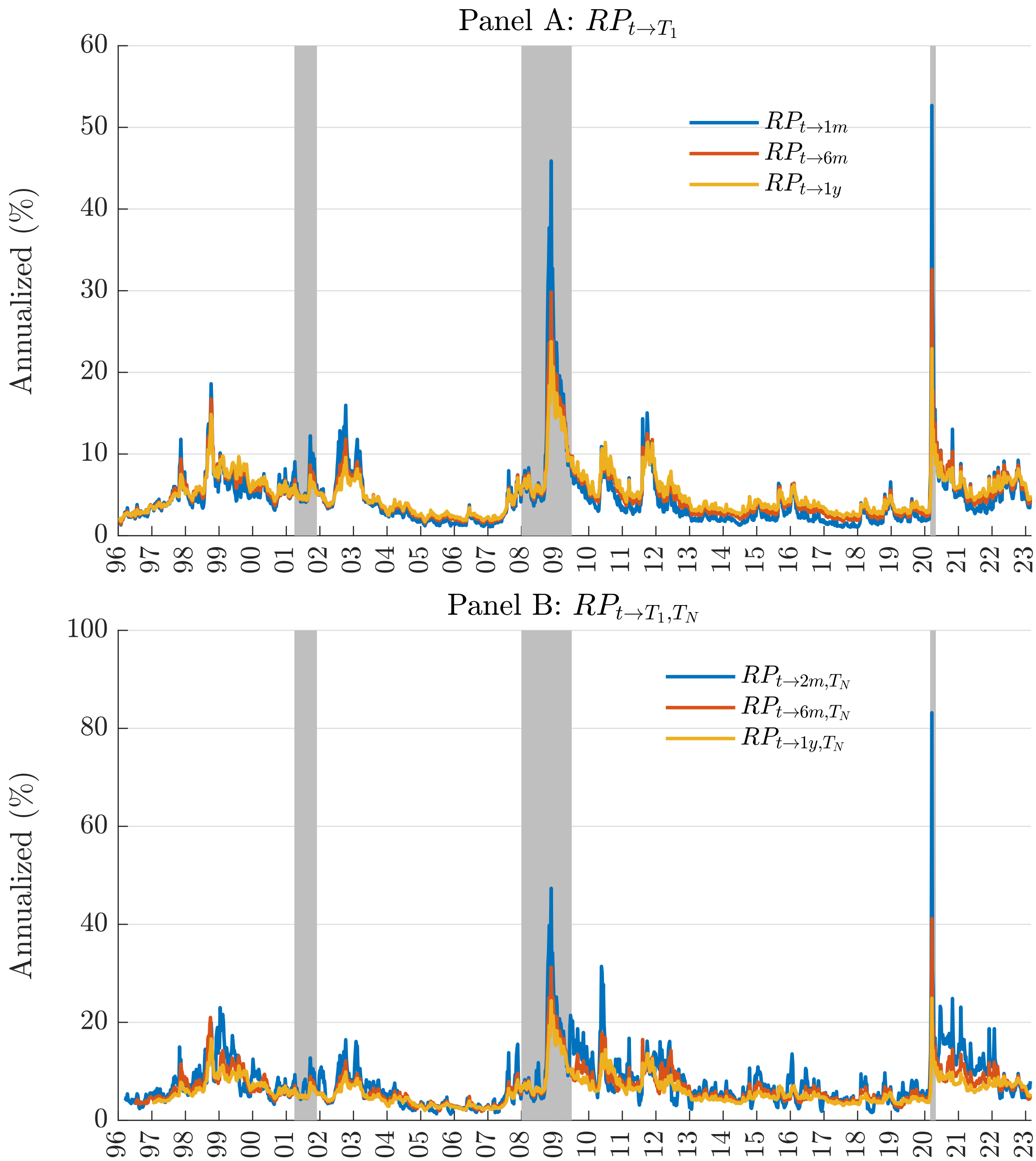

Figure 8: **Term structure of equity risk premium**
This graph represents the term structure of the equity risk premium bounds $RP_{t\to T_1}$, Chabi-Yo and Loudis (2020) (Panel A) and of our bound $RP_{t\to T_1,T_N}$ (Panel B). The forecast horizons are $T_1 = 1$ month, 6 months and 1 year, and T_N is set equal to $T^*_{N,t}$.

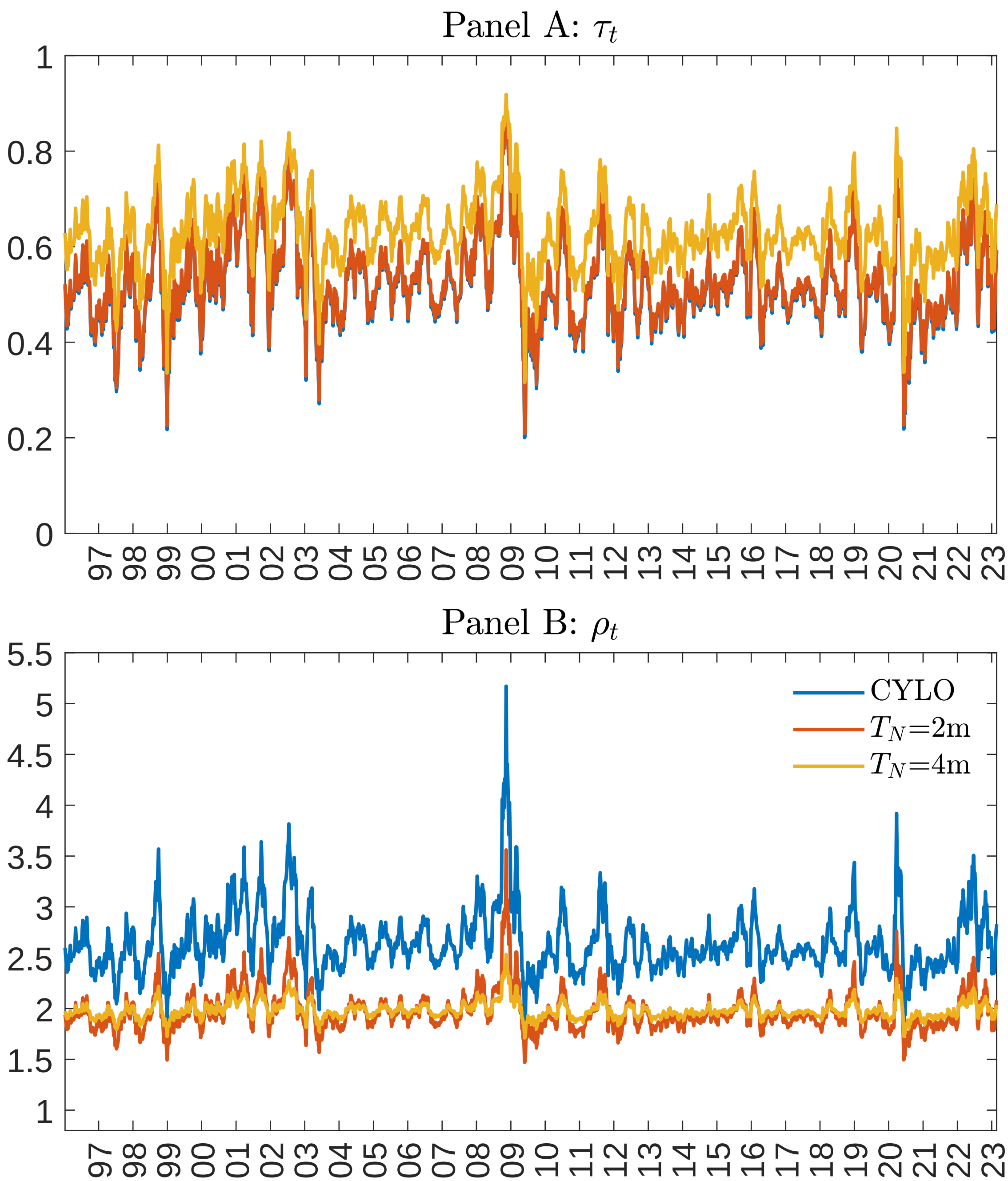

Figure 9: **Estimated preference parameters τ_t and ρ_t over the period 1996-2023.** This graph represents the estimated time series of risk aversion parameter τ_t and skewness tolerance parameter ρ_t, for $T_1 = 1$ month and varying T_N. Estimates are obtained by applying the estimation methodology described in Section 5.1 on the whole dataset, from 1996 to 2023.

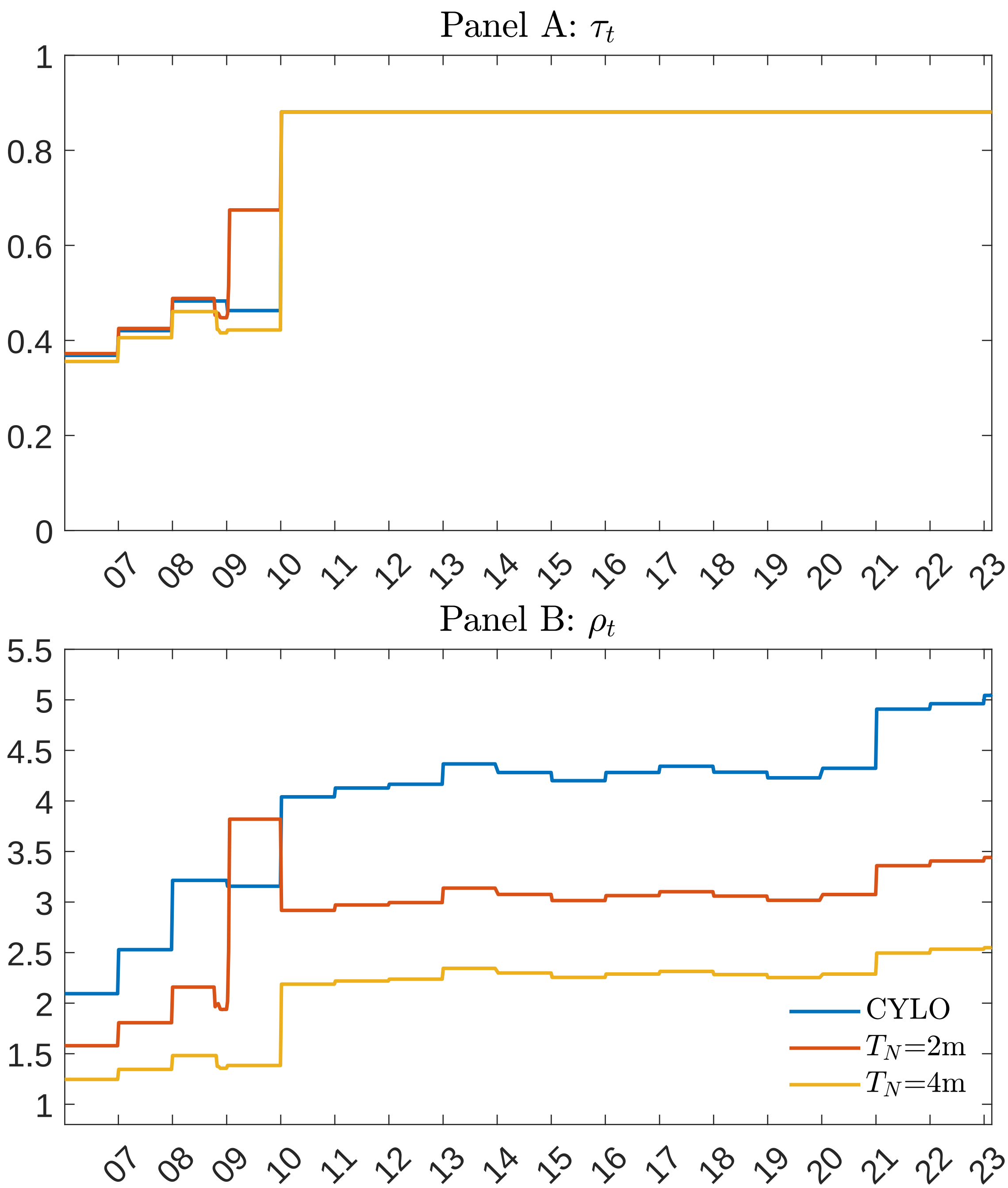

Figure 10: **Estimated preference parameters τ_t and ρ_t in telescopic estimation.** This graph represents the estimated time series of risk aversion parameter τ_t and skewness tolerance parameter ρ_t, for $T_1 = 1$ month and varying T_N. Estimates are obtained using the estimation methodology described in Section 5.1 on an expanding window of time. The initial window starts in 1996 until 2006.

Table 1: **Summary statistics for risk premia -** $T_N = 2y$

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. All values are annualized and in percent. Data are monthly from January 1996 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We use T_N = two years.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
$RP^{Log}_{t\to T_1}$	3.58	4.71	6.25	60.48	1.03	1.41	2.27	3.96	6.71
$RP_{t\to T_1}$	3.64	4.82	6.08	56.86	1.03	1.44	2.30	3.97	6.91
$RP_{t\to T_1,T_2}$	4.17	5.86	5.40	44.90	1.03	1.45	2.49	4.36	8.54
Panel B: One month									
$RP^{Log}_{t\to T_1}$	4.27	4.24	4.66	36.24	1.34	1.89	3.25	5.11	7.72
$RP_{t\to T_1}$	5.16	5.75	6.16	69.04	1.53	2.25	3.79	5.90	9.41
$RP_{t\to T_1,T_2}$	NaN.00	NaN.00	3.58	24.11	3.21	4.39	6.94	11.44	17.29
Panel C: One quarter									
$RP^{Log}_{t\to T_1}$	4.32	3.29	3.43	22.09	1.70	2.21	3.53	5.34	7.34
$RP_{t\to T_1}$	5.36	4.51	4.99	53.62	1.98	2.73	4.34	6.41	9.12
$RP_{t\to T_1,T_2}$	NaN.00	NaN.00	3.14	20.04	3.40	4.56	6.81	9.91	13.90
Panel D: Six months									
$RP^{Log}_{t\to T_1}$	4.30	2.62	2.60	14.30	1.95	2.45	3.71	5.35	6.94
$RP_{t\to T_1}$	5.46	3.59	3.11	21.27	2.36	3.12	4.68	6.57	8.99
$RP_{t\to T_1,T_2}$	7.37	4.26	2.29	11.95	3.38	4.52	6.41	9.01	12.05
Panel E: One year									
$RP^{Log}_{t\to T_1}$	4.23	2.18	2.16	11.02	2.15	2.65	3.83	5.20	6.65
$RP_{t\to T_1}$	5.25	2.72	2.13	10.90	2.62	3.38	4.73	6.35	8.19
$RP_{t\to T_1,T_2}$	5.92	2.73	1.96	10.66	3.02	4.02	5.44	7.22	9.11

Table 2: **Summary statistics for risk premia - $T_N = 1y$**

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. All values are annualized and in percent. Data are monthly from January 1996 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We use T_N = one year.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
$RP^{Log}_{t\to T_1}$	3.58	4.71	6.25	60.48	1.03	1.41	2.27	3.96	6.71
$RP_{t\to T_1}$	3.64	4.82	6.08	56.86	1.03	1.44	2.30	3.97	6.91
$RP_{t\to T_1,T_2}$	4.00	6.26	8.23	100.50	1.03	1.45	2.46	4.15	8.06
Panel B: One month									
$RP^{Log}_{t\to T_1}$	4.27	4.24	4.66	36.24	1.34	1.89	3.25	5.11	7.72
$RP_{t\to T_1}$	5.16	5.75	6.16	69.04	1.53	2.25	3.79	5.90	9.41
$RP_{t\to T_1,T_2}$	7.87	7.56	4.39	32.83	2.55	3.58	5.88	9.58	13.94
Panel C: One quarter									
$RP^{Log}_{t\to T_1}$	4.32	3.29	3.43	22.09	1.70	2.21	3.53	5.34	7.34
$RP_{t\to T_1}$	5.36	4.51	4.99	53.62	1.98	2.73	4.34	6.41	9.12
$RP_{t\to T_1,T_2}$	6.95	5.07	3.48	24.46	2.78	3.82	5.81	8.37	11.75
Panel D: Six months									
$RP^{Log}_{t\to T_1}$	4.30	2.62	2.60	14.30	1.95	2.45	3.71	5.35	6.94
$RP_{t\to T_1}$	5.46	3.59	3.11	21.27	2.36	3.12	4.68	6.57	8.99
$RP_{t\to T_1,T_2}$	6.22	3.83	2.57	14.19	2.83	3.71	5.39	7.53	10.13

Table 3: **Out-of-sample prediction and allocation performance**

We report the out-of-sample performance of different risk premium prediction methods. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). The results in the last column are based on predicted returns obtained by averaging $RP_{t\to T_1,T_N}$ across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. Negative certainty equivalents are not reported. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

			$RP_{t\to T_1,T_N}$ with $T_N =$ (in months)										
Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.08	−0.04	0.00	0.05	0.08	0.12	0.16	0.15	0.03	−0.28	−0.93	0.15
1	0.98	1.11	-	1.26	1.38	1.47	1.55	1.62	1.69	1.64	1.32	0.65	1.65
2	1.50	1.89	-	-	2.11	2.31	2.49	2.66	2.99	3.18	3.32	3.03	3.04
3	1.34	2.03	-	-	-	2.31*	2.57*	2.81*	3.37*	3.77	4.34	4.47	3.65*
4	1.91	2.94	-	-	-	-	3.22**	3.51**	4.20**	4.74*	5.55*	6.00	4.79*
5	2.66	4.07	-	-	-	-	-	4.38**	5.16**	5.82**	6.83*	7.54*	6.16**
6	2.84	4.56	-	-	-	-	-	-	5.39**	6.10**	7.25**	8.14**	6.87**
9	2.40	4.82	-	-	-	-	-	-	-	5.58**	6.84**	7.93**	6.86**
12	1.05	3.87	-	-	-	-	-	-	-	-	5.12	6.23	5.70
18	−2.21	1.43	-	-	-	-	-	-	-	-	-	2.74	2.74
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	4.56	4.69	4.96*	5.38*	5.86*	6.35*	6.83*	7.32*	8.26	8.18	-	-	8.19
1	4.71	4.90	-	5.12**	5.35**	5.56*	5.78*	5.99*	6.38	6.28	-	-	6.37
2	4.83	5.18	-	-	5.40**	5.63**	5.86**	6.09**	6.74**	7.29*	6.76	-	6.96**
3	5.03	5.51	-	-	-	5.72**	5.94**	6.15**	6.78**	7.20**	4.21	-	7.22**
4	5.21	5.80	-	-	-	-	5.99***	6.19***	6.78**	7.17**	5.10	-	7.40**
5	5.26	5.88	-	-	-	-	-	6.06***	6.62**	7.21**	8.52**	8.66	7.62**
6	5.21	5.89	-	-	-	-	-	-	6.31**	6.44	3.50	-	6.62
9	5.27	6.12	-	-	-	-	-	-	-	6.53**	7.38*	7.54	7.39*
12	5.51	5.38	-	-	-	-	-	-	-	-	2.16	-	0.05
18	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 4: **Out-of-sample prediction and allocation performance with T_N optimized**
We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$T_N=1$ year $RP_{t\to T_1,T_N}$	$T_N=2$ years $RP_{t\to T_1,T_N}$	Average across T_N $RP_{t\to T_1,T_N}$	T_N optimized $RP_{t\to T_1,T_N}$
Panel A: Out-of-sample R^2						
10d	−0.09	−0.07	0.04	−0.93	0.20	0.08
1	1.09	1.18	1.14	−0.37	1.39	1.73
2	1.34	1.59	2.16	1.32	2.41	3.84**
3	1.18	1.61	2.66	2.64	2.96	4.71***
4	2.16	2.86	4.09	4.72	4.41	5.47**
5	3.12	4.19	5.50	6.67	5.94*	6.44**
6	3.61	4.97	6.18*	7.76	6.93*	7.26**
9	4.32	6.37	7.01**	9.07*	8.32**	8.76**
12	4.00	6.54	-	8.78	8.32	8.44
18	2.29	6.17	-	7.66	7.66	7.66
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma=3$						
10d	4.56	4.69	8.18	13.90	8.44	5.81
1	3.55	3.68	4.16	0.48	4.80	3.52
2	3.69	3.96	5.55	2.92	5.70*	6.41
3	4.14	4.54	6.39*	5.63	6.64**	9.50***
4	4.27	4.75	6.21*	4.58	6.69**	8.46**
5	4.01	4.50	5.59*	5.60	6.16*	6.85
6	4.26	4.89	5.83*	6.78	6.78*	7.24
9	4.18	4.88	5.23*	6.53	6.15*	6.19
12	4.52	5.45	2.06	6.98**	6.72**	6.85**
18	4.59	5.62	2.14	6.11**	6.11**	6.11**

Table 5: **Out-of-sample crash prediction with T_N optimized**

We report the out-of-sample performance of different crash prediction methods. Each month, we use the crash probability from Martin (2017) ($\Pi^{Log}_{t\to T_1}[\alpha]$), the one from Chabi-Yo and Loudis (2020) ($\Pi_{t\to T_1}[\alpha]$ in Equation (19)), and the one from our methodology, $\Pi_{t\to T_1,T_N}[\alpha]$, defined in Equation (18) of Proposition 4. T_N is set equal to the implied investors' horizon $T^*_{N,t}$ at each time t. We compute the loss function for $\Pi_{t\to T_1,T_N}[\alpha]$ as $l_{t\to T_1,T_N} = -(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\log(\Pi_{t\to T_1,T_N}[\alpha]) + (1-\mathbb{1}_{R_{M,t\to T_1}<\alpha})(1-\log(\Pi_{t\to T_1,T_N}[\alpha])))$. Similarly, we compute a loss function for other methods. For each method in rows, we test whether the average loss functions are significantly larger than those of the method in columns using the Diebold and Mariano (1995) test. A significantly positive test statistic indicates that the column-method outperforms the row-method. We estimate the variance of the difference in loss functions using a Newey-West correction with 12 lags. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. We report on a 90% ($\alpha = 0.10$), and 80% ($\alpha = 0.20$) crash size. Data are from January 1996 to February 2023.

	10% crash		20% crash	
	$\Pi_{t\to T_1}[\alpha]$	$\Pi_{t\to T_1,T_N}[\alpha]$	$\Pi_{t\to T_1}[\alpha]$	$\Pi_{t\to T_1,T_N}[\alpha]$
Panel A: One week				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.56*	1.92**	1.29*	−0.92
$\Pi_{t\to T_1}[\alpha]$	-	2.06**	-	−0.92
Panel B: One month				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.76**	−0.97	5.71***	6.58***
$\Pi_{t\to T_1}[\alpha]$	-	−0.98	-	6.42***
Panel C: One quarter				
$\Pi^{Log}_{t\to T_1}[\alpha]$	4.42***	7.14***	2.67***	2.58***
$\Pi_{t\to T_1}[\alpha]$	-	6.75***	-	2.40***
Panel D: Six months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	3.91***	8.21***	3.36***	3.71***
$\Pi_{t\to T_1}[\alpha]$	-	10.54***	-	3.45***
Panel E: Nine months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.66***	5.10***	1.48*	2.18**
$\Pi_{t\to T_1}[\alpha]$	-	7.18***	-	2.36***
Panel F: One year				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.18**	2.79***	1.25	2.02**
$\Pi_{t\to T_1}[\alpha]$	-	3.34***	-	2.51***

Table 6: **Out-of-sample prediction and allocation performance with τ and ρ estimated**

We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In columns (4) and (5), they are kept constant over the time series of data, but the constants are estimated. In columns (6) and (7), they are modelled as linear functions of past 3-month returns. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ estimated constant		ρ, τ estimated linear in past returns	
	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel A: Out-of-sample R^2							
10d	−0.09	−0.07	0.08	−0.06	0.15	0.02	−0.03
1	1.09	1.18	1.73	1.23	1.11	1.49	1.56
2	1.34	1.59	3.84**	1.66	2.47	2.57	3.69*
3	1.18	1.61	4.71***	1.79	2.92*	4.66	5.80
4	2.16	2.86	5.47**	3.37	4.38	5.56	7.80**
5	3.12	4.19	6.44**	5.34	6.13	7.58	8.22
6	3.61	4.97	7.26**	6.64	7.40	8.61	9.00
9	4.32	6.37	8.76**	7.98	8.59*	9.79	10.15
12	4.00	6.54	8.44	7.68	8.68	9.36	10.10
18	2.29	6.17	7.66	8.06	9.44	9.33	11.16
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$							
10d	4.56	4.69	5.81	5.95	8.17	6.89	8.51
1	3.55	3.68	3.52	3.35	3.64	4.83	2.69
2	3.69	3.96	6.41	4.61	6.77*	5.76	5.01
3	4.14	4.54	9.50**	5.38	8.28**	7.42	8.74
4	4.27	4.75	8.46**	5.21	7.46*	7.38	8.12
5	4.01	4.50	6.85	4.76	6.11	5.24	1.62
6	4.26	4.89	7.24	4.80	4.02	0.28	3.88
9	4.18	4.88	6.19	5.04	5.71	3.13	4.51
12	4.52	5.45	6.85**	5.71	6.61**	5.81	3.80
18	4.59	5.62	6.11**	3.67	−0.47	2.56	1.18

Table 7: **Out-of-sample prediction and allocation performance with τ and ρ estimated, from 2006**

We report the out-of-sample performance of different risk premium prediction methods, from January 2006 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In columns (4) and (5), they are modelled constant and estimated on a telescopic window of time. In columns (6) and (7), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ estimated on telescopic window		ρ, τ estimated on rolling window	
	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel A: Out-of-sample R^2							
10d	-0.10	-0.08	0.07	0.07	0.01	-	-
1	0.74	0.87	1.97	-	-	-	-
2	1.03	1.41	4.55**	-	-	-	0.14*
3	0.29	0.97	5.16***	-	-	-	-
4	1.43	2.57	6.14**	-	-	-	0.65
5	2.65	4.35	7.39**	3.04	4.98	-0.52	4.67
6	2.95	5.13	8.29**	7.46	9.12*	5.62	5.98
9	3.11	6.55	10.01**	11.46	12.29	-	-
12	2.29	6.71	9.83	10.48	11.21	1.09	2.92
18	-0.67	6.08	8.44	10.72	12.38	17.97	19.67
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$							
10d	4.54	4.67	5.83	4.95	7.44	-2.79	-15.57
1	4.08	4.29	4.89	-3.04	-1.87	-17.92	-12.85
2	4.10	4.45	9.71*	2.91	-0.68	-13.41	-7.30**
3	4.71	5.28	11.89**	2.45	9.03***	-7.66	0.03**
4	4.38	4.99	8.42	4.44	7.02	-4.76	-2.33
5	4.96	5.76	8.92	2.27	7.55	0.99	5.45*
6	4.77	5.69	8.73*	5.12	4.60	-0.13	4.65
9	5.01	6.21	6.85	8.13	7.98	-19.57	-23.97
12	5.19	6.68	8.45	-1.94	1.18	-16.32	-12.62
18	5.31	7.41	8.08**	4.40	1.19	-0.75	-4.24

Table 8: **Out-of-sample prediction and allocation performance with τ and ρ estimated, from 2009**

We report the out-of-sample performance of different risk premium prediction methods, from January 2009 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In columns (4) and (5), they are modelled constant and estimated on a telescopic window of time. In columns (6) and (7), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ estimated on telescopic window		ρ, τ estimated on rolling window	
	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel A: Out-of-sample R^2							
10d	0.38	0.45	0.86	0.61	0.79	-	-
1	2.12	2.58	4.65*	4.57	4.95	2.57	3.58
2	4.07	5.30	10.26***	9.63	10.94	6.85	11.22
3	4.91	7.36	14.00***	15.13	17.16*	9.19	-
4	6.04	9.55	15.50**	19.71	20.88	18.95	18.10
5	6.42	10.72	16.07**	21.99	23.23	21.88	20.65
6	5.36	10.43	16.63***	22.56	24.52	25.60	21.27
9	2.98	10.32	17.03***	21.80	23.44*	17.06	15.31
12	-	8.33	15.20	19.28	21.35	17.47	21.16
18	-	1.88	6.73	12.49	16.27	20.82	23.77
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$							
10d	4.87	5.04	6.85	5.34	8.58	−3.03	−16.35
1	5.11	5.48	7.67	1.94	3.08	−9.22	−8.22
2	5.34	5.99	13.18*	9.96	6.25	−4.99	0.08**
3	6.14	7.13	15.56***	7.39	15.84***	2.63	5.84
4	5.71	6.80	11.45	9.51	13.29	1.55	5.39*
5	6.09	7.43	11.94*	11.20	14.59	5.26	13.43*
6	6.07	7.59	11.99*	9.76	10.83	5.65	13.86**
9	6.23	8.18	10.77	12.39	12.90	5.57	4.36
12	6.31	8.79	12.11**	1.84	7.08**	3.24	7.08*
18	6.14	9.47	11.51***	11.08	9.40	0.50	4.36**

Table 9: **Out-of-sample prediction and allocation performance with $\tau = 1$ and $\rho = 2$, with rebalancing**

We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023, setting the preference parameters to their default values $\tau = 1$ and $\rho = 2$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In columns (4) and (5), they are modelled constant and estimated on a telescopic window of time. In columns (6) and (7), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		No rebalancing		With rebalancing	
	$RP^{Log}_{t\to T_1}$ (1)	$RP_{t\to T_1}$ (2)	$RP_{t\to T_1,T^*_N}$ (3)	$RP_{t\to T_1}$ (4)	$RP_{t\to T_1,T^*_N}$ (5)
Panel A: Out-of-sample R^2					
10d	−0.09	−0.07	0.06	−0.07	0.16
1	1.09	1.18	1.73	1.18	1.65
2	1.34	1.59	3.84**	1.59	3.16
3	1.18	1.61	4.71***	1.61	3.76
4	2.16	2.86	5.47**	2.86	4.81
5	3.12	4.19	6.45**	4.19	5.94
6	3.61	4.97	7.26**	4.97	7.00
9	4.32	6.37	8.76**	6.37	8.75
12	4.00	6.54	8.44	6.54	8.89
18	2.29	6.17	7.66	6.17	7.66
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	4.56	4.69	5.75	4.69	5.34
1	3.55	3.68	3.52	3.68	2.78
2	3.69	3.96	6.40	3.96	6.51
3	4.14	4.54	9.50***	4.54	8.48
4	4.27	4.75	8.46**	4.75	7.96
5	4.01	4.50	6.85	4.50	6.69
6	4.26	4.89	7.24	4.89	7.23
9	4.18	4.88	6.19	4.88	6.18
12	4.52	5.45	6.85**	5.45	6.98
18	4.59	5.62	6.11**	5.62	6.11

Table 10: **Out-of-sample prediction and allocation performance of the third-order bound with $\tau = 1$ and $\rho = 2$**

We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023, setting the preference parameters to their default values $\tau = 1$ and $\rho = 2$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In columns (4) and (5), they are modelled constant and estimated on a telescopic window of time. In columns (6) and (7), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		No rebalancing		With rebalancing	
	$RP^{Log}_{t\to T_1}$ (1)	$RP_{t\to T_1}$ (2)	$RP_{t\to T_1,T^*_N}$ (3)	$RP_{t\to T_1}$ (4)	$RP_{t\to T_1,T^*_N}$ (5)
Panel A: Out-of-sample R^2					
10d	−0.10	−0.08	0.06	−0.08	-
1	0.74	0.87	1.97	0.91	-
2	1.03	1.41	4.56**	1.44	-
3	0.29	0.97	5.16***	0.92	-
4	1.43	2.57	6.14**	2.80	-
5	2.65	4.35	7.40**	5.14	-
6	2.95	5.13	8.29**	6.41	-
9	3.11	6.55	10.01**	9.23	-
12	2.29	6.71	9.83	10.51	-
18	−0.67	6.08	8.44	11.55	-
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	4.54	4.67	5.77	4.68	3.81
1	4.08	4.29	4.89	4.35	−0.99
2	4.10	4.45	9.72*	4.56	2.75
3	4.71	5.28	11.90***	5.48	2.29
4	4.38	4.99	8.42	5.23	1.15
5	4.96	5.76	8.92	6.08	1.21
6	4.77	5.69	8.73*	6.17	2.56
9	5.01	6.21	6.85	6.81	1.32
12	5.19	6.68	8.45	7.41	0.57
18	5.31	7.41	8.08**	4.59*	−12.76

A Proofs and derivations

This section contains the proofs and derivations of the main results presented in Section 2.

A.1 Proof of Equation (2)

Problem (1) can alternatively be written as

$$\max_{\omega_t} \mathbb{E}_t \left(\max_{\omega_{T_1}} \mathbb{E}_{T_1} \left(u \left[W_{t \to T_N} \right] \right) \right). \tag{A1}$$

Solving backward Problem (A1), the first step of (A1) is

$$\max_{\omega_{T_1}} \mathbb{E}_{T_1} \left(u \left[W_{t \to T_N} \right] \right). \tag{A2}$$

Equation (A2) produces an optimal weight $\omega^*_{T_1}$, and the corresponding SDF has the form

$$m_{T_1 \to T_N} = \delta_{T_1} u^{'} \left[W_{t \to T_1} \left(\omega^{*\intercal}_{T_1} R_{T_1 \to T_N} \right) \right]. \tag{A3}$$

Given the optimal value, $\omega^*_{T_1}$, the second step solves

$$\max_{\omega_t} \mathbb{E}_t \left(\mathbb{E}_{T_1} \left(u \left[W_{t \to T_N} \right] \right) \right) \tag{A4}$$

with $W_{t \to T_N} = W_{t \to T_1} \left(\omega^{*\intercal}_{T_1} R_{T_1 \to T_N} \right)$. This produces a SDF of the form

$$m_{t \to T_1} = \delta_t \mathbb{E}_{T_1} \left(u^{'} \left[W_{t \to T_N} \right] \left(\omega^{*\intercal}_{T_1} R_{T_1 \to T_N} \right) \right). \tag{A5}$$

From (A5), the constant δ_t can alternatively be written as

$$\delta_t = m_{t \to T_1} \left(\mathbb{E}_{T_1} \left(u^{'} \left[W_{t \to T_2} \right] \left(\omega^{*\intercal}_{T_1} R_{T_1 \to T_N} \right) \right) \right)^{-1}. \tag{A6}$$

Because parameter δ_t is a constant, we have $\delta_t = \mathbb{E}_t \delta_t$. Then, we exploit the no-arbitrage conditions that allows us to move from the physical measure to the risk-neutral measure to obtain,

$$\begin{aligned}
\delta_t &= \mathbb{E}_t \left(m_{t\to T_1} \left(\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \right)^{-1} \right) \\
&= \mathbb{E}_t \left(m_{t\to T_1} \right) \mathbb{E}_t \left(\frac{m_{t\to T_1}}{\mathbb{E}_t \left(m_{t\to T_1} \right)} \left(\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \right)^{-1} \right) \\
&= \mathbb{E}_t \left(m_{t\to T_1} \right) \mathbb{E}_t^{*} \left(\left(\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \right)^{-1} \right). \qquad \text{(A7)}
\end{aligned}$$

Next, we replace δ_t by its expression in (A5) and show that

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{1/\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right)}{\mathbb{E}_t^{*} \left(1/\mathbb{E}_{T_1} \left(u' \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \right)}. \qquad \text{(A8)}$$

Similarly, we can use the SDF (A3) and show that

$$\frac{\mathbb{E}_{T_1} m_{T_1\to T_N}}{m_{T_1\to T_N}} = \frac{1/u^{'} \left[W_{t\to T_N} \right]}{\mathbb{E}_{T_1}^{*} \left(1/u' \left[W_{t\to T_N} \right] \right)}. \qquad \text{(A9)}$$

Our next goal is to write $\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right)$ in terms of risk-neutral quantities. Note that:

$$\begin{aligned}
\mathbb{E}_{T_1} \left(u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) &= \mathbb{E}_{T_1} \left(\frac{m_{T_1\to T_N}}{\mathbb{E}_{T_1} m_{T_1\to T_N}} \frac{\mathbb{E}_{T_1} m_{T_1\to T_N}}{m_{T_1\to T_N}} u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \\
&= \mathbb{E}_{T_1}^{*} \left(\frac{\mathbb{E}_{T_1} m_{T_1\to T_N}}{m_{T_1\to T_N}} u^{'} \left[W_{t\to T_N} \right] \left(\omega_{T_1}^{*\intercal} R_{T_1\to T_N} \right) \right) \\
&= \frac{\omega_{T_1}^{*\intercal} \mathbb{E}_{T_1}^{*} R_{T_1\to T_N}}{\mathbb{E}_{T_1}^{*} \left(1/u' \left[W_{t\to T_N} \right] \right)}, \\
&= \frac{R_{f,T_1\to T_N}}{\mathbb{E}_{T_1}^{*} \left(1/u' \left[W_{t\to T_N} \right] \right)}, \qquad \text{(A10)}
\end{aligned}$$

where we have used the no-arbitrage conditions to move from the physical measure to the risk-neutral measure in the second equation, and Equation (A9) to obtain the third equation.

Finally, we replace (A10) in (A8) and obtain

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u'\left[W_{t\to T_N}\right]}\right)}{R_{f,T_1\to T_N}}}{\mathbb{E}^*_t\left(\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u'\left[W_{t\to T_N}\right]}\right)}{R_{f,T_1\to T_N}}\right)}$$

$$= \frac{\left(\left(1/R_{f,T_1\to T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right)}{\mathbb{E}^*_t\left(\left(\left(1/R_{f,T_1\to T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right)\right)}.$$

Since there is no interest rate risk, $1/R_{f,T_1\to T_N} = \mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)$, this last expression simplifies to

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right)\right)}. \tag{A11}$$

This ends the proof.

A.2 Proof of Equation (3)

We first use the identity

$$\mathbb{E}_t\left(R_{k,t\to T_1}\right) = \mathbb{E}_t\left(R_{k,t\to T_1}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\right). \tag{A12}$$

Hence, and

$$\mathbb{E}_t R_{t\to T_1} = \mathbb{COV}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}, R_{t\to T_1}\right) + \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right)\mathbb{E}^*_t\left(R_{t\to T_1}\right).$$

Notice that $\mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right) = 1$. This ends the proof.

A.3 Proof of Equation (4)

Assume that the gross return on the market can be used as the proxy for the return on the aggregate wealth, $R_{M,t\to T_N} = \frac{W_{t\to T_N}}{W_t}$ and $R_{M,T_1\to T_N} = \frac{W_{T_1\to T_N}}{W_{t\to T_1}}$ and adopt the following notations $x = R_{M,t\to T_1}$, $x_0 = R_{f,t\to T_1}$, $y = R_{M,T_1\to T_N}$, and $y_0 = R_{f,t\to T_N}/R_{f,t\to T_1} = R_{f,T_1\to T_N}$.

The inverse of the SDF (adjusted by its mean) is

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}\left(f\left[x,y\right]\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(f\left[x,y\right]\right)\right)}, \tag{A13}$$

where $m_{T_1\to T_N}$ is defined in Equation (A11) and,

$$f\left[x,y\right] = \frac{u'\left[W_t x_0 y_0\right]}{u'\left[W_t x y\right]}.$$

We adopt the following short notations. First, we use f_x and f_y to denote the first partial derivative, f_{xx} and f_{yy} the second partial derivatives, and f_{xy} the cross-derivative all evaluated at (x_0, y_0). Second, we denote as u', u'', and u''' the first, second, and third derivatives of $u\left[\cdot\right]$ evaluated at (x_0, y_0). We perform a second-order Taylor expansion series of $f\left[x,y\right]$ around $(x,y) = (x_0, y_0)$ as,

$$\begin{aligned} f\left[x,y\right] \approx\ & 1 + \frac{1}{1!}\left(x-x_0\right) f_x + \frac{1}{1!}\left(y-y_0\right) f_y + \frac{1}{2!}\left(x-x_0\right)^2 f_{xx} \\ & + \frac{1}{2!}\left(y-y_0\right)^2 f_{yy} + \frac{2}{2!}\left(x-x_0\right)\left(y-y_0\right) f_{xy}, \end{aligned}$$

where:

$$
\begin{aligned}
f_x &= \frac{y_0}{x_0} f_y = \frac{1}{x_0}\left(-\frac{(W_t x_0 y_0)\, u^{''}}{u^{'}}\right), \\
f_{xy} &= \frac{1}{x_0 y_0}\left(-\frac{W_t x_0 y_0 u^{''}}{u^{'}}\right) + \frac{1}{x_0 y_0}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{\left(u^{'}\right)^2}\left(2 - \frac{u^{'''} u^{'}}{\left(u^{''}\right)^2}\right), \\
f_{xx} &= \frac{y_0^2}{x_0^2} f_{yy} = \frac{1}{\left(x_0\right)^2}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{\left(u^{'}\right)^2}\left(2 - \frac{u^{'''} u^{'}}{\left(u^{''}\right)^2}\right).
\end{aligned}
$$

Note that $f_{xy} = f_{yx}$. Thus, we obtain,

$$
\begin{aligned}
f[x,y] \approx\ & 1 + \frac{1}{x_0}\frac{1}{\tau_t}\left(x - x_0\right) + \frac{1}{y_0}\frac{1}{\tau_t}\left(y - y_0\right) \\
& + \frac{1}{\left(x_0\right)^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(x - x_0\right)^2 + \frac{1}{\left(y_0\right)^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(y - y_0\right)^2 \\
& + \frac{1}{x_0 y_0}\left(\frac{1}{\tau_t} + \frac{2\left(1-\rho_t\right)}{\tau_t^2}\right)\left(x - x_0\right)\left(y - y_0\right), \qquad \text{(A14)}
\end{aligned}
$$

where τ_t and ρ_t are defined in Equation (7). Replacing x, x_0, y, and y_0 by their expressions and using preference parameters a_1 and a_2 defined in Equation (6), we obtain,

$$
\begin{aligned}
\mathbb{E}^*_{T_1}\left(f[x,y]\right) =\ & 1 + \frac{a_{1,t}}{R_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) + \frac{a_{1,t}}{R_{f,T_1\to T_N}}\left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right) \\
& + \frac{a_{2,t}}{\left(R_{f,t\to T_1}\right)^2}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{\left(R_{f,T_1\to T_N}\right)^2}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right) \\
& + \frac{a_{1,t} + 2a_{2,t}}{R_{f,t\to T_2}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right). \qquad \text{(A15)}
\end{aligned}
$$

Thus, $\mathbb{E}^*_{T_1} f[x,y]$ simplifies to

$$
\mathbb{E}^*_{T_1} f[x,y] = \mathbb{E}^*_{T_1}\left(\frac{u^{'}\left[W_t R_{f,t\to T_1} R_{f,T_1\to T_N}\right]}{u^{'}\left[W_t R_{M,t\to T_1} R_{M,T_1\to T_N}\right]}\right) = 1 + z_{T_1}, \qquad \text{(A16)}
$$

where

$$z_{T_1} = \frac{a_{1,t}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)}{R_{f,t\to T_1}} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^{*(2)}_{T_1\to T_N} \quad \text{(A17)}$$

We then replace Equation ((A16)) in (A13). This ends the proof.

A.4 Proof of Proposition 1

Given (A16), We replace (A13) in the expected return expression identity (3) and obtain.

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \frac{1+z_{T_1}}{1+\mathbb{E}^*_t z_{T_1}}\right).$$

We then replace (A17) in this expression, expand the covariance term and obtain the desired result. This ends the proof.

A.5 Proof of Corollary 2

The expected excess return can be decomposed into

$$\begin{aligned}\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) &= \frac{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\left(\frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}\right)\\ &+\frac{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\left(\frac{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\right).\end{aligned}$$

Setting

$$\pi^*_t = \frac{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{t\to T_1}}$$

ends the proof.

A.6 Proof of Proposition 4

Under no-arbitrage conditions, we use the Radon-Nikodym theorem. It allows us to move from the physical to the risk neutral measures and express the conditional crash probability as

$$\begin{aligned}\mathbb{P}_t\left(R_{M,t\to T_1}<\alpha\right) &= \mathbb{E}_t\left(\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)\\ &= \mathbb{E}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right).\end{aligned} \tag{A18}$$

We then replace the inverse of the SDF by Equation (4) in the conditional crash probability to obtain,

$$\mathbb{P}_t\left(R_{M,t\to T_1}<\alpha\right) = \frac{\mathbb{E}_t^*\left(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)+\mathbb{E}_t^*\left(z_{T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)+\mathbb{E}_t^*\left(z_{T_1}^{\upsilon}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{A19}$$

where

$$\begin{aligned}\mathbb{E}_t^*\left(z_{T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) &= \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)\\ &\quad+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right),\\ \mathbb{E}_t^*\left(z_{T_1}^{\upsilon}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) &= \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}_t^*\left(\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right).\end{aligned} \tag{A20}$$

B Estimation of moments

We provide closed-form solutions to the risk-neutral and physical moments used in our analysis. In many cases, we use the spanning formula of Carr and Madan (2001) and Bakshi and Madan (2000) to evaluate the risk-neutral expected value of a twice-differentiable function of the underlying asset price, $H(S_{T_1})$ as

$$\begin{aligned} \mathbb{E}_t^* H[S_{T_1}] &= H[S_t R_{f,t\to T_1}] + \mathbb{E}_t^* H_S[S_t R_{f,t\to T_1}] S_t (R_{M,t\to T_1} - R_{f,t\to T_1}) \\ &\quad + R_{f,t\to T_1}\left[\int_{S_t R_{f,t\to T_1}}^{\infty} H_{SS}[K] C_t[K] dK + \int_0^{S_t R_{f,t\to T_1}} H_{SS}[K] P_t[K] dK\right], \end{aligned} \tag{B21}$$

where H_S and H_{SS} are the first and second derivative of function $H(\cdot)$, respectively. We evaluate the integral terms via numerical integration using the 1,000-point moneyness grid described in Section 3.2.

B.1 Closed-form expressions for $\mathbb{M}_{t\to T_j}^{*(k)}$ and $\mathbb{E}_t^*\left(R_{M,t\to T_j}^k\right)$

To evaluate the risk-neutral moments of order k, $\mathbb{M}_{t\to T_j}^{*(k)}$ and $\mathbb{E}_t^*\left(R_{M,t\to T_j}^k\right)$, we set $H\left(S_{T_j}\right) = \left(\frac{S_{T_j}}{S_t} - R_{f,t\to T_j}\right)^k$ and $H\left(S_{T_j}\right) = \left(\frac{S_{T_j}}{S_t}\right)^k$ in Equation (B21), respectively. Then, we use options with maturity T_j to evaluate Equation (B21).

B.2 Physical variance

In this section, we provide expressions for the option-implied physical variance

$$\mathbb{E}_t\left(R_{M,t\to T_1} - \mathbb{E}_t R_{M,t\to T_1}\right)^2 = \mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 - \left(\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right)^2.$$

We already have an expression for $\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)$. Note that

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \mathbb{E}_t^*\left\{\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right\}.$$

Using the second-order approximation in Equation (4), we obtain

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \frac{\left\{\begin{array}{c}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(4)}_{t\to T_1} \\ +\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\left(\mathbb{LEK}^*_t + \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}. \tag{B22}$$

B.3 Closed-form expression of $\mathbb{LEK}^*_t$

Notice that

$$\begin{aligned}
\mathbb{LEK}^*_t &= \mathbb{COV}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2, \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right) \\
&= \mathbb{E}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right) \\
&\quad -\mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 \\
&= \theta_t\mathbb{VAR}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right)
\end{aligned}$$

because

$$\mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 = \theta_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2$$

Hence

$$\mathbb{LEK}^*_t = \theta_t\left(\mathbb{M}^{*(4)}_{t\to T_1} - \left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2\right)$$

B.4 Closed-form expression of $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}$ and $\mathbb{LES}_t^*$

We can write $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}$ and $\mathbb{LES}_t^*$ respectively as,

$$\begin{aligned}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)} &= \mathbb{E}_t^*\left((R_{M,T_1\to T_N}-R_{f,T_1\to T_N})^3\right)\\ &= \mathbb{E}_t^*\left(R_{M,T_1\to T_N}^3\right)-R_{f,T_1\to T_N}^3-3R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)},\end{aligned}\tag{B23}$$

and

$$\begin{aligned}\mathbb{LES}_t^* &= \mathbb{COV}_t^*\left(r_{M,t\to T_1},\mathbb{M}_{T_1\to T_N}^{*(3)}\right)\\ &= \mathbb{E}_t^*\left(r_{M,t\to T_1}\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^3\right)\\ &= -R_{f,t\to T_1}R_{f,T_1\to T_N}^3-R_{f,t\to T_1}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}-3R_{f,t\to T_1}R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}\\ &\quad -3R_{f,T_1\to T_N}\mathbb{LEV}_t^*+\mathbb{E}_t^*\left(R_{M,t\to T_1}R_{M,T_1\to T_N}^3\right).\end{aligned}\tag{B24}$$

To obtain $\mathbb{LES}_t^*$, we need to evaluate the terms $\mathbb{E}_t^*\left(R_{M,t\to T_1}R_{M,T_1\to T_N}^3\right)$ and $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}$ (The terms $\mathbb{M}_{T_1\to T_N}^{*(2)}$ and $\mathbb{LEV}_t^*$ have been derived in the main text). To do so, we assume that the term $\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N}^3\right)-R_{f,T_1\to T_N}^3$ is a nonlinear function of a function g of $R_{M,t\to T_1}-R_{f,t\to T_1}$ as

$$\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N}^3\right)-R_{f,T_1\to T_N}^3=\gamma_t g(R_{M,t\to T_1}-R_{f,t\to T_1})+\upsilon_t,\tag{B25}$$

with $\mathbb{E}_t^*\left(\upsilon_t|R_{M,t\to T_1}\right)=\mathbb{E}_t^*\left(\upsilon_t\right)=0$. Multiplying both sides of Equation (B25) by $R_{M,t\to T_1}^3$ and taking the time-t risk-neutral expectation, we obtain,

$$\gamma_t=\frac{\mathbb{M}_{t\to T_2}^{*(3)}+3R_{f,t\to T_2}\mathbb{M}_{t\to T_2}^{*(2)}-R_{f,T_1\to T_N}^3\left(\mathbb{M}_{t\to T_1}^{*(3)}+3R_{f,t\to T_1}\mathbb{M}_{t\to T_1}^{*(2)}\right)}{\mathbb{E}_t^*\left(R_{M,t\to T_1}^3 g(R_{M,t\to T_1}-R_{f,t\to T_1})\right)}.\tag{B26}$$

If we use $g(R_{M,t\to T_1}-R_{f,t\to T_1})=R_{M,t\to T_1}^3$, we obtain

$$\gamma_t=\frac{\mathbb{M}_{t\to T_2}^{*(3)}+3R_{f,t\to T_2}\mathbb{M}_{t\to T_2}^{*(2)}-R_{f,T_1\to T_N}^3\left(\mathbb{M}_{t\to T_1}^{*(3)}+3R_{f,t\to T_1}\mathbb{M}_{t\to T_1}^{*(2)}\right)}{\mathbb{E}_t^*\left(R_{M,t\to T_1}^6\right)},\tag{B27}$$

Taking the expectation of (B25) under the risk neutral measure,

$$\mathbb{E}_t^*\left(R_{M,T_1\to T_N}^3\right) - R_{f,T_1\to T_N}^3 = \gamma_t \mathbb{E}_t^*\left(R_{M,t\to T_1}^3\right), \tag{B28}$$

Multiplying both sides of Equation (B25) by $R_{M,t\to T_1}$ and taking the time-t risk-neutral expectation

$$\mathbb{E}_t^*\left(R_{M,t\to T_1} R_{M,T_1\to T_N}^3\right) = R_{f,t\to T_1} R_{f,T_1\to T_N}^3 + \gamma_t \mathbb{E}_t^*\left(R_{M,t\to T_1}^4\right). \tag{B29}$$

Therefore, using Equations (B23) and (B24) we obtain $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}$ and $\mathbb{LES}_t^*$ as,

$$\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)} = \gamma_t \mathbb{E}_t^*\left(R_{M,t\to T_1}^3\right) - 3R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)},$$

and

$$\mathbb{LES}_t^* \;=\; \gamma_t \mathbb{E}_t^*\left(R_{M,t\to T_1}^4\right) - R_{f,t\to T_1}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)} - 3R_{f,t\to T_1}R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)} - 3R_{f,T_1\to T_N}\mathbb{LEV}_t^*$$

To compute the physical variance, we also need the following moments which we obtain using a similar approach:

$$\mathbb{E}_t^*\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 = \mathbb{E}_t^*\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2$$

Using expression (20)

$$\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 = \theta_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \epsilon_{T_1},$$

it follows that

$$\mathbb{E}_t^*\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 = \theta_t\mathbb{E}_t^*\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^5$$

Further,

$$\begin{aligned}
& \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3 \\
= \ & \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R^3_{M,T_1\to T_N} \\
& -R^3_{f,T_1\to T_N} \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \\
& +3R^2_{f,T_1\to T_N} \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R_{M,T_1\to T_N} \\
& -3R_{f,T_1\to T_N} \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R^2_{M,T_1\to T_N}
\end{aligned}$$

which simplifies to

$$\begin{aligned}
& \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3 \\
= \ & \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{E}^*_{T_1} R^3_{M,T_1\to T_N} \\
& +R^3_{f,T_1\to T_N} \mathbb{M}^{*(2)}_{t\to T_1} \\
& -3R_{f,T_1\to T_N} \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}
\end{aligned}$$

Since

$$\mathbb{E}^*_{T_1} R^3_{M,T_1\to T_N} = \gamma_t R^3_{M,t\to T_1}$$

It follows that

$$\begin{aligned}
& \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3 \\
= \ & \gamma_t \mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R^3_{M,t\to T_1}\right) \\
& +R^3_{f,T_1\to T_N} \mathbb{M}^{*(2)}_{t\to T_1} \\
& -3R_{f,T_1\to T_N} \mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}\right)
\end{aligned}$$

where expression $\mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}\right)$ can be derived as follows:

$$\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N} = \theta_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^4 + \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \varepsilon_{T_1}$$

and

$$\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\mathbb{M}^{*(2)}_{T_1\to T_N}\right) = \theta_t\mathbb{E}_t^*\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^4+\mathbb{E}_t^*\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\varepsilon_{T_1}$$
$$= \theta_t\mathbb{E}_t^*\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^4$$

B.5 Closed-form expression of $\mathbb{M}^{*(k)}_{t\to T_1}[\alpha]$

Recall that $\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] = \mathbb{E}_t^*\left\{\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^k \mathbb{1}_{S_{T_1}<\alpha S_t}\right\}$. Therefore, we set $H[x] = \left(\frac{x}{S_t}-R_{f,t\to T_1}\right)^k$ in Equation (B21) and obtain,

$$\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] = H[\alpha S_t]\,\mathbb{P}_t^*[S_{T_1}<\alpha S_t]-H_S[\alpha S_t]\,R_{f,t\to T_1}P_t[\alpha S_t]+R_{f,t\to T_1}\int_0^{\alpha S_t}H_{SS}[K]\,P_t[K]\,dK.$$

B.6 Closed-form expression of $\mathbb{E}_t^*\left(r^j_{M,t\to T_1}\mathbb{M}^{*(k)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)$

We use Equation (22) to obtain the required expressions when $k=2$. First, we have

$$\mathbb{M}^*_{t,\upsilon}[\alpha]\equiv\mathbb{E}_t^*\left(\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) = \theta_t\mathbb{M}^{*(2)}_{t\to T_1}[\alpha], \tag{B30}$$

and

$$\mathbb{M}^*_{t,s\upsilon}[\alpha]\equiv\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) = \theta_t\mathbb{M}^{*(3)}_{t\to T_1}[\alpha]. \tag{B31}$$

Next, we can write the future third central moment as,

$$\mathbb{M}^{*(3)}_{T_1\to T_N} = \mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N}\right)-R^3_{f,T_1\to T_N}-3R_{f,T_1\to T_N}\mathbb{E}^*_{T_1}\left(R^2_{M,T_1\to T_N}\right)+3R^3_{f,T_1\to T_N} \tag{B32}$$

Multiplying this expression by $\mathbb{1}_{R_{M,t\to T_1}<\alpha}$ and computing the time-t risk-neutral expectation, we obtain,

$$\begin{aligned}
\mathbb{M}^*_{t,s}[\alpha] \equiv \mathbb{E}^*_t\left(\mathbb{M}^{*(3)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) &= \gamma_t\mathbb{E}^*_t\left(R^3_{M,t\to T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) \\
&\quad -3R_{f,T_1\to T_N}\mathbb{E}^*_t\left(\left(\mathbb{M}^{*(2)}_{T_1\to T_N}+R^2_{f,T_1\to T_N}\right)\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) \\
&\quad +3R^3_{f,T_1\to T_N}\mathbb{E}^*_t\left(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right), \\
&= \gamma_t\Big(\mathbb{M}^{*(3)}_{t\to T_1}[\alpha]+3R_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha]+3R^2_{f,t\to T_1}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] \\
&\quad +R^3_{f,t\to T_1}\mathbb{M}^{*(0)}_{t\to T_1}[\alpha]\Big)-3R_{f,T_1\to T_N}\mathbb{M}^*_{t,\upsilon}[\alpha],
\end{aligned}$$

where the second equality is obtained using Equation (B25).

B.7 Closed-form expressions for $\mathbb{N}^{*(n,j)}_{t\to T_1}[k_{0,t}]$ and $\mathbb{N}^{*(n,j)}_{\upsilon,t\to T_1}[k_{0,t}]$

Observe that

$$\begin{aligned}
\mathbb{N}^{*(n,c)}_{t\to T_1}[k_{0,t}] &= \mathbb{E}^*_t\left\{(R_{M,t\to T_1}-R_{f,t\to T_1})^n G^{(c)}\right\} \\
&= \mathbb{E}^*_t\left\{\left(\frac{S_{T_1}}{S_t}-R_{f,t\to T_1}\right)^n\left(\frac{S_{T_1}}{S_t}-\frac{K_0}{S_t}\right)1_{S_{T_1}>K_0}\right\} \\
&= \mathbb{E}^*_t\left\{\mathcal{H}^{(c)}[S_{T_1}]\,1_{S_{T_1}>K_0}\right\}
\end{aligned}$$

where

$$\mathcal{H}^{(c)}[S_{T_1}] = \left(\frac{S_{T_1}}{S_t}-R_{f,t\to T_1}\right)^n\left(\frac{S_{T_1}}{S_t}-\frac{K_0}{S_t}\right)$$

The Spanning formula allows us to write for $j\in\{c,p\}$

$$\begin{aligned}
\mathcal{H}^{(j)}[S_{T_1}] &= \mathcal{H}^{(j)}[K_0]+\mathcal{H}^{(j)}_S[K_0](S_{T_1}-K_0)+\int_{K_0}^{\infty}\mathcal{H}^{(j)}_{SS}[K](S_{T_1}-K)^+dK \\
&\quad +\int_0^{K_0}\mathcal{H}^{(j)}_{SS}[K](K-S_{T_1})^+dK.
\end{aligned} \tag{B33}$$

where $\mathcal{H}_S^{(c)}[x]$ is the first derivative of $\mathcal{H}^{(c)}[x]$ and $\mathcal{H}_{SS}^{(c)}[x]$ is the second derivative of $\mathcal{H}^{(c)}[x]$. Multiplying this expression by $1_{S_{T_1}>K_0}$ produces

$$\begin{aligned}\mathcal{H}^{(c)}[S_{T_1}]1_{S_{T_1}>K_0} &= \mathcal{H}^{(c)}[K_0]1_{S_{T_1}>K_0}+\mathcal{H}_S^{(c)}[K_0](S_{T_1}-K_0)^+\\ &\quad+\int_{K_0}^{\infty}\mathcal{H}_{SS}^{(c)}[K](S_{T_1}-K)^+dK.\end{aligned}$$

We then take the expected value of this expression and obtain

$$\begin{aligned}\mathbb{N}_{t\to T_1}^{*(n,c)}[k_{0,t}] &= \mathbb{E}_t^*\left\{\mathcal{H}^{(c)}[S_{T_1}]1_{S_{T_1}>K_0}\right\}\\ &= \mathcal{H}^{(c)}[K_0]\mathbb{E}_t^*1_{S_{T_1}>K_0}+\mathcal{H}_S^{(c)}[K_0]\mathbb{E}_t^*(S_{T_1}-K_0)^+\\ &\quad+\int_{K_0}^{\infty}\mathcal{H}_{SS}^{(c)}[K]\mathbb{E}_t^*(S_{T_1}-K)^+dK\end{aligned}$$

which simplifies to

$$\begin{aligned}\mathbb{N}_{t\to T_1}^{*(n,c)}[k_{0,t}] &= \mathcal{H}^{(c)}[K_0]\mathbb{P}_t^*[S_{T_1}>K_0]+R_{f,t\to T_1}\mathcal{H}_S^{(c)}[K_0]C_t[K_0]\\ &\quad+R_{f,t\to T_1}\int_{K_0}^{\infty}\mathcal{H}_{SS}^{(c)}[K]C_t[K]dK.\end{aligned}\tag{B34}$$

Next, multiplying (B33) by $1_{S_{T_1}<S_t}$ and using $j=p$ (focusing on put) produces

$$\begin{aligned}\mathcal{H}^{(p)}[S_{T_1}]1_{S_{T_1}<K_0} &= \mathcal{H}^{(p)}[K_0]1_{S_{T_1}<K_0}-\mathcal{H}_S^{(p)}[K_0](K_0-S_{T_1})^+\\ &\quad+\int_0^{K_0}\mathcal{H}_{SS}^{(p)}[K](K-S_{T_1})^+dK.\end{aligned}$$

We then take the expected value of this expression and obtain

$$\begin{aligned}\mathbb{N}_{t\to T_1}^{*(n,p)}[k_{0,t}] &= \mathbb{E}_t^*\mathcal{H}^{(p)}[S_{T_1}]1_{S_{T_1}<K_0}\\ &= \mathcal{H}^{(p)}[K_0]\mathbb{E}_t^*1_{S_{T_1}<K_0}-\mathcal{H}_S^{(p)}[K_0]\mathbb{E}_t^*(K_0-S_{T_1})^+\\ &\quad+\int_0^{K_0}\mathcal{H}_{SS}^{(p)}[K]\mathbb{E}_t^*(K-S_{T_1})^+dK\end{aligned}\tag{B35}$$

which simplifies to

$$
\begin{aligned}
\mathbb{N}^{*(n,p)}_{t\to T_1}[k_{0,t}] &= \mathbb{E}^*_t \mathcal{H}^{(p)}[S_{T_1}] 1_{S_{T_1}<K_0} \\
&= \mathcal{H}^{(p)}[K_0]\mathbb{P}_t[S_{T_1}<K_0] - R_{f,t\to T_1}\mathcal{H}^{(p)}_S[K_0]P_t[K_0] \\
&\quad + R_{f,t\to T_1}\int_0^{K_0}\mathcal{H}^{(p)}_{SS}[K]P_t[K]\,dK.
\end{aligned}
$$

Next, let's compute $\mathbb{N}^{*(n,j)}_{\upsilon,t\to T_1}[k_{0,t}]$. Observe that

$$
\mathbb{N}^{*(n,c)}_{\upsilon,t\to T_1}[k_{0,t}] = \mathbb{E}^*_t\left[\mathbb{M}^{*(n)}_{T_1\to T_N} 1_{S_{T_1}>K_0}\right] \tag{B36}
$$

If

$$
\mathbb{M}^{*(n)}_{T_1\to T_N} = \zeta_t\mathcal{G}^{(n)}(R_{M,t\to T_1} - R_{f,t\to T_1}) + \eta_{T_1} \text{ with } \mathbb{E}^*_t[\eta_{T_1}|R_{M,t\to T_1}] = 0 \tag{B37}
$$

where ζ_t and the function $\mathcal{G}^{(n)}(x)$ is known, it follows that, using (B37), (B36) can alternatively be written as

$$
\mathbb{N}^{*(n,c)}_{\upsilon,t\to T_1}[k_{0,t}] = \zeta_t\mathbb{E}^*_t\left[\mathcal{G}^{(n)}\left(\frac{S_{T_1}}{S_t} - R_{f,t\to T_1}\right) 1_{S_{T_1}>K_0}\right] \tag{B38}
$$

Since ζ_t is known, the risk neutral quantity $\mathbb{N}^{*(n,c)}_{\upsilon,t\to T_1}[k_{0,t}]$ can be computed as

$$
\mathbb{N}^{*(n,c)}_{\upsilon,t\to T_1}[k_{0,t}] = \zeta_t\left\{\begin{array}{c} \mathcal{G}^{(c)}[K_0]\mathbb{P}^*_t[S_{T_1}>K_0] + R_{f,t\to T_1}\mathcal{G}^{(c)}_S[K_0]C_t[K_0] \\ +R_{f,t\to T_1}\int_{K_0}^{\infty}\mathcal{G}^{(c)}_{SS}[K]C_t[K]\,dK \end{array}\right\}
$$

The proof is similar to the proof of (B34). Similarly

$$
\mathbb{N}^{*(n,p)}_{\upsilon,t\to T_1}[k_{0,t}] = \zeta_t\mathbb{E}^*_t\left[\mathcal{G}^{(n)}\left(\frac{S_{T_1}}{S_t} - R_{f,t\to T_1}\right) 1_{S_{T_1}<K_0}\right] \tag{B39}
$$

and

$$\mathbb{N}^{*(n,p)}_{\upsilon,t\to T_1}\left[k_{0,t}\right] = \zeta_t \left\{ \begin{array}{c} \mathcal{G}^{(p)}\left[K_0\right]\mathbb{P}_t\left[S_{T_1} < K_0\right] - R_{f,t\to T_1}\mathcal{G}^{(p)}_S\left[S_t K_0\right] P_t\left[K_0\right] \\ +R_{f,t\to T_1}\int_0^{K_0}\mathcal{G}^{(p)}_{SS}\left[K\right] P_t\left[K\right] dK \end{array} \right\}$$

C Implications of high-order leverage terms

C.1 Conditional expected return with high-order leverages

Proof. The expected excess market return is

$$\mathbb{E}_t\left(R_{t\to T_1} - R_{f,t\to T_1}\right) = \mathbb{COV}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}, \left(R_{t\to T_1} - R_{f,t\to T_1}\right)\right).$$

We then replace the inverse SDF by its expression and obtain

$$\begin{aligned}
\mathbb{E}_t\left(R_{t\to T_1} - R_{f,t\to T_1}\right) &= \mathbb{COV}^*_t\left(\frac{1 + z_{T_1} + z^{\upsilon}_{T_1}}{1 + \mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}}, \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right) \\
&= \frac{\mathbb{COV}^*_t\left(z_{T_1}, r_{M,t\to T_1}\right) + \mathbb{COV}^*_t\left(z^{\upsilon}_{T_1}, r_{M,t\to T_1}\right)}{1 + \mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}}
\end{aligned}$$

Setting $r_{M,t\to T_1} = R_{M,t\to T_1} - R_{f,t\to T_1}$ and using the definitions of z_{T_1} and $z^{\upsilon}_{T_1}$, it follows that

$$\begin{aligned}
\mathbb{E}^*_t z_{T_1} &= \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{E}^*_t r^2_{M,t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T_1}}\mathbb{E}^*_t r^3_{M,t\to T_1} \\
\mathbb{E}^*_t z^{\upsilon}_{T_1} &= \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N} + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{E}^*_t \mathbb{M}^{*(3)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{E}^*_t r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^*_t z_{T_1}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) &= \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{E}^*_t r^2_{M,t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T}}\mathbb{E}^*_t r^3_{M,t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T}}\mathbb{E}^*_t r^4_{M,t\to T_1} \\
&= \frac{a_{1,t}}{R_{f,t\to T}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T}}\mathbb{M}^{*(4)}_{t\to T_1}
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_t^* z_{T_1}^{\upsilon}\left(R_{M,t\to T} - R_{f,t\to T_1}\right) &= \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{COV}_t^*\left(r_{M,t\to T_1}, \mathbb{M}_{T_1\to T_N}^{*(2)}\right) + \frac{a_{3,t}}{R_{f,T_1\to T_N}^3}\mathbb{COV}_t^*\left(r_{M,t\to T_1}, \mathbb{M}_{T_1\to T_N}^{*(3)}\right) \\ &+ \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\left(\mathbb{COV}_t^*\left(r_{M,t\to T_1}^2, \mathbb{M}_{T_1\to T_N}^{*(2)}\right) + \mathbb{M}_{t\to T_1}^{*(2)}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}\right)\end{aligned}$$

This ends the proof. ■

C.2 Conditional crash probability with high-order leverages

Proof. The probability of crash is

$$\Pi_{t\to T_1}^{3rd}[\alpha] = \mathbb{E}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} 1_{R_{M,t\to T}<\alpha}\right)$$

We then replace the inverse SDF by its expression and obtain

$$\begin{aligned}\Pi_{t\to T_1}^{3rd}[\alpha] &= \frac{\mathbb{E}_t^*\left(\left(1 + z_{T_1} + z_{T_1}^{\upsilon}\right)1_{R_{M,t\to T}<\alpha}\right)}{1 + \mathbb{E}_t^* z_{T_1} + \mathbb{E}_t^* z_{T_1}^{\upsilon}} \\ &= \frac{\mathbb{E}_t^*\left(1_{R_{M,t\to T}<\alpha}\right) + \mathbb{E}_t^*\left(z_{T_1}1_{R_{M,t\to T}<\alpha}\right) + \mathbb{E}_t^*\left(z_{T_1}^{\upsilon}1_{R_{M,t\to T}<\alpha}\right)}{1 + \mathbb{E}_t^* z_{T_1} + \mathbb{E}_t^* z_{T_1}^{\upsilon}} \\ &= \frac{\left\{\begin{array}{c}\mathbb{M}_{t\to T_1}^{*(0)}[\alpha] + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,t\to T_1}^k}\mathbb{M}_{t\to T_1}^{*(k)}[\alpha] + \\ \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{M}_{t,\upsilon}^*[\alpha] + \frac{a_{3,t}}{R_{f,T_1\to T_N}^3}\mathbb{M}_{t,s}^*[\alpha] + \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\mathbb{M}_{t,s\upsilon}^*[\alpha]\end{array}\right\}}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,t\to T_1}^k}\mathbb{M}_{t\to T_1}^{*(k)} + \sum_{k=2}^{3}\frac{a_{k,t}}{R_{f,T_1\to T_N}^k}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(k)} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R_{f,T_1\to T_N}^2}\mathbb{E}_t^* r_{M,t\to T_1}\mathbb{M}_{T_1\to T_N}^{*(2)}}\end{aligned}$$

This ends the proof ■

C.3 Proof of Equation (33)

Consider the partial derivatives

$$
\begin{aligned}
f_{xxy} &= \frac{2}{(x_0)^2 y_0} \frac{\left(W_t x_0 y_0 u''\right)^2}{(u')^2} \left(2 - \frac{u''' u'}{(u'')^2}\right) \\
&+ \frac{1}{(x_0)^2 y_0} \left\{ 6 \frac{\left(W_t x_0 y_0\right)^3 u'' u' u'''}{(u')^3} - \left(W_t x_0 y_0\right)^3 \frac{u''''}{u'} - 6 \frac{\left(W_t x_0 y_0 u''\right)^3}{(u')^3} \right\}, \\
f_{xxx} &= \frac{y_0^3}{x_0^3} f_{yyy} = \frac{1}{(x_0)^3} \left(6 \frac{\left(W_t x_0 y_0\right)^3 u'' u'''}{(u')^2} - \frac{\left(W_t x_0 y_0\right)^3 u''''}{u'} - 6 \frac{\left(W_t x_0 y_0\right)^3 \left(u''\right)^3}{(u')^3} \right).
\end{aligned}
$$

Thus, a third order Taylor expansion-series yields

$$
\begin{aligned}
f[x,y] &= f[x,y]^{2nd} \\
&+ \frac{1}{(x_0)^3} \frac{(\kappa_t + 1 - 2\rho_t)}{\tau_t^3} (x - x_0)^3 + \frac{1}{(y_0)^3} \frac{(\kappa_t + 1 - 2\rho_t)}{\tau_t^3} (y - y_0)^3 \\
&+ \frac{1}{(x_0)^2 y_0} \left(\frac{2(1-\rho_t)}{\tau_t^2} + \frac{3(\kappa_t + 1 - 2\rho_t)}{\tau_t^3} \right) (x - x_0)^2 (y - y_0) \\
&+ \frac{1}{x_0 (y_0)^2} \left(\frac{2(1-\rho_t)}{\tau_t^2} + \frac{3(\kappa_t + 1 - 2\rho_t)}{\tau_t^3} \right) (y - y_0)^2 (x - x_0), \qquad \text{(C40)}
\end{aligned}
$$

where $f[x,y]^{2nd}$ is the second order Taylor expansion-series in Equation (A14).

Replacing x, x_0, y, and y_0 by their expressions and using preference parameters a_1, a_2, and a_3 defined in Equation (6), we obtain,

$$
\begin{aligned}
\mathbb{E}_{T_1}^{*}\left(f[x,y]\right) &= 1 + \frac{a_{1,t}}{R_{f,t\to T_1}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) + \frac{a_{1,t}}{R_{f,T_1\to T_N}} \left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right) \\
&+ \frac{a_{2,t}}{\left(R_{f,t\to T_1}\right)^2} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{\left(R_{f,T_1\to T_N}\right)^2} \mathbb{E}_{T_1}^{*}\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right) \\
&+ \frac{a_{1,t} + 2a_{2,t}}{R_{f,t\to T_2}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right) \\
&+ \frac{a_{3,t}}{\left(R_{f,t\to T_1}\right)^3} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3 + \frac{a_{3,t}}{\left(R_{f,T_1\to T_N}\right)^3} \mathbb{E}_{T_1}^{*}\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3\right) \\
&+ \frac{2a_{2,t} + 3a_{3,t}}{\left(R_{f,t\to T_1}\right)^2 R_{f,T_1\to T_N}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right) \\
&+ \frac{2a_{2,t} + 3a_{3,t}}{R_{f,t\to T_1}\left(R_{f,T_1\to T_N}\right)^2} \mathbb{E}_{T_1}^{*}\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right)\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) \qquad \text{(C41)}
\end{aligned}
$$

which gives the desired result when interest rates are deterministic.

D Online Appendix

D.1 Volatility Dynamic Implied by (20)

To further show that our formulation (20) is different from the GARCH (1,1), we use the closed-form expression of θ_t displayed in (21) and show that

$$\mathbb{M}^{*(2)}_{t\to T_N} = \theta_t \mathbb{E}^*_t R^2_{M,t\to T_1}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + R^2_{f,T_1\to T_2}\mathbb{M}^{*(2)}_{t\to T_1}. \tag{D42}$$

Since $R^2_{M,t\to T_1} = \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + 2R_{M,t\to T_1}R_{f,t\to T_1} - R^2_{f,t\to T_1}$, it follows that

$$\mathbb{E}^*_t R^2_{M,t\to T_1}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1}\mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}.$$

We then replace this expression in the RHS of (D42) and obtain

$$\mathbb{M}^{*(2)}_{t\to T_N} = \theta_t \mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1}\theta_t \mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,T_1\to T_N}\left(\theta_t + 1\right)\mathbb{M}^{*(2)}_{t\to T_1}.$$

This shows that the process of $\mathbb{M}^{*(2)}_{t\to T_N}$ is different from a GARCH dynamic. To check similarities with the GARCH process, let's assume for illustration purpose that $\mathbb{M}^{*(3)}_{t\to T_1} = 0$ and $\mathbb{M}^{*(4)}_{t\to T_1} = 3\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2$ then

$$\mathbb{M}^{*(2)}_{t\to T_N} = 3\theta_t\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2 + R^2_{f,T_1\to T_N}\left(\theta_t + 1\right)\mathbb{M}^{*(2)}_{t\to T_1}. \tag{D43}$$

Expression (D43) is reminiscent but distinct from the GARCH process.

D.2 The case with consumption

In this section, we introduce consumption in the representative agent problem. Under the minimal assumption that (i) odd risk neutral moments are negative, (ii) preference parameters satisfy the restrictions $a_{2,t} > 0$, $a_{2,t} \leq 0$, $a_{3,t} \geq 0$, $a_{2,3,t} \geq 0$ (see Eq), (iii) consumption-wealth ratio is positively related to the market return and (iv) the correlation of the square of the consumption wealth ratio and market return is negative (condition reminiscent of market coskewness), our measure of expected excess return remains a lower bound to the true measure of market expected excess return.

To proceed, we start by having the representative agent solve the problem

$$\max_{\omega_t, c_t} \mathbb{E}_t \left\{ \max_{\omega_{T_1}, c_{T_1}} \left\{ \mathbb{E}_{T_1} u\left[W_{t\to T_N}\right]\right\} \right\},$$

where the terminal wealth is

$$W_{t\to T_N} = \left(1 - c_{T_1}\right) W_{T_1} \left(\omega_{T_1}^{\intercal} R_{T_1\to T_N}\right) \text{ with } W_{T_1} = \left(1 - c_t\right) W_t \left(\omega_t^{\intercal} R_{t\to T_1}\right)$$

and c_t is the consumption wealth ratio. The terminal wealth can alternatively be written as

$$W_{t\to T_N} = \left(1 - c_{T_1}\right)\left(1 - c_t\right) W_t \left(\omega_t^{\intercal} R_{t\to T_1}\right)\left(\omega_{T_1}^{\intercal} R_{T_1\to T_N}\right).$$

For simplicity, we assume no interest rate risk. Notice that the SDF is given by the identity:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\upsilon_{T_1}}{\mathbb{E}_t^{*}\left(\upsilon_{T_1}\right)},$$

where

$$\upsilon_{T_1} = \mathbb{E}_{T_1}^{*}\left(\frac{u'\left[\overline{W}_{t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}\right) \text{ with } \overline{W}_{t\to T_N} = W_t R_{f,t\to T_1} R_{f,T_1\to T_N}. \tag{D44}$$

We set

$$R_{M,t\to T_1} = \omega_t^\intercal R_{t\to T_1},\, R_{M,T_1\to T_N} = \omega_{T_1}^\intercal R_{T_1\to T_N},\, cc_{tT_1} = \left(1 - c_{T_1}\right)\left(1 - c_t\right). \tag{D45}$$

Next, we define

$$\mathbf{x} = cc_{tT_1},\, \mathbf{y} = \omega_t^\intercal R_{t\to T_1},\, \mathbf{z} = \omega_{T_1}^\intercal R_{T_1\to T_N} \tag{D46}$$

$$\mathbf{x}_0 = 1,\, \mathbf{y}_0 = R_{f,t\to T_1},\, \mathbf{z}_0 = R_{f,T_1\to T_N} \tag{D47}$$

and set

$$\mathbf{X} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \text{ and } \mathbf{X}_0 = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0).$$

Notice that $0 < cc_{tT_1} \leq 1$ since $0 < c_{T_1} \leq 1$ and $0 < c_t \leq 1$. Now, assume that the utility function is well-behaved and admits high-order derivatives that exist. Denote

$$\mathbf{G} = \frac{u'\left[\overline{W}_{t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}$$

D.2.1 Second-order Taylor expansion-series

A second-order Taylor expansion of $\mathbf{G}$ around $\mathbf{X} = \mathbf{X}_0$ gives

$$\begin{aligned}
\mathbf{G} &= 1 - (\mathbf{x} - \mathbf{x}_0) \frac{W_t \mathbf{y}_0 \mathbf{z}_0 u'' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} - (\mathbf{y} - \mathbf{y}_0) \frac{W_t \mathbf{z}_0 u'' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} \\
&\quad - (\mathbf{z} - \mathbf{z}_0) \frac{W_t \mathbf{y}_0 u'' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} + \frac{1}{2} W_t^2 \mathbf{y}_0^2 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t\to T_N}\right]\right)^2} \right) (\mathbf{x} - \mathbf{x}_0)^2 \\
&\quad + \frac{1}{2} W_t^2 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t\to T_N}\right]\right)^2} \right) (\mathbf{y} - \mathbf{y}_0)^2 \\
&\quad + \frac{1}{2} W_t^2 \mathbf{y}_0^2 \left(-\frac{u''' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t\to T_N}\right]\right)^2} \right) (\mathbf{z} - \mathbf{z}_0)^2 \\
&\quad + W_t^2 \mathbf{y}_0 \mathbf{x}_0 \mathbf{z}_0^2 \left(-\frac{u''' \left[\overline{W}_{t\to T_N}\right]}{u' \left[\overline{W}_{t\to T_N}\right]} + \frac{2 \left(u'' \left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u' \left[\overline{W}_{t\to T_N}\right]\right)^2} \right) (\mathbf{x} - \mathbf{x}_0)(\mathbf{y} - \mathbf{y}_0) \\
&\quad + \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{x} \partial \mathbf{z}} \right)_{\mathbf{X}=\mathbf{X}_0} (\mathbf{x} - \mathbf{x}_0)(\mathbf{z} - \mathbf{z}_0) + \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{y} \partial \mathbf{z}} \right)_{\mathbf{X}=\mathbf{X}_0} (\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0).
\end{aligned}$$

Notice that

$$\mathbb{E}^*_{T_1} (\mathbf{z} - \mathbf{z}_0) = 0$$

and

$$\begin{aligned}
\mathbb{E}^*_{T_1} (\mathbf{x} - \mathbf{x}_0)(\mathbf{z} - \mathbf{z}_0) &= (\mathbf{x} - \mathbf{x}_0) \mathbb{E}^*_{T_1} (\mathbf{z} - \mathbf{z}_0) = 0, \\
\mathbb{E}^*_{T_1} (\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0) &= (\mathbf{y} - \mathbf{y}_0) \mathbb{E}^*_{T_1} (\mathbf{z} - \mathbf{z}_0) = 0.
\end{aligned}$$

We use these expressions to simplify (D44) as

$$
\begin{aligned}
\upsilon_{T_1} &= 1-\frac{W_t\mathbf{y}_0\mathbf{z}_0u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}\left(\mathbf{x}-\mathbf{x}_0\right)-\left(\mathbf{y}-\mathbf{y}_0\right)\frac{W_t\mathbf{z}_0u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} \\
&\quad +\frac{1}{2}W_t^2\mathbf{y}_0^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)^2 \\
&\quad +\frac{1}{2}W_t^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{y}-\mathbf{y}_0\right)^2 \\
&\quad +\frac{1}{2}W_t^2\mathbf{y}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2 \\
&\quad +W_t^2\mathbf{y}_0\mathbf{x}_0\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\upsilon_{T_1} &= 1+\frac{1}{\tau_t}\mathbb{E}_{T_1}^*\left(cc_{tT_1}-1\right)+\frac{1}{\tau_tR_{f,t\to T_1}}\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right)+\frac{(1-\rho_t)}{\tau_t^2}\left(cc_{tT_1}-1\right)^2 \\
&\quad +\frac{(1-\rho_t)}{\tau_t^2R_{f,t\to T_1}^2}\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right)^2+\frac{(1-\rho_t)}{\tau_t^2R_{f,T_1\to T_2}^2}\mathbb{E}_{T_1}^*\left(\omega_{T_1}^\intercal R_{T_1\to T_2}-R_{f,T_1\to T_N}\right)^2 \\
&\quad +\frac{2(1-\rho_t)}{\tau_t^2R_{f,t\to T_1}}\mathbb{E}_{T_1}^*\left(cc_{tT_1}-1\right)\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right).
\end{aligned}
$$

We then exploit the notation $R_{M,t\to T_1}=\omega_t^\intercal R_{t\to T_1}$, $R_{M,T_1\to T_N}=\omega_{T_1}^\intercal R_{T_1\to T_N}$ and express the expected value of υ_{T_1} under the risk neutral measure as

$$
\begin{aligned}
\mathbb{E}_t^*\upsilon_{T_1} &= 1+\frac{1}{\tau_t}\mathbb{E}_t^*\left(cc_{tT_1}-1\right)+\frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}_t^*\left(cc_{tT_1}-1\right)^2 \\
&\quad +\frac{(1-\rho_t)}{\tau_t^2R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{(1-\rho_t)}{\tau_t^2R_{f,T_1\to T_N}^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)} \\
&\quad +\frac{2(1-\rho_t)}{\tau_t^2R_{f,t\to T_1}}\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t\to T_1}\right).
\end{aligned}
\tag{D48}
$$

where

$$\mathbb{M}^{*(n)}_{t\to T_1} = \mathbb{E}^*_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^n$$
$$\mathbb{M}^{*(2)}_{T_1\to T_N} = \mathbb{E}^*_{T_1} \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2$$

The expected excess market return is

$$\begin{aligned}\mathbb{E}_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) &= \mathbb{E}_t \left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right] \\ &= \mathbb{E}^*_t \left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right] \\ &= \frac{\mathbb{COV}^*_t \left[\upsilon_{T_1}, R_{M,t\to T_1}\right]}{\mathbb{E}^*_t \upsilon_{T_1}}.\end{aligned}$$

Observe that

$$\begin{aligned}\mathbb{COV}^*_t \left[\upsilon_{T_1}, R_{M,t\to T_1}\right] &= 1 + \frac{1}{\tau_t}\mathbb{COV}^*_t \left(cc_{tT_1}, R_{M,t\to T_1}\right) + \frac{1}{\tau_t R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} \\ &+ \frac{(1-\rho_t)}{\tau_t^2}\mathbb{COV}^*_t \left(\left(cc_{tT_1} - 1\right)^2, R_{M,t\to T_1}\right) \\ &+ \frac{(1-\rho_t)}{\tau_t^2 R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{(1-\rho_t)}{\tau_t^2 R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t \\ &+ \frac{2(1-\rho_t)}{\tau_t^2 R^2_{f,t\to T_1}}\mathbb{E}^*_t \left(\left(cc_{tT_1} - 1\right)\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right). \end{aligned} \tag{D49}$$

Notice that $\mathbb{E}^*_t \left(\left(cc_{tT_1} - 1\right)\left(\omega_t^\intercal R_{t\to T_1} - R_{f,t\to T_1}\right)^2\right) < 0$ because $cc_{tT_1} - 1 < 0$. In addition, $\mathbb{M}^{*(3)}_{t\to T_1} \leq 0$, $\mathbb{LEV}^*_t \leq 0$, and $\mathbb{COV}^*_t \left(R_{M,t\to T_1}, \mathbb{LEV}^*_t\right) \leq 0$. Recall that

$$\frac{1}{\tau_t} > 0 \text{ and } 1 - \rho_t \leq 0. \tag{D50}$$

In theory, each factor risk factor in υ_{T_1} positively contributes to the risk premium. Thus each term in (D49) is positive. Assuming (D50) is satisfied, one should expect

$$\mathbb{COV}^*_t \left(cc_{tT_1}, R_{M,t\to T_1}\right) > 0 \text{ and } \mathbb{COV}^*_t \left(\left(cc_{tT_1} - 1\right)^2, R_{M,t\to T_1}\right) \leq 0. \tag{D51}$$

Since $1 - c_{T_1} = \frac{W_{T_1} - C_{T_1}}{W_{T_1}}$ is the fraction of wealth W_{T_1} invested at T_1, it follows that

$$\begin{aligned}\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t\to T_1}\right) &= (1-c_t)\,\mathbb{COV}_t^*\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}}, R_{M,t\to T_1}\right) \text{ and} \\ \mathbb{COV}_t^*\left(\left(cc_{tT_1}-1\right)^2, R_{M,t\to T_1}\right) &= (1-c_t)^2\,\mathbb{COV}_t^*\left(\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}}\right)^2, R_{M,t\to T_1}\right).\end{aligned}$$

The positive sign of $\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t\to T_1}\right)$ is motivated by the positive impact of wealth-consumption ratio on the market expected excess return. Conditions (D51) are reminiscent of the dependence between the wealth-consumption ratio and the return on the market under the physical measure. Under the physical measure, the wealth-consumption ratio is positively correlated to the market. Under conditions (D50) and (D51), the covariance $\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t\to T_1}\right]$ is bounded:

$$\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t\to T_1}\right] \geq \frac{1}{R_{f,t\to T_1}}\frac{1}{\tau_t}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(3)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{LEV}_t^*. \qquad \text{(D52)}$$

Next, since $cc_{tT_1} \leq 1$, we use (D48) and exploit (D50) and (D51) to obtain

$$\mathbb{E}_t^*\upsilon_{T_1} \leq 1 + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}.$$

Therefore,

$$\frac{1}{\mathbb{E}_t^*\upsilon_{T_1}} \geq \frac{1}{1 + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}}. \qquad \text{(D53)}$$

Combining (D52) and (D53), the expected excess return is bounded

$$\mathbb{E}_t\left[R_{M,t\to T_1} - R_{f,t\to T_1}\right] \geq \underbrace{\frac{\frac{1}{R_{f,t\to T_1}}\frac{1}{\tau_t}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(3)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{LEV}_t^*}{1 + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}}}_{\text{This is our measure of expected excess return}}$$

This shows that under minimal conditions, our measure of expected excess return is a bound on the true expected excess return when consumption is taken into account.

Next, we focus on the third-order Taylor expansion-series of the inverse marginal utility function.

D.2.2 Third-order Taylor expansion-series

A Third-order Taylor expansion of $\frac{u'\left[\overline{W}_{t\to T_2}\right]}{u'\left[W_{t\to T_2}\right]}$ arround $\mathbf{X} = \mathbf{X}_0$ gives

$$
\begin{aligned}
\mathbf{G} \;=\;& 1 + (\mathbf{x} - \mathbf{x}_0)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t} + (\mathbf{y} - \mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t} + (\mathbf{z} - \mathbf{z}_0)\frac{1}{\mathbf{z}_0}\frac{1}{\tau_t} + \frac{1}{\mathbf{x}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{x} - \mathbf{x}_0)^2 \\
& + \frac{1}{\mathbf{y}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{y} - \mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{z} - \mathbf{z}_0)^2 + \frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}\right)(\mathbf{x} - \mathbf{x}_0)(\mathbf{y} - \mathbf{y}_0) \\
& + \left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{x} - \mathbf{x}_0)(\mathbf{z} - \mathbf{z}_0) + \left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0). \\
& + \frac{1}{\mathbf{x}_0^3}\frac{(\kappa_t - 2\rho_t + 1)}{\tau_t^3}(\mathbf{x} - \mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3}\frac{(\kappa_t - 2\rho_t + 1)}{\tau_t^3}(\mathbf{z} - \mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3}\frac{(\kappa_t - 2\rho_t + 1)}{\tau_t^3}(\mathbf{y} - \mathbf{y}_0)^3 \\
& + \frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t - 2\rho_t + 1)}{\tau_t^3}\right)(\mathbf{x} - \mathbf{x}_0)^2(\mathbf{y} - \mathbf{y}_0) \\
& + \frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t - 2\rho_t + 1)}{\tau_t^3}\right)(\mathbf{y} - \mathbf{y}_0)^2(\mathbf{x} - \mathbf{x}_0) \\
& + \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t - 2\rho_t + 1)}{\tau_t^3}\right)(\mathbf{z} - \mathbf{z}_0)^2(\mathbf{x} - \mathbf{x}_0) \\
& + \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t - 2\rho_t + 1)}{\tau_t^3}\right)(\mathbf{z} - \mathbf{z}_0)^2(\mathbf{y} - \mathbf{y}_0) \\
& + 6\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{z} - \mathbf{z}_0)(\mathbf{y} - \mathbf{y}_0)(\mathbf{x} - \mathbf{x}_0) \\
& + 3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{x} - \mathbf{x}_0)^2(\mathbf{z} - \mathbf{z}_0) \\
& + 3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{y} - \mathbf{y}_0)^2(\mathbf{z} - \mathbf{z}_0)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1+(\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t}+(\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}+\frac{1}{\mathbf{x}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{x}-\mathbf{x}_0)^2 \\
&\quad +\frac{1}{\mathbf{y}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{y}-\mathbf{y}_0)^2+\frac{1}{\mathbf{z}_0^2}\frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2+\frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}\right)(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&\quad +\frac{1}{\mathbf{x}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{x}-\mathbf{x}_0)^3+\frac{1}{\mathbf{z}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3+\frac{1}{\mathbf{y}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{y}-\mathbf{y}_0)^3 \\
&\quad +\frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) \\
&\quad +\frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&\quad +\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&\quad +\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2}+\frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2
\end{aligned}
$$

Using Eq (6) in the main text of the paper, it follows that

$$
\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1+(\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}a_{1,t}+(\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}+\frac{1}{\mathbf{x}_0^2}a_{2,t}(\mathbf{x}-\mathbf{x}_0)^2 \\
&\quad +\frac{1}{\mathbf{y}_0^2}a_{2,t}(\mathbf{y}-\mathbf{y}_0)^2+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2+\frac{2}{\mathbf{x}_0\mathbf{y}_0}a_{2,t}(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&\quad +\frac{1}{\mathbf{x}_0^3}a_{3,t}(\mathbf{x}-\mathbf{x}_0)^3+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3+\frac{1}{\mathbf{y}_0^3}a_{3,t}(\mathbf{y}-\mathbf{y}_0)^3 \\
&\quad +\frac{6}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0)+\frac{6}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&\quad +\frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&\quad +\frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \qquad \text{(D54)}
\end{aligned}
$$

We then compute the expected value $\mathbb{E}_t^* \upsilon_{T_1}$ to obtain

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} &= 1 + (\mathbf{x} - \mathbf{x}_0) \frac{1}{\mathbf{x}_0} a_{1,t} + \frac{1}{\mathbf{x}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^2 \\
&+ \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 + \frac{2}{\mathbf{x}_0 \mathbf{y}_0} a_{2,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)(\mathbf{y} - \mathbf{y}_0) \\
&+ \frac{1}{\mathbf{x}_0^3} a_{3,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 \\
&+ \frac{1}{\mathbf{y}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 (\mathbf{x} - \mathbf{x}_0) + \frac{1}{\mathbf{z}_0^2 \mathbf{x}_0} a_{2,3,t} \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \\
&+ \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \right)
\end{aligned}$$

Notice that

$$\mathbf{x} - \mathbf{x}_0 \leq 0,$$

and the following inequalities hold:

$$a_{2,t} > 0, \; a_{2,t} \leq 0, \; a_{3,t} \geq 0, \; a_{2,3,t} \geq 0, \tag{D55}$$

and

$$\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 \leq 0, \; \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 \leq 0, \mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)^3 \leq 0,$$

and

$$\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0)(\mathbf{y} - \mathbf{y}_0) = \mathbb{COV}_t^* (\mathbf{x} - \mathbf{x}_0, \mathbf{y}) \geq 0$$

and

$$\begin{aligned}
\mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^2 (\mathbf{x} - \mathbf{x}_0) &\leq 0 \text{ (because } (\mathbf{x} - \mathbf{x}_0) \leq 0) \\
\mathbb{E}_t^* (\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 &\leq 0 \text{ (because } (\mathbf{x} - \mathbf{x}_0) \leq 0) \\
\mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2 \right) &= \mathbb{LEV}_t^* \leq 0.
\end{aligned}$$

This allows us to bound $\mathbb{E}_t^* \upsilon_{T_1}$ as

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} \leq\ & 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2 \\
& + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^3 \\
& + \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2\right).
\end{aligned}$$

As a result,

$$\frac{1}{\mathbb{E}_t^* \upsilon_{T_1}} \geq \frac{1}{\left\{\begin{array}{c} 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2 \\ + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^3 \\ + \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2\right) \end{array}\right\}} \tag{D56}$$

Next, our goal is to bound $\mathbb{COV}_t^*(\upsilon_{T_1}, \mathbf{y}-\mathbf{y}_0) = \mathbb{COV}_t^*(\upsilon_{T_1}, R_{M,t\to T_1} - R_{f,t\to T_1})$. We then use (D54) to compute this covariance as

$$
\begin{aligned}
& \mathbb{COV}_t^*(\upsilon_{T_1}, \mathbf{y}-\mathbf{y}_0) \\
= \; & \frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*(\mathbf{y}) + \frac{1}{\mathbf{x}_0^2} a_{2,t} \mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^2, \mathbf{y}-\mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{2}{\mathbf{x}_0\mathbf{y}_0} a_{2,t} \mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{x}_0^3} a_{3,t} \mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^3, \mathbf{y}-\mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0} a_{2,3,t} \mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0} a_{2,3,t} \mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2, \mathbf{y}-\mathbf{y}_0\right).
\end{aligned}
$$

Notice that

$$
\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0), \mathbf{y}-\mathbf{y}_0\right) = \mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0)^2 \leq 0 \text{ (since } \mathbf{x} \leq \mathbf{x}_0\text{)},
$$

and

$$
\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0), \mathbf{y}-\mathbf{y}_0\right) = \mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0)^2 \geq 0.
$$

We assume

$$
\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^2, \mathbf{y}-\mathbf{y}_0\right) \leq 0 \tag{D57}
$$

and

$$\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)^3,\mathbf{y}-\mathbf{y}_0\right) \geq 0, \tag{D58}$$

$$\mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0),\mathbf{y}-\mathbf{y}_0\right) \geq 0, \tag{D59}$$

$$\mathbb{COV}_t^*\left((\mathbf{x}-\mathbf{x}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2,\mathbf{y}-\mathbf{y}_0\right) \geq 0. \tag{D60}$$

These conditions are reminiscent of the sign of coskewness and cokurtosis when random variables of interest are return. While $\mathbf{y}-\mathbf{y}_0$ and $\mathbf{z}-\mathbf{z}_0$ are realized excess returns, $\mathbf{x}-\mathbf{x}_0$ is a function of wealth-consumption ratio (See (D45)-(D47)). Because coskewness is negative (see Harvey and Siddique (2000)) and cokurtosis is positive (Dittmar (2002)) and the wealth-consumption ratio is positively correlated to the market return, one should expect (D58)-(D60) to hold.

Under conditions (D57)-(D60), it follows that

$$\begin{aligned}\mathbb{COV}_t^*\left(\upsilon_{T_1},\mathbf{y}-\mathbf{y}_0\right) \geq\ & \frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*(\mathbf{y})+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^3+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2,\mathbf{y}-\mathbf{y}_0\right)\\ &+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^3,\mathbf{y}-\mathbf{y}_0\right)+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)^3,\mathbf{y}-\mathbf{y}_0\right)\\ &+\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left((\mathbf{y}-\mathbf{y}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2,\mathbf{y}-\mathbf{y}_0\right)\end{aligned} \tag{D61}$$

Combining (D56) and (D61) leads to

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right) = \frac{\mathbb{COV}_t^*\left[\upsilon_{T_1},R_{M,t\to T_1}\right]}{\mathbb{E}_t^*\upsilon_{T_1}}$$

$$\geq \frac{\left\{\begin{array}{c}\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*\left(\mathbf{y}\right)+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^3\\ \frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2,\mathbf{y}-\mathbf{y}_0\right)\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3,\mathbf{y}-\mathbf{y}_0\right)\\ +\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^3,\mathbf{y}-\mathbf{y}_0\right)\end{array}\right\}}{\left\{\begin{array}{c}1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^2+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{E}_t^*\left(\mathbf{y}-\mathbf{y}_0\right)^3\\ +\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\mathbf{y},\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2\right)\end{array}\right\}}$$

which simplifies to

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\geq\frac{\left(\begin{array}{c}\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}_{t\to T_1}^{*(3)}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}_{t\to T_1}^{*(4)}\\ +\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{LEV}_t^*+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{LES}_t^*\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}_{t\to T_1}^{*(3)}\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}+\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{LEV}_t^*\end{array}\right)}$$

We, thereafter, replace $\mathbf{y}_0$ and $\mathbf{z}_0$ by their expressions

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\geq\underbrace{\frac{\left(\begin{array}{c}\frac{1}{R_{f,t\to T_1}}\frac{1}{\tau_t}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{1}{R_{f,t\to T_1}^2}a_{2,t}\mathbb{M}_{t\to T_1}^{*(3)}+\frac{1}{R_{f,t\to T_1}^3}a_{3,t}\mathbb{M}_{t\to T_1}^{*(4)}\\ +\frac{1}{R_{f,T_1\to T_N}^2}a_{2,t}\mathbb{LEV}_t^*+\frac{1}{R_{f,T_1\to T_N}^3}a_{3,t}\mathbb{LES}_t^*\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{R_{f,t\to T_1}^2}a_{2,t}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{1}{R_{f,T_1\to T_2}^2}a_{2,t}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}+\frac{1}{R_{f,t\to T_1}^3}a_{3,t}\mathbb{M}_{t\to T_1}^{*(3)}\\ +\frac{1}{R_{f,T_1\to T_N}^3}a_{3,t}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(3)}+\frac{1}{R_{f,T_1\to T_N}^2R_{f,t\to T_1}}a_{2,3,t}\mathbb{LEV}_t^*\end{array}\right)}}$$

D.3 Is our Market Expected Return a Lower Bound to the Expected Return?

Setting consumption-wealth ratio to 1 in Section D.2 and using reasonable minimal assumptions that (i) odd risk neutral moments are negative, (ii) preference parameters satisfy the restrictions (D50) proves that our measure of expected excess market return (8) remains a lower bound to the true expected excess market return.

Setting consumption-wealth ratio to 1 in Section D.2 and using reasonable assumptions that (i) odd risk neutral moments are negative, (ii) preference parameters satisfy the restrictions (D55) proves that our measure of expected excess market return (39) remains a lower bound to the true expected excess market return.

E Additional performance tests

Table A1: **Out-of-sample prediction and allocation performance reached by fixing τ and ρ and estimating it, from 2000 (using 1-month returns as determinant for preference parameters)**

We report the out-of-sample performance of different risk premium prediction methods. $RP_{t\to T_1}^{Log}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

T_1		$\tau = 1$ and $\rho = 2$		$\rho = 2$		ρ, τ estimated	
	$RP_{t\to T_1}^{Log}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$
Panel A: Out-of-sample R^2							
10d	−0.09	−0.07	0.08	0.37	−1.44	0.11	0.12
1	1.09	1.18	1.73	1.15	−0.39	1.15	1.46
2	1.34	1.59	3.84**	1.30	2.09	1.42	1.98
3	1.18	1.61	4.71***	1.76	4.05	2.09	3.59*
4	2.16	2.86	5.47**	3.85	5.38	4.01	6.18**
5	3.12	4.19	6.44**	5.92	6.38	6.10	8.08**
6	3.61	4.97	7.26**	6.89	6.79	7.17	7.92
9	4.32	6.37	8.76**	8.98	10.35	8.59	9.35
12	4.00	6.54	8.44	9.23	9.09	8.27	9.24
18	2.29	6.17	7.66	9.70	10.65	7.72	9.29
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$							
10d	4.56	4.69	5.81	8.50	−8.88	7.96	6.79
1	3.55	3.68	3.52	4.91	−13.24	4.40	2.94
2	3.69	3.96	6.41	4.39	−6.90	2.96	4.49
3	4.14	4.54	9.50**	4.93	1.44	5.23	7.88*
4	4.27	4.75	8.46**	5.71	1.05	5.39	6.75
5	4.01	4.50	6.85	5.80	5.17	5.66	4.41
6	4.26	4.89	7.24	4.91	1.95	4.92	2.50
9	4.18	4.88	6.19	3.58	3.91	5.48	5.03
12	4.52	5.45***	6.85**	−2.50***	−17.46	5.91***	6.45*
18	4.59	5.62***	6.11**	−27.26***	−30.67	3.86***	5.30**

Table A2: **Out-of-sample prediction and allocation performance reached by fixing τ and ρ and estimating it, from 2000 (using 12-month returns as determinant for preference parameters)**

We report the out-of-sample performance of different risk premium prediction methods. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (8). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

T_1		$\tau=1$ and $\rho=2$		$\rho=2$		ρ, τ estimated	
	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$
Panel A: Out-of-sample R^2							
10d	−0.09	−0.07	0.08	0.60	0.33	0.52	−0.02
1	1.09	1.18	1.73	2.24	2.13	1.91	2.01
2	1.34	1.59	3.84**	2.45	2.69	2.05	2.75*
3	1.18	1.61	4.71***	2.78	3.20	2.57	3.66*
4	2.16	2.86	5.47**	4.47	5.26	3.81	5.39**
5	3.12	4.19	6.44**	6.27	7.37**	6.07	7.45**
6	3.61	4.97	7.26**	6.94	5.06	6.83	8.30*
9	4.32	6.37	8.76**	8.71	9.10	8.85	9.29
12	4.00	6.54	8.44	8.44	9.16	8.43	9.21
18	2.29	6.17	7.66	7.36	9.85	8.47	10.51
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma=3$							
10d	4.56	4.69	5.81	9.33	4.50	8.40	6.65
1	3.55	3.68	3.52	3.10	1.72	2.51	−0.10
2	3.69	3.96	6.41	3.69	4.27	3.36	2.85
3	4.14	4.54	9.50***	6.49	5.74	6.38	6.45
4	4.27	4.75	8.46**	7.03	5.96	5.47	5.88
5	4.01	4.50	6.85	4.57	3.77	4.03	3.05
6	4.26	4.89	7.24	−1.76	−4.24	−2.67	−1.10
9	4.18	4.88	6.19	1.15	4.65	0.57	6.57*
12	4.52	5.45***	6.85**	3.74***	1.80	3.34***	0.20
18	4.59	5.62***	6.11**	5.36***	−25.16	2.29***	−5.67

Table A3: **Out-of-sample prediction and allocation performance of $RP^{3rd}_{t\to T_1,T_N}$, with fixed preference parameters**
We report the out-of-sample performance of different risk premium prediction methods. The preference parameters are set to $\tau = 1$ and $\rho = 2$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (12). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (39). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (25)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (27)). The physical variances are computed using option prices (see Appendix B.2). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

Horizon T_1			$RP_{t\to T_1,T_N}$ with $T_N =$ (in months)										
(in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.08	−0.08	−0.53	−1.23	−2.31	−3.30	−4.35	−9.35	−17.77	−37.19	−62.73	−7.76
1	0.98	1.16	-	1.64	1.64	1.27	0.62	−0.24	−5.24	−14.28	−39.93	−80.07	−6.44
2	1.50	1.98	-	-	2.70	3.01	2.98	2.68	−1.09	−9.49	−36.27	−83.30	−4.51
3	1.34	2.15	-	-	-	3.09	3.70	4.03	2.18	−4.17	−27.41	−71.73	−2.41
4	1.91	3.25	-	-	-	-	4.30	5.16	5.16	1.14	−17.31	−54.74	0.37
5	2.66	4.76	-	-	-	-	-	5.97**	7.58	5.86	−7.12	−36.29	3.38
6	2.84	5.56	-	-	-	-	-	-	8.23*	8.10	−0.62	−22.98	4.15
9	2.40	6.67	-	-	-	-	-	-	-	9.08*	7.73	−2.58	7.98
12	1.05	6.14	-	-	-	-	-	-	-	-	8.44	4.67	7.50
18	−2.21	3.87	-	-	-	-	-	-	-	-	-	7.26	7.26
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	4.56	4.71	5.80	−2.88	−25.18	−51.02	-	-	-	-	-	-	-
1	4.71	4.95	-	5.62	−70.31	−132.18	-	-	-	-	-	-	-
2	4.83	5.30	-	-	6.05*	-	-	-	-	-	-	-	-
3	5.03	5.66	-	-	-	−12.35	-	-	-	-	-	-	-
4	5.21	5.87	-	-	-	-	3.61	-	-	-	-	-	-
5	5.26	6.14	-	-	-	-	-	6.99**	−84.35	-	-	-	-
6	5.21	4.90	-	-	-	-	-	-	-	-	-	-	-
9	5.27	6.55	-	-	-	-	-	-	-	−24.40	-	-	-
12	5.51	-	-	-	-	-	-	-	-	-	-	-	-
18	-	-	-	-	-	-	-	-	-	-	-	-	-