Option-Implied Risk Premia with Intertemporal Hedging *

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August 2025

Abstract

The equity and variance risk premia at a horizon T_1 depend on the risk of changes in the future economic environment beyond T_1. We derive novel estimates of these risk premia that account for intertemporal risk hedging and embed information on the market's term structure of risk. Crucially, our risk premia can be measured ex ante using option prices. We find that intertemporal hedging accounts for up to 80% of the equity and variance risk premia. In particular, intertemporal hedging increases the equity risk premium and decreases the variance risk premium in times of market expansion, characterized by long investors' horizons. Our estimates improve the out-of-sample R^2 of market return prediction by a factor of up to 2.

JEL classification: G11, G12, G13, G17.

Keywords: equity risk premium, intertemporal hedging, term structure.

*We are grateful to Patrick Augustin, Martin Boyer, Jean-Édouard Colliard, Kevin Crotty (discussant), Mathieu Fournier, Johan Hombert, Piotr Orlowski, Alexei Ovtchinnikov, Olivier Scaillet, Grigory Vilkov, Irina Zviadadze and conference and seminar participants at the University of California Riverside, University of Cincinnati, HEC Paris, CICF 2022, and HEC Montreal for helpful comments and suggestions. This research has received support from a grant of the French National Research Agency (ANR), "Investissements d'Avenir" (LabEx Ecodec/ANR-11-LABX-0047). This paper subsumes an earlier draft titled "Conditional leverage and the term structure of option-implied risk premia."

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1 Introduction

The equity risk premium—the expected return on the equity market over the risk-free rate—is a crucial input for corporate valuation and portfolio allocation. Unfortunately, it is also notoriously hard to estimate *ex ante*. Martin (2017a) shows how the risk-neutral market variance discounted at the risk-free rate provides a lower bound for the equity risk premium, in a one-period economy that ignores the higher-order moments of market returns. A major benefit of his approach is that the risk-neutral variance can be computed in real-time from observed option prices. Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024) extend the approach of Martin (2017a) and provide estimates for the equity risk premium that account for higher-order risks, still in a one-period model.

Restricting the economy to a one-period economy simplifies the analysis, but at the expense of strong assumptions. In particular, the risks of future shifts in the economic environment, e.g., changes in the expected returns or return volatility, are ignored. Consider, for example, a forecast horizon $T_1 > t$. A one-period model assumes that investors choose their portfolio allocation at time t ignoring the risks beyond time T_1. These risks, however, impact future consumption. Merton (1973) shows that investors optimally seek to hedge these risks by tilting their portfolio allocation towards assets that deliver higher returns when consumption is negatively affected. Intertemporal hedging of risk materializing after time T_1 therefore affects demand as well as equilibrium prices and risk premia at horizon T_1.

We derive novel estimates for the equity and variance risk premia, which take into account both higher order risks and intertemporal hedging. Our model features a multi-period economy, in which the representative investor chooses the optimal allocation to the market, to maximize the expected utility of the wealth accumulated between time t and their investment horizon $T_N \geq T_1$. In this economy, we derive an estimate for the equity and variance risk premia at horizon T_1, using a Taylor expansion of the inverse marginal utility. The resulting premia depend on the conditional moments of the horizon T_1-market returns, but also on time-t expected conditional moments of returns over $[T_1, T_N]$. Whereas the equity risk premium estimates of Martin (2017a), Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024) only need options expiring at T_1 to forecast market returns at horizon T_1, our method extracts information from options at horizons T_1 and T_N. Furthermore, we show that the stochastic discount factor of our multi-period model nests the ones of Martin (2017a), Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024).

Using minimal assumptions on investors' preferences, we compute estimates for the equity and variance risk premia on the S&P 500 from 1996 to 2023, over horizons ranging from 10 days to 18 months. These estimates can be readily computed from option prices, in real-time. We show that accounting for intertemporal hedging leads to an increase of the equity risk premium, in particular during times of market calm. Intertemporal hedging accounts for up to 80% of the total equity risk premium during these periods, and around 30% during NBER recessions. Furthermore, our risk premium achieves higher out-of-sample R^2 of return prediction than the bounds of Martin (2017a), Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024). For all forecast horizons T_1 from 10 days to 18 months, the out-of-sample R^2 increases with the investors' horizon T_N, up to a given T_N. For example, for T_1 at 10 days, the maximum out-of-sample R^2 is achieved at 6 months. For T_1 larger than two months, the maximum R^2 is obtained for the longest horizon for which we have available option maturities, namely $T_N = 2$ years. This result suggests that having options with maturity longer than two years would help us better capture intertemporal hedging, and would improve the forecast of returns at horizons three months and more. We also construct market-timing strategies and compute realized mean-variance certainty equivalents. These certainty equivalents indicate that our risk premium reaches better forecasts of both the first and second return moments, and that the improvements upon the forecasts of Chabi-Yo and Loudis (2020) are statistically significant.

We define the implied investors' horizon $T^*_{N,t}$, as the investment horizon which at each time t maximizes the fit of our equity risk premium estimate to the data. Specifically, $T^*_{N,t}$ is chosen so that it maximizes the R^2 of returns over a window of three months $[t - 3m, t]$.

We find that the implied investors' horizon switches between the longest available horizon T_N, e.g., two years, and the shortest horizon $T_N > T_1$. We further show that the investors' implied horizon varies with the probability of a crash, which we are able to calculate in real-time, from option prices. When the probability of a crash is high (above 10%), the implied investors' horizon is short. When this probability is low, the representative agent behaves as a long-term investor. This result provides empirical evidence to the theory of Hirshleifer and Subrahmanyam (1993), which predicts that investors' time horizons shorten during periods of uncertainty due to increased risk aversion and limited attention. It is also in line with Campbell and Vuolteenaho (2004), who find that in volatile markets, investors become more sensitive to "bad beta" – short-term cash flow shocks–, than to "good beta" – long-term discount rate changes–.

The equity risk premium using this implied investors' horizon $T^*_{N,t}$ is higher than the one of Chabi-Yo and Loudis (2020) under normal market conditions, and achieves an out-of-sample R^2 of return prediction that is about twice the one of Chabi-Yo and Loudis (2020). This difference is statistically significant. Intertemporal hedging accounts for up to 70% of the total premium. During market stress, the implied investors' horizon shrinks to the forecast horizon, and intertemporal hedging becomes negligible. The equity risk premium hence remains roughly unchanged. Similarly, intertemporal hedging makes the variance risk premium more negative during market calm, and accounts for up to 80% of the total premium. During market turmoil, the variance risk premium overlaps with the one without intertemporal hedging.

Our main results are based on preference parameters that are fixed. These results are however robust to changes in this assumption. We estimate the preference parameters over the period 1996-2023, in-sample. We test two different specifications: one in which the preference parameters are constant and one in which they vary over time, as linear functions of past returns. We show that the preference parameters in the second specification generate the largest out-of-sample R^2. However, such estimation over the full time period yields a look-ahead bias. We overcome this issue by estimating these parameters over a telescopic window of data, initially ranging from 1996 to 2006, and expanding with time. We find that the resulting equity risk premium estimates do not improve upon our main estimates in terms of out-of-sample R^2, over the period 2006-2023. We also extend our estimates of the equity and variance risk premia to third-order extensions, and study an extension of our setup that allows the representative investor to rebalance their portfolio between times T_1 and T_N. These extensions do not allow reaching significantly higher out-of-sample R^2. Our results thus indicate that our risk premia with fixed preference parameters provide the best overall performance: they can be computed from available option prices in real-time and achieve high out-of-sample forecast performance. Finally, in all of the extensions, our results on the magnitude and effect of intertemporal hedging on the equity and variance risk premia still hold.

We contribute to different strands of literature.

First, we contribute to the growing literature that constructs *bounds* on and estimates of physical return moments. Building on Martin (2017a), Crescini, Trojani, and Vedolin (2025) use investors' option holdings to recover investor-specific expected returns and risks. Martin and Wagner (2019), Kadan and Tang (2020), and Chabi-Yo, Dim, and Vilkov (2021) build bounds for the expected return on individual stocks and Kremens and Martin (2019) provide a bound for currency expected exchange rate appreciation using Quanto index options. See Back, Crotty, and Kazempour (2022) for a discussion and empirical tests of bounds for individual stocks and the stock market. Our novel bound for the equity risk premium involves intertemporal hedging demand implied from options prices.

Our intertemporal hedging term includes a risk-neutral leverage effect that is closely related to the asymmetric volatility implied correlation studied by Jackwerth and Vilkov (2019). They use short- and long-term options on the S&P 500 Index and options on VIX futures to calibrate the risk-neutral correlation between returns and future volatility. As options on VIX futures are available only starting in 2006, data availability prevents us from using their methodology.

Second, our work is related to the Recovery Theorem of Ross (2015), which shows how to disentangle the physical probability distribution from the pricing kernel and risk-neutral probabilities. This work has been challenged on theoretical and empirical grounds.[1] Instead of making assumptions about the pricing kernel process, Schneider and Trojani (2019) impose sign restrictions on the risk premia of return moments and find predictive power for future returns. Our approach differs in that we express, in a multi-period economy, the equity risk premium and the market's conditional variance under the phyisical measure as functions of risk-neutral moments of returns at different horizons.

Close to us, Gormsen and Jensen (2020) use the Martin (2017b) approach with the assumption that investors have power utility to estimate ex ante higher-order moments of market returns under the physical measure, from their risk-neutral counterparts. Their focus is on the behavior of estimated ex ante physical moments and their relation to proposed macroeconomic risk mechanisms that have been proposed in the literature. Unlike Gormsen and Jensen (2020), we do not make any assumption on the utility function.

Third, we build on the vast literature on the importance of the variance risk premium—the difference between the physical and risk-neutral variance—for predicting the equity risk premium (see, Bollerslev, Tauchen, and Zhou, 2009). Hu, Jacobs, and Seo (2021) show that the leverage effect, measured under the physical probability measure, has a strong positive relation with the variance risk premium. We derive an expression that relates the equity risk premium to the variance and leverage effect under the risk-neutral measure.

Finally, our paper is related to the literature on the equity term structure. van Binsbergen, Brandt, and Koijen (2012) show that the expected one-period return on claims on

[1] Borovička, Hansen, and Scheinkman (2016) show that Ross' assumptions rule out realistic models. Bakshi, Chabi-Yo, and Gao (2018) do not find support for the implications of the Recovery Theorem using U.S. Treasury bond futures. While Audrino, Huitema, and Ludwig (2019) find some forecasting power, Jensen, Lando, and Pedersen (2019) generalize the assumptions of Ross' (2015) model and find weak predictive power for future realized returns.

dividends decreases in the maturity of the dividend. Gormsen (2020) shows that this slope is countercyclical (see also, van Binsbergen, Hueskes, Koijen, and Vrugt, 2013; van Binsbergen and Koijen, 2017; Bansal, Miller, Song, and Yaron, 2021; Ulrich, Florig, and Seehuber, 2022; Giglio, Kelly, and Kozak, 2024). While the main object in this literature is the expected one-period return on claims on dividends several years in the future, we focus on the term structure of expected total market return with maturity of up to one year.

Recently, Knox, Londono, Samadi, and Vissing-Jorgensen (2025) identify equity premium days, defined as days with significantly elevated equity premia relative to the daily equity term structure. They show that these days coincide with macroeconomic releases, including FOMC, CPI, and nonfarm payrolls. Equity premium events are identified using estimates of the equity risk premium at short maturity (up to one month). Instead, we show that our method provides improvements upon the standard estimates for forecast horizons from one to six months and beyond.

Our paper proceeds as follows. Section 2 presents our theoretical results based on a second-order approximation, Section 3 discusses our empirical framework to build equity risk premium forecasts. Section 4 presents our main empirical results. In Section 5 we show the results when estimating the preference parameters of our model. Sections 6 and E study the robustness of our results to two extensions. Finally, Section 7 concludes.

2 Theoretical framework

In this section, we provide our main theoretical results. We derive a lower bound on the equity risk premium in a multi-period economy, accounting for the risks of future intertemporal shifts in the economic environment. We further use our methodology to derive the probability of a crash under the physical measure. We highlight the new components of the equity risk premium and crash probabilities, compared to estimates that do not account for intertemporal hedging. These components capture conditional moments of market returns beyond the forecast horizon. All proofs are provided in Appendix A.

2.1 Equity risk premium in a multi-period economy

We consider a three-date (two-period) economy with dates t, T_1, and T_N such that $t < T_1 < T_N$.[2], and a representative agent. T_1 is the forecast horizon at which we aim to build a lower bound for the equity risk premium. T_N is the representative agent's investment horizon. We assume that this economy is arbitrage-free, which guarantees the existence of a stochastic discount factor (SDF) and of a risk-neutral measure. For simplicity, we assume no interest rate risk.

At time t, the representative agent invests their wealth W_t in an asset delivering the risk-free gross return $R_{f,t\to T_1}$, and in a set of risky assets delivering gross returns $R_{k,t\to T_1}, k = 1, ..., N$. Under no-arbitrage conditions, the expected excess return on each risky asset from time t to time T_1 can be expressed as the risk-neutral covariance between the asset return and the inverse of the one-period SDF from t to T_1, $m_{t\to T_1}$:

$$\mathbb{E}_t\left(R_{k,t\to T_1} - R_{f,t\to T_1}\right) = \mathbb{COV}_t^*\left(R_{k,t\to T_1}, \frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right). \tag{1}$$

See Appendix A.1 for the proof of this identity, also used by Chabi-Yo and Loudis (2019).

Let us aggregate the gross returns on risky assets between t and T_1 in the vector $R_{t\to T_1}$, and between T_1 and T_N in the vector $R_{T_1\to T_N}$. At time T_1, the agent rebalances their portfolio. The terminal wealth of the representative agent at their investment horizon T_N is

$$\begin{aligned} W_{T_N} &= W_t\left(R_{f,t\to T_1} + \omega_t^\intercal(R_{t\to T_1} - R_{f,t\to T_1})\right)\left(R_{f,T_1\to T_N} + \omega_{T_1}^\intercal(R_{T_1\to T_N} - R_{f,T_1\to T_N})\right) \\ &= W_t\left(\omega_t^\intercal R_{t\to T_1}\right)\left(\omega_{T_1}^\intercal R_{T_1\to T_N}\right), \end{aligned} \tag{2}$$

[2] We use the notation $T_0 = t$ for simplicity.

where ω_t and ω_{T_1} are the vectors of portfolio weights in risky assets at times t and T_1, respectively. $R_{f,T_1\to T_N}$ is the risk-free gross return from time T_1 to time T_N.

The investor chooses the portfolio weights $\{\omega_t, \omega_{T_1}\}$ so as to maximize their expected utility of terminal wealth[3] over the period $[t, T_N]$.[4] The main innovation of our approach is that the investor considers what happens beyond the forecast horizon T_1, up to the representative agent's investment horizon T_N, when solving the portfolio allocation problem. In contrast, the equity risk premium estimates of Martin (2017a); Chabi-Yo and Loudis (2020); Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025) are derived in an economy in which the investor maximizes the expected utility of wealth over $[t, T_1]$.

Provided that no-arbitrage conditions hold in this economy, and assuming that the gross return on the market can be used as proxy for the return on aggregate wealth, we show in Appendix A.2 that we can express the one-period stochastic discount factor (SDF) from t to T_1, $m_{t\to T_1}$ as,

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\upsilon_{T_1}}{\mathbb{E}_t^*\left(\upsilon_{T_1}\right)} \text{ with } \upsilon_{T_1} = \mathbb{E}_{T_1}^*\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_t R_{M,t\to T_N}\right]}\right), \tag{3}$$

where $\mathbb{E}_{T_1}^*(\cdot)$ denotes the expected value at time T_1 under the risk-neutral measure, $R_{f,t\to T_N}$ is the risk-free gross return from t to T_N and $R_{M,t\to T_N}$ is the gross market return.

The one-period SDF thus depends on the marginal utility of wealth at the representative agent's investment horizon T_N. This result stands in contrast to the SDF of Martin (2017a), Chabi-Yo and Loudis (2020), Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025), which do not depend on any quantity beyond the forecast horizon T_1.

[3]The utility function $u[.]$ is well-defined, its derivatives up to order four exist, and their signs obey the following economic theory restriction: $\text{sign}\left(u^{(i)}[\cdot]\right) = \text{sign}(-1)^{i+1}$ (Eeckhoudt and Schlesinger, 2006; Deck and Schlesinger, 2014).

[4]We exclude consumption for simplicity. In the Internet Appendix G, we show that under minimal assumptions regarding the sign of the correlation between the consumption wealth ratio and the market return, the equity risk premium derived in this section still holds.

We do not assume that we know the functional form of the marginal utility function.[5] We use a Taylor expansion series of the inverse of the marginal utility to produce a one-period SDF of the form[6]

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \approx \frac{(1+z_{T_1})}{\mathbb{E}_t^*(1+z_{T_1})}, \tag{4}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t\to T_1}}(R_{M,t\to T_1}-R_{f,t\to T_1}) + \frac{a_{2,t}}{R^2_{f,t\to T_1}}(R_{M,t\to T_1}-R_{f,t\to T_1})^2 + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^{*(2)}_{T_1\to T_N} \tag{5}$$

and $\mathbb{M}^{*(2)}_{T_1\to T_N} = \mathbb{E}^*_{T_1}(R_{M,T_1\to T_N}-R_{f,T_1\to T_N})^2$ is the risk-neutral variance at time T_1. The coefficients $a_{1,t}$, $a_{2,t}$ and $a_{3,t}$ in the Taylor expansion series are functions of the investor's risk, skewness and kurtosis tolerance parameters τ_t, ρ_t and κ_t:

$$a_{1,t} = \frac{1}{\tau_t}, \qquad a_{2,t} = \frac{(1-\rho_t)}{\tau_t^2}, \qquad a_{3,t} = \frac{(\kappa_t+1-2\rho_t)}{\tau_t^3}, \tag{6}$$

where

$$\tau_t = -\frac{u^{(1)}[W_t R_{f,t\to T_N}]}{W_t R_{f,t\to T_N} u^{(2)}[W_t R_{f,t\to T_N}]}, \tag{7}$$

$$\rho_t = \frac{1}{2!}\frac{u^{(3)}[W_t R_{f,t\to T_N}]\,u^{(1)}[W_t R_{f,t\to T_N}]}{\left(u^{(2)}[W_t R_{f,t\to T_N}]\right)^2}, \tag{8}$$

$$\kappa_t = \frac{1}{3!}\frac{u^{(4)}[W_t R_{f,t\to T_N}]\left(u^{(1)}[W_t R_{f,t\to T_N}]\right)^2}{\left(u^{(2)}[W_t R_{f,t\to T_N}]\right)^3}. \tag{9}$$

The proof of Equations (4) to (9) is in Appendix A.3.[7]

[5]In contrast, the equity risk premium estimates of Martin (2017a), Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025) assume log utility. As logarithmic utility implies that the representative agent is myopic, this assumption would not be suitable in our setting to study the effect of intertemporal hedging.

[6]In Section 6, we extend our framework to allow the representative agent to rebalance her portfolio at discrete times t such as $T_1 \le t < T_N$.

[7]Our baseline results do not involve kurtosis preference, but we define the kurtosis preference parameter together with the risk aversion and skewness preference parameters for completeness. We will use the kurtosis preference parameter in Section E, where we apply third-order Taylor expansion series.

Equations (4) and (5) show that the inverse of the SDF is a function of three terms: the excess market return, the squared excess market return, and the market risk-neutral variance $\mathbb{M}^{*(2)}_{T_1\to T_N}$ at time T_1. This risk-neutral variance term is new and only arises in a two-period economy.[8] In contrast, the risk, skewness and kurtosis tolerance parameters in Equations (7)–(9) differ from those derived by Chabi-Yo and Loudis (2020) but we expect this difference to be small. They indeed involve risk-free returns between t and T_N, instead of these returns between t and T_1. Due to the shape of the yield curve, the risk-free returns from T_1 to T_N tend to be close to 1 empirically.

We present our main theoretical result in Proposition 1 below. In this proposition, we combine the risk premium expression in Equation (1) with the SDF expression (4) to provide a closed-form solution to the conditional expected excess market return in terms of risk-neutral moments.

Proposition 1 *Up to a second-order expansion-series, consistent with (4), under no-arbitrage conditions, the equity risk premium is a function of risk neutral return moments:*

$$RP_{t\to T_1,T_N} \equiv \mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{10}$$

where

$$\mathbb{LEV}^*_t = \mathbb{COV}^*_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}, \mathbb{M}^{*(2)}_{T_1\to T_N}\right), \tag{11}$$

and

$$\mathbb{M}^{*(n)}_{T_i\to T_j} = \mathbb{E}^*_{T_i}\left(R_{M,T_i\to T_j} - R_{f,T_i\to T_j}\right)^n, \text{ with } i<j,\ i=0,1,\ T_0=t, \text{ and } n>1. \tag{12}$$

[8] We know from Merton's ICAPM that shocks to risk can generate hedging demand and so can be priced. But Merton's ICAPM shows that market physical volatility is determinant in explaining the expected excess return on a stock. Merton's model argument is not about risk neutral market volatility. Strong evidence of time-varying volatility risk premium suggests that the risk neutral market variance and the physical market variance are distinct and carry different sets of information. Thus, our theoretical results are distinct from implications from Merton's ICAPM model. Further, Merton's ICAPM was not intended to derive closed-form expression of the risk premium on the market as a function of risk neutral correlation between market return and market risk neutral volatility.

Proof. See Appendix A.4. ■

Two new terms contribute to the equity risk premium in a two-period economy, compared to a one-period economy: the risk-neutral leverage effect $\mathbb{LEV}_t^*$ and the expected future variance $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}$. Our conjecture is that the risk-neutral leverage effect, $\mathbb{LEV}_t^*$, is negative. Provided that $\rho_t \geq 1$, that is, $a_{2,t}$ is negative, a negative risk-neutral leverage contributes positively to the conditional equity risk premium. This increase captures the compensation required by investors for exposure to the future risk-neutral variance.[9]

Furthermore, under the assumptions: (i) odd market risk neutral moments are negative and (ii) conditions $1/\tau_t \geq 1$ and $\rho_t \geq 2$ hold, we can further restrict bound (10):

$$RP_{t\to T_1,T_N} \geq \frac{\frac{1}{R_{f,t\to T_1}}\mathbb{M}_{t\to T_1}^{*(2)} - \frac{1}{R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(3)} - \frac{1}{R_{f,T_1\to T_N}^2}\mathbb{LEV}_t^*}{1 - \frac{1}{R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(2)} - \frac{1}{R_{f,T_1\to T_N}^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}}. \tag{13}$$

The proof is in Appendix A.5.

Finally, we show in Appendix G.2 that, when consumption is introduced in the representative agent problem, under minimal realistic assumptions, our restricted measure of risk premium remains a lower bound to the expected market return.

2.2 Comparison to existing bounds

The computation of the risk-neutral leverage effect $\mathbb{LEV}_t^*$ and of the expected future variance $\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}$ relies on information from options of maturities T_1 and T_N. In contrast, the existing bounds of Martin (2017a) and Chabi-Yo and Loudis (2020) and the equity risk premium estimates of Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025) only rely on options with maturity T_1.

The bound in Martin (2017a) corresponds to the expected excess return when the representative agent is endowed with a myopic log utility. The log utility assumption corresponds to $\tau_t = 1$ ($a_{1,t} = 1$) and $\rho_t = 1$ ($a_{2,t} = 0$), making higher-order moments and the lever-

[9]There is a vast literature on leverage under the physical measure. But to our knowledge, this paper is the first to show the relevance of the risk-neutral leverage for computing the equity risk premium.

age under the risk-neutral measure irrelevant in a two-period economy. In case of a CRRA utility with relative risk aversion α, an equivalent expression of (10) can be obtained by recognizing that Equations (7)-(9) reduce to $\frac{1}{\tau_t} = \alpha$, $\rho_t = \frac{1}{2}\frac{(\alpha+1)}{\alpha}$, and $\kappa_t = \frac{1}{6}\frac{(\alpha+1)(\alpha+2)}{\alpha^2}$. In case of a CARA utility with absolute risk aversion $\widetilde{\alpha}$, an equivalent expression of (10) can be obtained by recognizing that Equations (7)-(9) reduce to $\frac{1}{\tau_t} = \alpha_t$, $\rho_t = \frac{1}{2}$, and $\kappa_t = \frac{1}{6}$ with $\alpha_t = \widetilde{\alpha} W_t R_{f,t\to T_N}$.

To compare our measure to the one of Chabi-Yo and Loudis (2020), we first introduce Corollary 2, which expresses the conditional expected excess market return as a weighted average of two risk premia.

Corollary 2 *Up to a second-order expansion-series, consistent with (4), the expected excess market return is a weighted average of two premia:*

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \pi_t^* RP_{t\to T_1} + \left(1-\pi_t^*\right)\mathbb{RP}^{\upsilon}_{t\to T_N}, \tag{14}$$

where

$$RP_{t\to T_1} = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}, \tag{15}$$

and

$$\mathbb{RP}^{\upsilon}_{t\to T_N} = \frac{\mathbb{LEV}^*_t}{\mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{16}$$

with

$$\pi_t^* = \frac{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}. \tag{17}$$

Proof. See Appendix A.6. ■

The first risk premium $RP_{t\to T_1}$ in Equation (15), which corresponds to the measure obtained by Chabi-Yo and Loudis (2020) in a one-period economy, involves the risk-neutral variance and skewness of market returns.[10] The novelty of decomposition is the contribution

[10] Chabi-Yo and Loudis (2020) derive their expression using a third-order expansion-series of the inverse marginal utility. The expression provided in Equation (15) is the counterpart of the one given by Chabi-Yo and Loudis (2020) when using a second-order expansion-series of the inverse marginal utility.

of the risk-neutral leverage effect $\mathbb{LEV}_t^*$ and expected future variance $\mathbb{E}_t^* \mathbb{M}_{T_1 \to T_N}^{*(2)}$ to the conditional risk premium.

To derive their equity risk premium estimate, Tetlock, McCoy, and Shah (2024) assume that the representative agent can trade the market, as well as derivatives securities that make higher order moments of market returns tradable. While this approach seems more general than ours, we show in Appendix A.7 that their SDF is equivalent to the one of Chabi-Yo and Loudis (2020), and therefore a special case of ours. We further show in Appendix A.8 that this SDF can be rewritten in the form of the SDF used by Crescini, Trojani, and Vedolin (2025), i.e., as excess returns on a portfolio of index and options with maturity T_1. Similarly, our SDF can be written as excess returns on a portfolio of index, options with maturity T_1 but also options with maturity T_N, sold at T_1. The additional returns on options with maturity T_N arise due to intertemporal hedging. The decomposition can be found in Appendix A.9.

2.3 Intertemporal hedging demand premium

Building on Corollary 2, we define the intertemporal hedging demand premium as the difference between the equity risk premium from t to T_1 in our multi-period economy and the premium in a one-period (two-date) economy.

Corollary 3 *Up to a second-order expansion-series, the intertemporal hedging premium is*

$$IHP_{t \to T_1, T_N} = \underbrace{\pi_t^* RP_{t \to T_1} + (1 - \pi_t^*) \, \mathbb{RP}_{t \to T_N}^{\upsilon}}_{\substack{\text{One-period expected excess} \\ \text{return in a two-period economy}}} - \underbrace{RP_{t \to T_1},}_{\substack{\text{One-period expected excess} \\ \text{return in a one-period economy}}} \tag{18}$$

and can be alternatively written as

$$IHP_{t \to T_1, T_N} = (\pi_t^* - 1) \left(RP_{t \to T_1} - \mathbb{RP}_{t \to T_N}^{\upsilon} \right), \tag{19}$$

where $RP_{t\to T_1}$, $\mathbb{RP}^{v}_{t\to T_N}$, π^*_t *are defined in* (15), (A28) *and* (17), *respectively.*

Under the assumption that $\rho_t \geq 1$ and $\mathbb{LEV}^*_t \leq 0$, the intertemporal hedging demand premium $IHP_{t\to T_1,T_N}$ is positive, i.e., our risk premium, $RP_{t\to T_1,T_N}$, is higher than $RP_{t\to T_1}$. The differences in the shape of the term structure of risk premia depend on how $IHP_{t\to T_1,T_N}$ varies across T_1.

2.4 Physical variance

Similar to the equity risk premium, the conditional variance can be written as a function of risk-neutral moments between t and the forecast horizon T_1, but also intertemporal hedging terms using information up to the representative agent's investment horizon T_N.

Proposition 4 *Up to a second-order expansion-series, consistent with (4), under no-arbitrage conditions, the conditional variance of returns under the physical measure is a function of risk neutral return moments:*

$$\begin{aligned}\mathbb{V}ar_t &\equiv \mathbb{E}_t\left(R_{M,t\to T_1} - \mathbb{E}_t R_{M,t\to T_1}\right)^2 \\ &= \mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 - \left(\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right)^2 \end{aligned} \tag{20}$$

where $\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)$ *is given by Equation* (10),

$$E_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \frac{\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(4)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\left(\mathbb{LEK}^*_t + \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}\right)}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{21}$$

and

$$\mathbb{LEK}^*_t = \mathbb{COV}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2, \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right). \tag{22}$$

Proof. See Appendix A.10. ■

This estimate of the physical variance presents two major advantages. First, it is computable readily from available options and does not require high-frequency data, as in Tetlock, McCoy, and Shah (2024) for example. Second, it is model-free and relies on minimal assumptions, similar to our estimate of the equity risk premium.

In a two-period economy (without intertemporal hedging), the conditional variance (20) reduces to

$$\mathbb{E}_t \left(R_{M,t\to T_1} - R_{f,t\to T_1} \right)^2 = \frac{\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{1,t}}{R_{f,t\to T_1}} \mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{M}^{*(4)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{M}^{*(2)}_{t\to T_1}}. \tag{23}$$

2.5 Probability of a market crash

We finally use our methodology to obtain the probability of a market crash under the physical measure. We define the probability of a crash as $\mathbb{P}_t \left(R_{M,t\to T_1} < \alpha \right)$ where α is given. For example, $\alpha = 0.8$ for a 20% market crash. We then exploit the no-arbitrage assumption that allows us to move from the physical measure to the risk-neutral measure. While the coefficient α could be time-varying or constant, we remove the time subscript on α to ease notations.

Proposition 5 *Up to a second-order expansion-series of the inverse marginal utilities, the conditional crash probability defined as* $\Pi_{t\to T_1,T_N}[\alpha] \equiv P_t \left(R_{M,t\to T} < \alpha \right)$ *can be expressed in terms of risk neutral quantities*

$$\Pi_{t\to T_1,T_N}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}} \mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{M}^{*(2)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{M}^{*}_{t,v}[\alpha]}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{24}$$

where $\mathbb{M}^{*(n)}_{t\to T_1}[\alpha] = \mathbb{E}^*_t \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1} \right)^n \mathbb{1}_{R_{M,t\to T_1} < \alpha} \right)$ *and* $\mathbb{M}^*_{t,v}[\alpha] = \mathbb{E}^*_t \left(\mathbb{M}^{*(2)}_{T_1\to T_N} \mathbb{1}_{R_{M,t\to T_1} < \alpha} \right)$.

Proof. See Appendix A.11. ■

Proposition 5 shows that truncated market moments matter for extracting the probability of the market crash. But more importantly, it shows that when the SDF is a function of future risk-neutral volatility as in (5), the tail of the distribution of risk-neutral volatility, captured by $\mathbb{M}^*_{t,v}[\alpha]$, has an impact on the probability of a crash. When the expected future volatility is not present in the SDF (4), i.e., with no intertemporal hedging, the probability of a market crash reduces to

$$\Pi_{t\to T_1}[\alpha] \equiv P_t\left(R_{M,t\to T} < \alpha\right) = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha]}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}. \tag{25}$$

3 Empirical framework

We show in this section how the theoretical expressions derived in Section 2 can be brought to the data.

3.1 Leverage and future risk-neutral variance

The equity risk premium and crash probabilities are functions of risk-neutral moments, including the risk-neutral leverage effect $\mathbb{LEV}^*_t$ and the expected future risk-neutral variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$. These moments involve T_1- and T_N-horizon quantities. While closed-form expressions of risk-neutral moments $\mathbb{M}^{*(n)}_{t\to T_1}$, for a given horizon T_1, are readily available from option prices using the spanning formula of Carr and Madan (2001a) and Bakshi and Madan (2000), similar closed-form expressions are not directly available for the risk-neutral leverage effect and the expected future risk-neutral variance.

We propose a method to compute $\mathbb{LEV}^*_t$ and $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$ using options with maturity T_1 and T_N. The key objective here is to isolate the component of $\mathbb{M}^{*(2)}_{T_1\to T_N}$ that is correlated with the excess market return from t to T_1, $R_{M,t\to T_1} - R_{f,t\to T_1}$. We assume that this component can be written as a nonlinear function f of $R_{M,t\to T_1} - R_{f,t\to T_1}$:

$$\mathbb{M}^{*(2)}_{T_1\to T_N} = \theta_t f[R_{M,t\to T_1} - R_{f,t\to T_1}] + \epsilon_t, \tag{26}$$

with $\mathbb{E}^*_t\left(\epsilon_t | R_{M,t\to T_1}\right) = \mathbb{E}^*_t\left(\epsilon_t\right) = 0.$[11]

[11] Note that Equation (26) is distinct from the assumption that the risk neutral volatility follows a GARCH process. The returns in the left- and right-handsides are different: the risk-neutral variance on the left

Multiplying both sides of Equation (26) by $R^2_{M,t\to T_1}$ and taking the time-t risk-neutral expectation, we obtain

$$\theta_t = \frac{\mathbb{M}^{*(2)}_{t\to T_N} - R^2_{f,T_1\to T_N}\mathbb{M}^{*(2)}_{t\to T_1}}{\mathbb{E}^*_t\left(R^2_{M,t\to T_1} f[R_{M,t\to T_1} - R_{f,t\to T_1}]\right)}, \tag{27}$$

The expected future variance $\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}$ is obtained by taking the time-t risk-neutral expectation of (26), and the leverage $\mathbb{LEV}^*_t$ from the time-t risk-neutral covariance (11).

The final step consists to choose the function $f[\cdot]$. We use $(R_{M,t\to T_1} - R_{f,t\to T_1})^2$ for two reasons. First, the numerator of θ_t is always positive in the data. Therefore, our choice of function $f[\cdot]$ ensures that the expected future variance is a positive number. Second, as $(R_{M,t\to T_1} - R_{f,t\to T_1})^2$ is a proxy for the first period conditional variance, this function captures the well-documented fact that conditional variances are highly positively correlated over time.

With this choice for the function $f[\cdot]$, we have,

$$\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N} = \frac{\mathbb{M}^{*(2)}_{t\to T_N} - R^2_{f,T_1\to T_N}\mathbb{M}^{*(2)}_{t\to T_1}}{\mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1}\mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}, \tag{28}$$

and,

$$\mathbb{LEV}^*_t = \frac{\mathbb{M}^{*(2)}_{t\to T_N} - R^2_{f,T_1\to T_N}\mathbb{M}^{*(2)}_{t\to T_1}}{\mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1}\mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}. \tag{29}$$

Substituting Equations (28) and (29) in Equation (10) highlights that our expression for the equity risk premium is a non-linear function of T_1-return moments and the T_N-return variance. Using this decomposition, we show in Appendix A.12 that $RP_{t\to T_1,T_N}$ is increasing in the investment horizon T_N.

handside depends on the return from T_1 to T_N, while right-handside is a function of the return from t to T_1. We further show in the Internet Appendix G.1, that the key risk-neutral volatility dynamics implied by (26) is distinct from that of a GARCH process. Hence, a direct comparison cannot be made with a GARCH process.

3.2 Data

We use the S&P 500 index as the market portfolio. We obtain volatility surfaces, index levels, and forward term structures for the S&P 500 Index and the zero-coupon rate term structures from Ivy DB OptionMetrics. The data cover the period January 1996 to February 2023. When computing the excess returns on the S&P 500 index before January 1996, we use its level and the Fama term structures on U.S. Treasuries from the Center for Research in Security Prices (CRSP).

Implementing our risk premia requires the evaluation of different functions of risk-neutral expected values. We estimate these expected values at the end of each month and for each maturity provided in OptionMetrics' Volatility Surface File (10, 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 days). We refer to these maturities as one week, one month, two months, one quarter, four, five, six, and nine months, one year, 18 months, and two years.

We import annualized continuously-compounded zero-coupon yields from Jing Cynthia Wu's website, Liu and Wu (2021). We interpolate the term structure of zero-coupon rates using Nelson and Siegel (1987) model to find each maturity's risk-free rate.

Following Chabi-Yo, Dim, and Vilkov (2021), we define a moneyness grid of 1,000 equally spaced points from 1/3 to 3. We use a piecewise cubic Hermite polynomial to interpolate the implied volatility surface to the moneyness grid. We extrapolate the implied volatility using the closest value for moneyness points outside the implied volatility surface. Finally, we use the Black-Scholes formula to convert implied volatilities to call and put prices for each moneyness level.

3.3 Risk-neutral moments

We compute the risk-neutral moments of market returns and excess returns using the spanning formula of Carr and Madan (2001a) and Bakshi and Madan (2000), as described in Appendix B.1. We report in Figure 1 excess return moments over time for horizons of one week to two years. To compare values across horizons, we report the annualized volatility in the top graph $\left(\sqrt{(365/T_1)\,\mathbb{M}^{*(2)}_{t\to T_1}}\right)$, skewness in the middle graph $\left(\mathbb{M}^{*(3)}_{t\to T_1}/\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^{\frac{3}{2}}\right)$, and kurtosis in the bottom graph $\left(\mathbb{M}^{*(4)}_{t\to T_1}/\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^{2}\right)$. We also report the expected future second moments and leverage in Figure 2, using Equations (28) and (29).

Risk-neutral volatilities and expected future volatilies vary over time, reaching a peak during the financial crisis of 2008. Risk-neutral skewness values are almost always negative and decrease over the sample period. Risk-neutral kurtosis values range between three and eight and trend upward over the sample period. The risk-neutral leverage effect is always negative and exhibits large time variations.

3.4 Preference parameters

The expressions for the one-period equity risk premium and crash probabilities provided in Section 2 are all functions of the investor's preference parameters τ_t and ρ_t.

Following Chabi-Yo and Loudis (2020), we first set these parameters to $\tau_t = 1$ and $\rho_t = 2$ for all t, which is equivalent to $a_{1,t} = 1$ and $a_{2,t} = -1$. Setting these parameters to constants yields tractable equity and variance risk premia, which can be computed instantaneously using readily available options. We derive our main results in Section 4 based on these values. In Section 5, we attempt to estimate the preference parameters but find little improvement in out-of-sample results. We further show that our main findings do not change.

4 Results

In this section, we describe our estimates of equity risk premium $RP_{t\to T_1,T_N}$ and discuss their ability to capture future returns. We show that $RP_{t\to T_1,T_N}$ outperforms the existing premia for most horizons T_N, and underline the existence of an implied investors' horizon, which corresponds to the value of T_N that best matches the data. This horizon is long in quiet times, when the probability of crash is low, and short during market turmoil, when the probability of crash is high.

4.1 Estimated equity risk premium

We report in Figure 3 the time series of equity risk premia for horizons of T_1 equal to one and six months, using investment horizons T_N of one and two years. In line with theory, $RP_{t\to T_1,T_N}$ is larger than $RP_{t\to T_1}$ over the entire sample period, for both forecast horizons T_1. Furthermore, $RP_{t\to T_1,2y}$ is always larger than $RP_{t\to T_1,1y}$, in line with $RP_{t\to T_1,T_N}$being increasing in the investment horizon T_N. The summary statistics in Table A3 further show that the difference between $RP_{t\to T_1,2y}$ and $RP_{t\to T_1,1y}$ is larger for shorter forecast horizons T_1, indicating that intertemporal hedging matters most for short maturity forecasts. This result can be due to two reasons. Either the intertemporal hedging premium becomes smaller for longer forecast horizons, or we only capture part of it because of our maximum T_N of two years. For long forecast horizons, longer-maturity options may be needed to fully capture the intertemporal hedging premium.

We further compare our equity risk premium estimate to the Implied Equity Risk Premium (IERP) of Tetlock, McCoy, and Shah (2024) in Figure 4.[12] The investment horizons are chosen such that the two time series be as close to each other as possible. This results in T_N equal to one year for $T_1 =$ one month, and T_N equal to two years for $T_1 =$ six months.

[12] We thank Paul Tetlock for providing us with the growth optimal weights needed to calculate the IERP, from January 1997 to December 2021. Based on these weights, we computed the IERP for all forecast horizons over this time period. For comparability, all tables and graphs with the IERP are from January 1997 to December 2021.

Using these values of T_N, the two risk premium estimates are close during quiet times. During NBER recessions, the IERP is larger than our premium. The summary statistics in Table A3 confirm and complement these results. For short forecast horizons T_1, the IERP is on average close to our premium with $T_N =$ one year, and smaller than our premium with $T_N =$ two years. As our estimate is a lower bound for the equity risk premium, whereas the IERP is a point estimate, the gap between the IERP and our premium is a lower bound to the intertemporal hedging premium. When the forecast horizon increases, the gap reduces and then disappears. This result further indicates that for forecast horizons of six months and more, we would need options with maturity longer than two years to produce a more precise measure of the intertemporal hedging premium.

Figure 5, Panel A, displays the intertemporal hedging premium, estimated by the difference between our risk premium $RP_{t\to T_1,T_N}$ and $RP_{t\to T_1}$, for a forecast horizon of one month. Intertemporal hedging accounts for about half of the total equity risk premium using an investment horizon of one year, and up to 70% of the equity risk premium using an investment horizon of two years. These ratios are larger outside NBER recessions. During these recessions, intertemporal hedging is about a quarter of the total premium.

Panel B shows that for longer forecast horizons (six months), intertemporal hedging accounts for a smaller fraction of the total risk premium. But this fraction with $T_N = 2$ years is about twice the fraction with $T_N = 1$ year, indicating that it would grow further had we access to longer maturity options.

4.2 Conditional variance and variance risk premium

Figure 6, compares the conditional physical variance obtained when ignoring intertemporal hedging, to the its analogue with intertemporal hedging. Panel A corresponds to forecast horizon T_1 at 1 month, and Panel B to T_1 at six months. For both forecast horizons, the physical variance is lower with intertemporal hedging throughout the time period than without intertemporal hedging. We observe large differences in times of market turmoil.

Figure 7 displays the corresponding variance risk premium, computed as the difference between the conditional variance under the physical measure, and under the risk-neutral measure. As the risk-neutral variance is computed from options, it does not depend on the investment horizon. Therefore, the lower physical variance with intertemporal hedging translates directly into a variance risk premium that is larger in magnitude, and more negative than without intertemporal hedging.

We conclude that intertemporal hedging yields increases both in the equity and in the variance risk premium.

4.3 Out-of-sample performance

We study whether accounting for intertemporal hedging improves the out-of-sample performance of the equity risk premium. To assess the change in performance, we use two different metrics.

First, we follow Welch and Goyal (2008) and Campbell and Thompson (2008) in computing the out-of-sample R^2 measure as,

$$R^2_{OOS} = 1 - \frac{\sum_t \left(r_{M,t\to T_1} - \tilde{r}_{M,t\to T_1}\right)^2}{\sum_t \left(r_{M,t\to T_1} - \bar{r}_{M,t\to T_1}\right)^2}, \tag{30}$$

where $r_{M,t\to T_1} = R_{M,t\to T_1} - R_{f,t\to T_1}$ capture the ex-post realized excess returns at horizon T_1, $\bar{r}_{M,t\to T_1}$ is the sample average of excess returns at horizon T_1 prior to week t and $\tilde{r}_{M,t\to T_1}$ is a risk premium forecast at time t. A positive R^2_{OOS} indicates that the prediction $\tilde{r}_{M,t\to T_1}$ is more accurate than the past average realized returns, while a negative R^2_{OOS} would favor the past average realized returns.

We report in Panel A of Table 2 the R^2_{OOS}, in percent, for $\tilde{r}_{M,t\to T_1} = RP^{Log}_{t\to T_1}$, $RP_{t\to T_1}$, and $RP_{t\to T_1,T_N}$ over the period 1997 to 2021.[13] Forecast horizons T_1 range from one month to 18 months and all available investment horizons $T_N > T_1$ up to two years are considered.

For all forecast horizons T_1, $RP_{t\to T_1}$ outperforms $RP^{Log}_{t\to T_1}$, and $RP_{t\to T_1,T_N}$ outperforms $RP_{t\to T_1}$ for almost all investment horizons T_N. In particular, for the 10-day forecast horizon, $RP_{t\to T_1}$and $RP^{Log}_{t\to T_1}$both perform worse, out-of-sample, than a forecast based on the past average realized returns, as they have negative R^2_{OOS}. In contrast, $RP_{t\to T_1,T_N}$ exhibits positive R^2_{OOS} for T_N between three months and one year. We test whether the differences in performance between $RP_{t\to T_1}$and $RP_{t\to T_1,T_N}$ are statistically significant, using the Diebold and Mariano (1995) test. The outperformance of $RP_{t\to T_1,T_N}$ is significant for forecast horizons T_1 between three and nine months, and for most T_N. Therefore, our results indicate that accounting for intertemporal hedging in the equity risk premium leads to a large and significant increase in out-of-sample forecast performance.

Inspection of the R^2_{OOS} achieved by $RP_{t\to T_1,T_N}$ in Table 2 reveals the importance of T_N on the performance of our risk premium. For all forecast horizons T_1, the R^2_{OOS} increases with T_N, up to a given T_N. For $T_1 = 10$ days, it reaches its maximum at $T_N = 9$ months, for $T_1 = 1$ month at $T_N = 9$ and 12 months, and for $T_N = 2$ months at $T_N = 18$ months. For all T_1 equal to 10 days, 1 month and 2 months, the R^2_{OOS} drops after reaching it maximum value, when increasing T_N. For T_1 larger than two months, the R^2_{OOS} increases up to $T_N =$ 24 months. The pattern of R^2_{OOS} that we observe for $T_1 \leq 2$ months suggests that for $T_1 > 2$ months, there exists an optimal T_N beyond 24 months. Overall, the R^2_{OOS} suggests the existence of an optimal $T_N > T_1$. The past column indicates the performance of a prediction based on the average prediction across investment horizons T_N. Such prediction achieves R^2_{OOS} that are all larger than those of $RP_{t\to T_1}$.

[13] The results are reported over the period 1997 to 2021 as this is the period over which we have the IERP. Out-of-sample R^2 have been computed for $RP^{Log}_{t\to T_1}$, $RP_{t\to T_1}$ and $RP_{t\to T_1,T_N}$ over the full period from 1996 fo February 2023. They are comparable to those reported in Table 2.

The comparison of our risk premium estimates to the IERP of Tetlock, McCoy, and Shah (2024) is less straightforward. For short forecast horizons, up to 4 months, both $RP_{t\to T_1}$ and $RP_{t\to T_1,T_N}$ outperform the IERP, for almost all values of T_N. For T_1 at 10 days, the IERP yields a negative R^2, like other estimates that do not account for intertemporal hedging. Our estimate is the only one to yield a positive R^2, suggesting that even for short forecast horizons, intertemporal hedging is important. For forecast maturities at 4 months and more, the IERP outperforms $RP_{t\to T_1}$, and $RP_{t\to T_1,T_N}$ outperforms the IERP for T_N above a given threshold. For $T_1 = 4$ months, $RP_{t\to T_1,T_N}$ performs better than the IERP for all T_N larger than 9 months. For $T_1 = 6$ months, $RP_{t\to T_1,T_N}$ only performs better than the IERP for $T_N = 2$ years. For $T_1 = 1$ year, the IERP outperforms $RP_{t\to T_1,T_N}$. These results confirm the need for options with maturity longer than 2 years, to accurately estimate the intertemporal hedging premium at forecast horizons of more than 4 months.

Second, we construct market-timing strategies and compute realized mean-variance certainty equivalents. While the R^2_{OOS} reported in Panel A of Table 2 show that our methodology captures the expected excess market return, results in Panel B combine both first and second moment predictions. For each forecasting method, we compute the weight of the market portfolio in the optimal portfolio at time t as,

$$\omega_{t\to T_1} = \frac{\tilde{r}_{M,t\to T_1}}{\gamma \tilde{\sigma}^2_{t\to T_1}} \tag{31}$$

where γ is a risk aversion parameter and $\tilde{\sigma}^2_{t\to T_1}$ is the physical variance of returns computed for each method, as described in Section 4.2. Then, we compute the realized mean-variance certainty equivalent as,

$$CE = E(r_{p,t\to T_1}) - \frac{\gamma}{2}\text{Var}(r_{p,t\to T_1}), \tag{32}$$

where $r_{p,t} = r_{f,t\to T_1} + \omega_{t\to T_1} r_{M,t\to T_1}$ are portfolio returns. The certainty equivalent is estimated using the sample return average and variance using non-overlapping returns over horizon T_1.

We report realized certainty equivalents annualized in percent for $\gamma = 3$. We find better performance of $RP_{t\to T_1,T_N}$, compared to $RP_{t\to T_1}$ and $RP^{Log}_{t\to T_1}$, for investment horizons T_N up to one year. In line with the results reported in Panel A of Table 2, the certainty equivalents increase with T_N, reaching a maximum for T_N between 9 months and 24 months. Negative values are not displayed. They are obtained for $T_N = 18$ and 24 months due to estimates of the physical variance that are close to zero. We block-bootstrap the time-series of realized portfolio returns to compute the significance of the certainty equivalent differences for each strategy, compared to the one based on $RP_{t\to T_1}$ (see Politis and Romano, 1994).[14] We find that almost all differences between $RP_{t\to T_1,T_N}$-based and $RP_{t\to T_1}$-based strategies are statistically significant at the 5% level, when T_N is less or equal than the investment horizon at which the out-of-sample R^2 reaches its maximum.

Both out-of-sample performance metrics –out-of-sample R^2 and realized certainty equivalents– thus indicate that accounting for intertemporal hedging in the construction of the equity risk premium allows reaching better forecasts of the first and second return moments. Most differences are statistically significant.

These equity risk premium measures are lower bounds for the equity risk premium. As a last analysis, we follow the methodology of Back, Crotty, and Kazempour (2022) and test for the validity and tightness of these bounds. Results are provided in Online Appendix C. For all measures and horizons T_1, we do not reject that any of the bounds are valid lower bounds for T_N up to one year. When increasing T_N, validity gets rejected more often. It is rejected in half of the cases for T_N at two years, confirming that using the longest investment horizon is not always optimal. We furthermore reject, for most bounds, that they are tight. However, as expected, the magnitude of the error from our bound is lower than either $RP_{t\to T_1}$ and $RP^{Log}_{t\to T_1}$.

[14] We use 10,000 bootstrap samples and a mean block length equivalent to three years.

4.4 Implied investors' horizon

We have shown that the out-of-sample performance of the equity risk premium depends on the choice of the investment horizon T_N, for all forecast horizons T_1. Increasing T_N, up to a threshold, improves the out-of-sample performance of our risk premium. The forecast however deteriorates when increasing T_N beyond that threshold. Furthermore, for about half of the forecast horizons T_1, the validity of the bound using $T_N = 2$ years is rejected. In this section, we study whether the optimal threshold is time-dependent, by optimizing the investment horizon T_N used to make the prediction at each time t.

We select the optimal T_N at each time t in sample, by maximizing the R^2 of the forecast over a window of 90 days. This window covers the interval $t - T_1 - 90$ days, up to $t - T_1$, ensuring that there is no look-ahead bias. We denote this optimal time-varying horizon by $T^*_{N,t}$.

Table 3 reports the out-of-sample R^2_{OOS} achieved with $T^*_{N,t}$, and compares them to the R^2_{OOS} achieved with T_N at one and two years, and with the one obtained with the prediction averaged across T_N. Comparing the first three columns ($RP^{Log}_{t\to T_1}$, the IERP of Tetlock, McCoy, and Shah (2024) and $RP_{t\to T_1}$) to the next two columns ($T_N = 1$ year and $T_N = 2$ years) confirms that $RP_{t\to T_1}$ and the IERP outperform $RP^{Log}_{t\to T_1}$ for most T_1, indicating that higher order moments are important for out-of-sample performance. But none of the two $RP_{t\to T_1,T_N}$ outperforms the other systematically. The $T_N = 1$ year estimate tends to perform better for shorter forecast horizons, whereas the $T_N = 2$ years tends to outperform for longer horizons. The average prediction in column 6 yields a more stable outperformance across forecast horizons. The largest gain, for all T_1 except 10 days, is achieved when optimizing upon T_N (last column). The R^2 is 1.5 to twice that of Chabi-Yo and Loudis (2020) and 1.25 to 2 times that of Tetlock, McCoy, and Shah (2024) for maturities up to five months. This increase is statistically significant. Similarly, the largest realized certainty equivalents are obtained when optimizing T_N, for most forecast horizons.

Figure 8 displays in Panel A the estimated risk premium obtained with $T^*_{N,t}$, for T_1 at four months. Panel B depicts the time series of $T^*_{N,t}$. It oscillates between the smallest possible value of T_N (five months) and its largest value (two years). In particular, it is at five months during the two NBER recession periods, and tends to be at two years at most other times. This result is robust to varying the forecast horizon T_1. We thus conclude that in quiet times, the implied investors' horizon is long (here, at its maximum of two years). In contrast, in turbulent times, the implied investors' horizon is short. This conclusion provides empirical evidence in line with the asset pricing model of Hirshleifer and Subrahmanyam (1993), in which investors' time horizon decreases in periods of high uncertainty, due to heightened risk aversion and liquidity needs. It also echoes the results of Campbell and Vuolteenaho (2004), who use a VAR approach to show that investors' horizons shorten in volatile or declining markets because they become more sensitive to "bad beta", i.e., short-term negative cash flow news.

In turbulent times, the short-term horizon implies that intertemporal hedging has a small effect. As a result, the equity risk premium remains close to the one of $RP_{t\to T_1}$. In contrast, it is important in calm times, and pushes the equity risk premium up, since $RP_{t\to T_1,T_N}$ increases with T_N.

Figure 9 shows the variance risk premium resulting from using the investors' implied horizon T^*_N, for forecast horizon T_1. The gap between the blue and the red line is due to intertemporal hedging. This gap is the largest during quiet times. Panel B displays how much this gap is as a the percentage of the total premium. In quiet times, intertemporal hedging accounts for more than 80% of the variance risk premium. This percentage drops during turbulent times, as expected.

To better understand these punctual switches between long and short implied investors' horizon, we investigate the crash probabilities implied by our methodology.

4.5 Crash probabilities

Figure 10, Panel A, displays the conditional probabilities of a $1-\alpha = 10\%$ crash over a horizon of four months. We present the probabilities without intertemporal hedging ($\Pi_{t\to T_1}[\alpha]$), and those obtained with our methodology ($\Pi_{t\to T_1,T_N}[\alpha]$), with an investment horizon T_N of one and two years. Crash probabilities obtained with our method are lower than those without intertemporal hedging. The longer the investment horizon, the lower the crash probabilities.

In Panel B, we compare the crash probabilities from Martin (2017a) ($\Pi^{Log}_{t\to T_1}[\alpha]$) to ours using the implied investors' horizon $T_N = T_N^*$. During turmoil periods, the implied investment horizon is short, and the resulting crash probabilities are close to those without intertemporal hedging. During quiet times, in contrast, the implied investment horizon is long. As the crash probabilities are a decreasing function of the investment horizon, they are thus lower, equal to about half the crash probabilities without intertemporal hedging.

To determine whether these lower probabilities are more accurate, we assess in Table 4 out-of-sample prediction performances. For each horizon, we compute the loss function of our prediction as the negative of the log-likelihood function as,

$$l_{t\to T_1,T_N} = -\left(\mathbb{1}_{r_{M,t\to T_1}<\alpha}\log\left(\Pi_{t\to T_1,T_N}[\alpha]\right) + (1-\mathbb{1}_{r_{M,t\to T_1}<\alpha})(1-\log\left(\Pi_{t\to T_1,T_N}[\alpha]\right))\right).$$

Similarly, we compute the loss function for $\Pi_{t\to T_1}[\alpha]$ and $\Pi^{Log}_{t\to T_1}[\alpha]$, which we respectively denote $l_{t\to T_1}$ and $l^{Log}_{t\to T_1}$. Next, we test the significance of the average difference in loss functions using the Diebold and Mariano (1995) test. We find that our probabilities for a 10% crash, reported in the third column, lead to significantly lower losses (i.e., higher realized log-likelihoods) than other benchmark probabilities for most horizons. Finally, we similarly find significantly superior predictions for a crash size of 20% for all horizons except one week.

5 Estimating preference parameters

Using fixed preference parameters $\tau = 1$ and $\rho = 2$, we find that intertemporal hedging has a large impact on the equity and the variance risk premia. This result holds true during market calm, as the investors' implied horizon is long. Intertemporal hedging then accounts for up to 70% of the equity risk premium, and 80% of the variance risk premium.

In this section, we attempt to estimate the preference parameters, and study the robustness of our results to the choice of these parameters.

5.1 Methodology

We estimate the preference parameters ρ_t and τ_t using a two-stage non-linear least squares approach, similar to Chabi-Yo and Loudis (2020). Specifically, we estimate the coefficients τ_t, ρ_t, $\beta_0^{(1)}$, and $\beta_0^{(2)}$ by minimizing the weighted sum of squared errors $w_1 \epsilon_{t\to T_1}^{(1)\intercal} \epsilon_{t\to T_1}^{(1)} + w_2 \epsilon_{t\to T_1}^{(2)\intercal} \epsilon_{t\to T_1}^{(2)}$ in the following equations,

$$
\begin{aligned}
R_{M,t\to T_1} - R_{f,t\to T_1} &= \beta_0^{(1)} + RP_{t\to T_1,T_N} + \epsilon_{t\to T_1}^{(1)}, \\
\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 &= \beta_0^{(2)} + \mathbb{E}_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \epsilon_{t\to T_1}^{(2)}.
\end{aligned} \tag{33}
$$

In the first stage, we set $w_1 = w_2 = 1$. In the second stage, we weigh each sum of squared errors by the inverse of the standard deviations of first-stage errors. Note that parameters τ_t and ρ_t enter the above equations through $RP_{t\to T_1,T_N}$ and $\mathbb{E}_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2$. We estimate parameters separately for each horizon T_1 and T_N. We restrict the parameter space such that the resulting risk premiums be positive.

5.2 Performance with in-sample estimation

We first estimate the preference parameters over our time sample from 1996 to 2023.[15] We find estimates of τ that range between 0.86 and 0.88 across forecast horizons T_1 and investment horizons T_N. There is therefore very little variation in the estimated τ coefficient, when estimated over the whole period of data. In contrast, the estimates of ρ vary more. Specifically, the estimated ρ for the bound $RP_{t\to T_1}$ decreases sharply with T_1, from 5.06 to 1.20. The estimate of ρ also decreases with T_N. The estimate for $T_N = 2$ years is quite stable, between 1.20 and 1.60 for all T_1.

Figure 11 displays the equity risk premium estimate with these values of τ and ρ. It shows that the resulting risk premium (dotted line) overlaps with the risk premium with τ and ρ set to 1 and 2 (dashed line), for most dates in the time series. This overlap happens because the optimal T_N is two years most of the time. For T_N at two years, the values of τ and ρ are rather close to the fixed values we chose in Section 4, so that the resulting equity risk premium estimates nearly overlap. Discrepancies are observed during turmoil periods, in which the optimal T_N is short, and the estimated ρ is thus equal to values close to or above 5. This high ρ translates into a higher equity risk premium.

Table A6 compares the out-of-sample R^2 achieved when setting $\tau = 1$ and $\rho = 2$, as in Section 4, to those obtained when estimating these parameters. This comparison is informative as to whether the high values of ρ during turmoil periods improve the return forecasts. Column (4) contains the R^2 for our new bound, with T_N optimized, using estimated preference parameters. Estimating these parameters yields R^2 that are still larger than those of $RP_{t\to T_1}$ for all forecast horizons, but they are smaller than those obtained when setting $\tau = 1$ and $\rho = 2$. This lack of forecast performance indicates the high ρ values estimated during market turmoil overfit the data.

[15] As in Chabi-Yo and Loudis (2020), this estimation introduces a look-ahead bias when computing the out-of-sample performance measures. The main goal of this exercise is not to provide an estimation method for the preference parameters, but to question whether the results we obtained in Section 4 still hold with optimal preference parameters. We eliminate this bias in Section 5.3.

Alternatively, we model τ and ρ as linear functions of past three-month returns, and estimate the loadings on these returns and on a constant term over the whole data period. The estimated time series of τ_t are displayed in Panel A of Figure 12, for a forecast horizon T_1 of 1 month. The estimate for $T_N = 1$ year oscillates around a value close to 0.85, i.e., close to the constant estimate found above. τ_t increases and gets closer to 1 during recessions. Decreasing the investment horizon T_N decreases the estimated τ_t, which oscillates around 0.65 for $T_N = 6$ months. This estimate however also increases during turmoil periods. The two estimates therefore nearly overlap during these periods. As the implied investors' horizon is long in quiet times and short in stress times, the resulting time series of τ_t oscillates around a level close to 0.85, throughout the time period, with volatility that is nearly constant. A similar result is achieved with the estimated time series of ρ_t, displayed in Panel B. ρ_t exhibits time series variation, and oscillates around 2. It has low volatility for shorter investment horizons, and higher volatility for longer horizons. This, combined with the long investors' implied horizon during market calm, leads to the conclusion that ρ_t is volatile, with volatility nearly constant over time. Setting τ_t to 1 and ρ_t to 2 thus eliminates the noise of the estimation, improving the signal-to-noise ratio.

Column (5) of Table A6 reports the out-of-sample prediction results obtained when modelling τ and ρ as linear functions of past three-month returns. This additional degree of flexibility improves the performance of our bound for most forecast horizons T_1. This is however at the expense of realized portfolio returns' volatility. Certainty equivalents are for most of them negative, because of the increased volatility.

These results show that a more precise estimation of the preference parameters, using a time series as large as possible, leads to mixed results in terms of out-of-sample performance.

5.3 Telescopic and rolling window estimations

In order to avoid a look-ahead bias, we now estimate a set of parameters using windows of data that do not include any observation beyond prediction time. We consider two types of windows: first, a telescopic window of past observations, which starts in 1996 and expands with time, and second, a rolling window of the past most recent five years. We start measuring performance in 2006, after ten years of observations.

Table A7 reports the out-of-sample prediction results when the parameters τ and ρ are estimated over a telescopic and rolling window. These results are expected to be worse than those of the in-sample estimation described in Section 5.2. As this estimation did not yield satisfactory results with constant preference parameters, but did achieve higher performance with the preference parameters set linear in past 3-month returns, we only report the results of this second specification. The bounds perform better than $RP_{t\to T_1}$ but not as well as with $\tau = 1$ and $\rho = 2$. Furthermore, the improvement compared to $RP_{t\to T_1}$ is no longer significant. Furthermore, the certainty equivalents are for most of them negative. These results illustrate the challenge of achieving good out-of-sample performance when estimating the preference parameters.

Figure 13 displays the fraction of intertemporal hedging demand relative to the total equity risk premium estimate, for forecast horizons of 1 month (blue line) and 6 months (red line), with the different estimation methods of τ and ρ. In all estimations, the intertemporal hedging demand represents up to 60% of the total equity risk premium in quiet times. During recessions, intertemporal hedging disappears, as the implied investors' horizon shrinks to the forecast horizon.

The importance of intertemporal hedging is thus independent of the estimation method chosen for the preference parameters.

In Appendix E, we extend all our results using a third-order approximation. Our results do not change.

6 Multi-period model with portfolio rebalancing

The results derived so far were under the assumption that the representative agent could only rebalance her portfolio at time T_1. In this section, we relax this assumption and let the representative agent rebalance her portfolio at any time t such that $T_1 < t < T_N$. We assess whether this extension changes our main results.

As before, we use a second-order Taylor expansion-series of the inverse marginal utility (term inside the conditional expectation in (3). The novelty is that the Taylor-expansion uses the information that the agent re-balances her portfolio at any time t such that $T_1 < t < T_N$.

We denote

$$R_{M,t\to T_N} = \prod_{j=1}^{N} R_{M,T_{Q_{j-1}}\to T_{Q_j}} \text{ and } R_{f,t\to T_N} = \prod_{j=1}^{N} R_{f,T_{Q_{j-1}}\to T_{Q_j}}$$

with $T_0 = t$ and

$$x_j = R_{M,T_{Q_{j-1}}\to T_{Q_j}} \text{ and } x_{0,j} = R_{f,T_{Q_{j-1}}\to T_{Q_j}}$$

where $Q_{j-1} \in \{0, 1, ..., N-1\}$ and $Q_j \in \{1, ..., N\}$ with $Q_{j-1} < Q_j$. A second-order Taylor expansion-series of the inverse marginal utility (term inside the conditional expectation in (3)) around $(x_1, ..., x_N) = (x_{0.1,}, ..., x_{0.N})$ and taking the expectation under the risk neutral measure at time T_1 allows us to write (3) as

$$\upsilon_{T_1} = 1 + \frac{1}{\tau_t x_{0,1}} (x_1 - x_{0.1}) + \frac{1}{x_{0,1}^2} \frac{(1-\rho_t)}{\tau_t^2} (x_1 - x_{0,1})^2 + \frac{(1-\rho_t)}{\tau_t^2} \sum_{j>1}^{N} \frac{1}{x_{0,j}^2} \mathbb{E}_{T_1}^* (x_j - x_{0.j})^2 .$$

We replace this expression in (1) and derive the expected excess return on the market:

$$RP_{t\to T_1,T_N} = \frac{\frac{1}{\tau_t R_{f,t\to T_1}} \mathbb{M}_{t\to T_1}^{*(2)} + \frac{1}{R_{f,t\to T_1}^2} \frac{(1-\rho_t)}{\tau_t^2} \mathbb{M}_{t\to T_1}^{*(3)} + \frac{(1-\rho_t)}{\tau_t^2} \mathcal{LEV}_t^*}{1 + \frac{1}{R_{f,t\to T_1}^2} \frac{(1-\rho_t)}{\tau_t^2} \mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{\tau_t^2} \mathbb{E}_t^* \mathcal{M}_{t,T_N}^{*(2)}}. \quad (34)$$

where

$$\begin{aligned} \mathcal{LEV}_t^* &= \mathbb{COV}_t^*\left(R_{M,t\to T_1}, \mathcal{M}_{t,T_N}^{*(2)}\right), \\ \mathcal{M}_{t,T_N}^{*(2)} &= \sum_{j>1}^{N} \frac{1}{R_{f,T_{Q_{j-1}}\to T_{Q_j}}^2} \mathbb{M}_{T_{Q_{j-1}}\to T_{Q_j}}^{*(2)}. \end{aligned}$$

Provided that preference parameters are estimated, expression (34) enables us to extract the risk premium from option prices if the risk neutral quantities $\mathbb{M}_{T_{Q_{j-1}}\to T_{Q_j}}^{*(2)}$ can be recovered from option prices with various maturities. We discuss the implementation of this approach in section D.

6.1 Empirical results

Table 7 summarizes the results when portfolio rebalancing is allowed. The new bound is very close to the bound obtained without rebalancing, for all forecast horizons T_1. Therefore, it still outperforms the bound $RP_{t\to T_1}$ and our results do not change.

7 Conclusion

Given its importance in financial applications, there is considerable interest in improving our measurement of the conditional expected return on the market portfolio. Several methods using forward-looking information embedded in option prices have been proposed in recent years. Martin (2017a), Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024) measure a one-period expected excess return in a one-period, two-date economy. We contribute to the literature by deriving an expression accounting for intertemporal hedging.

We, theoretically and empirically, show a significant difference between a static and a dynamic estimation. In a dynamic economy, the SDF is a nonlinear function of the market return as in a one-period economy. But it also depends on novel risk-neutral quantities such as the expected future variance and skewness and the covariances between market returns and future variance and skewness, namely the leverage effects. We show how these quantities significantly impact the one-period conditional expected excess return on the market from the perspective of an investor who holds the market portfolio in a multi-period economy. We also derive expressions for the one-period conditional probability of a crash, in a multi-period economy, in terms of risk-neutral quantities.

Our methodology provides significantly better risk premium and crash predictions and market-timing allocations in empirical tests. We further use our measure to shed light on the shape and time variations of the term structure of equity risk premia, which we define as the expected excess market return as a function of the investment horizon. In a one-period economy, Chabi-Yo and Loudis (2020) find that the term structure is upward sloping on average and downward sloping during recessions. Our term structure slope is essentially flat during normal market conditions and downward sloping during recessions.

While we have used the S&P 500 index to proxy for the market portfolio, our methodology can be extended to individual assets and international markets. We leave these endeavors for future research.

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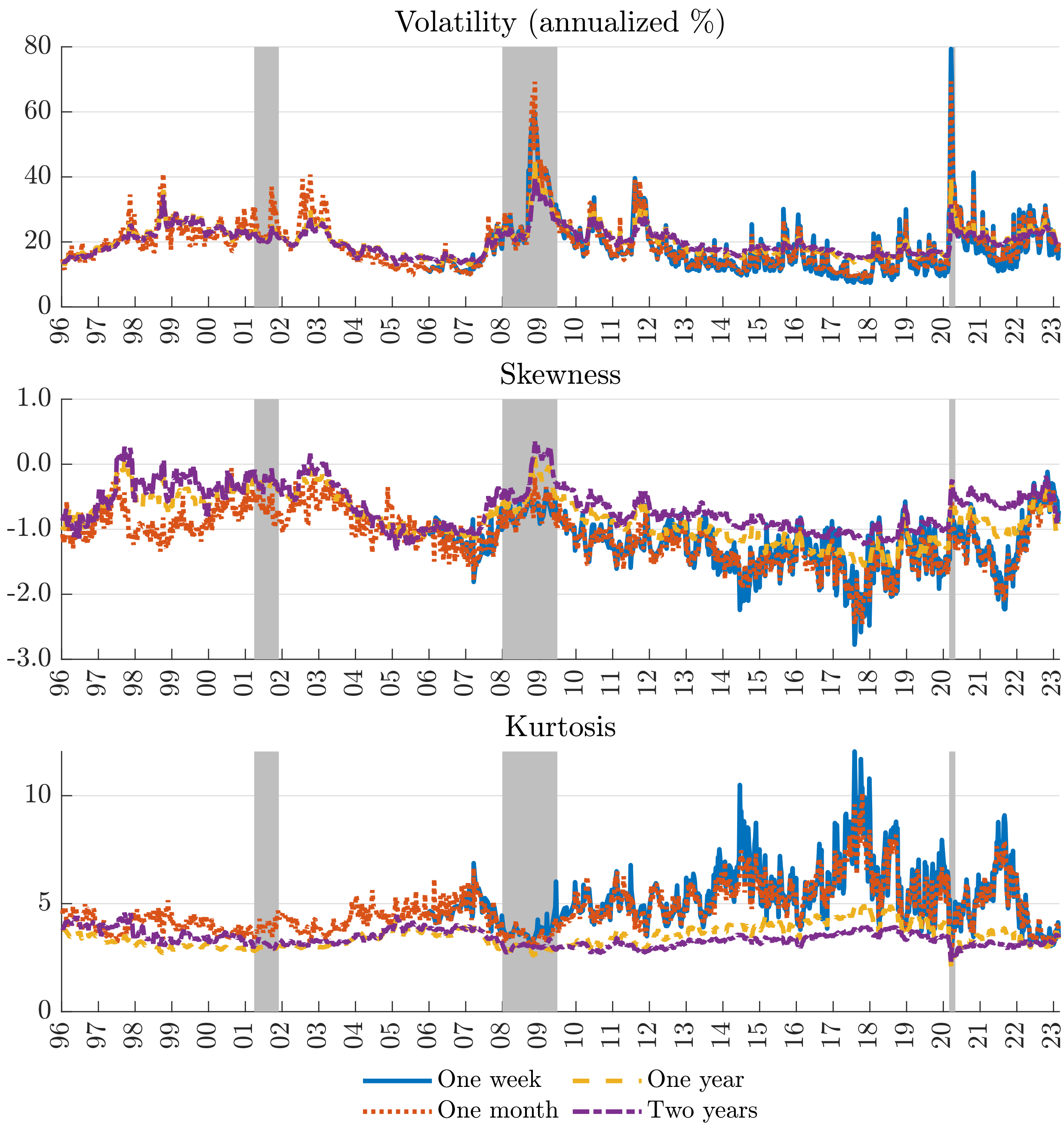

Figure 1: **Risk-neutral moments.**
We report option-implied risk-neutral volatility, skewness, and kurtosis for the S&P 500 index at a horizon of one week, one month, one year, and two years. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

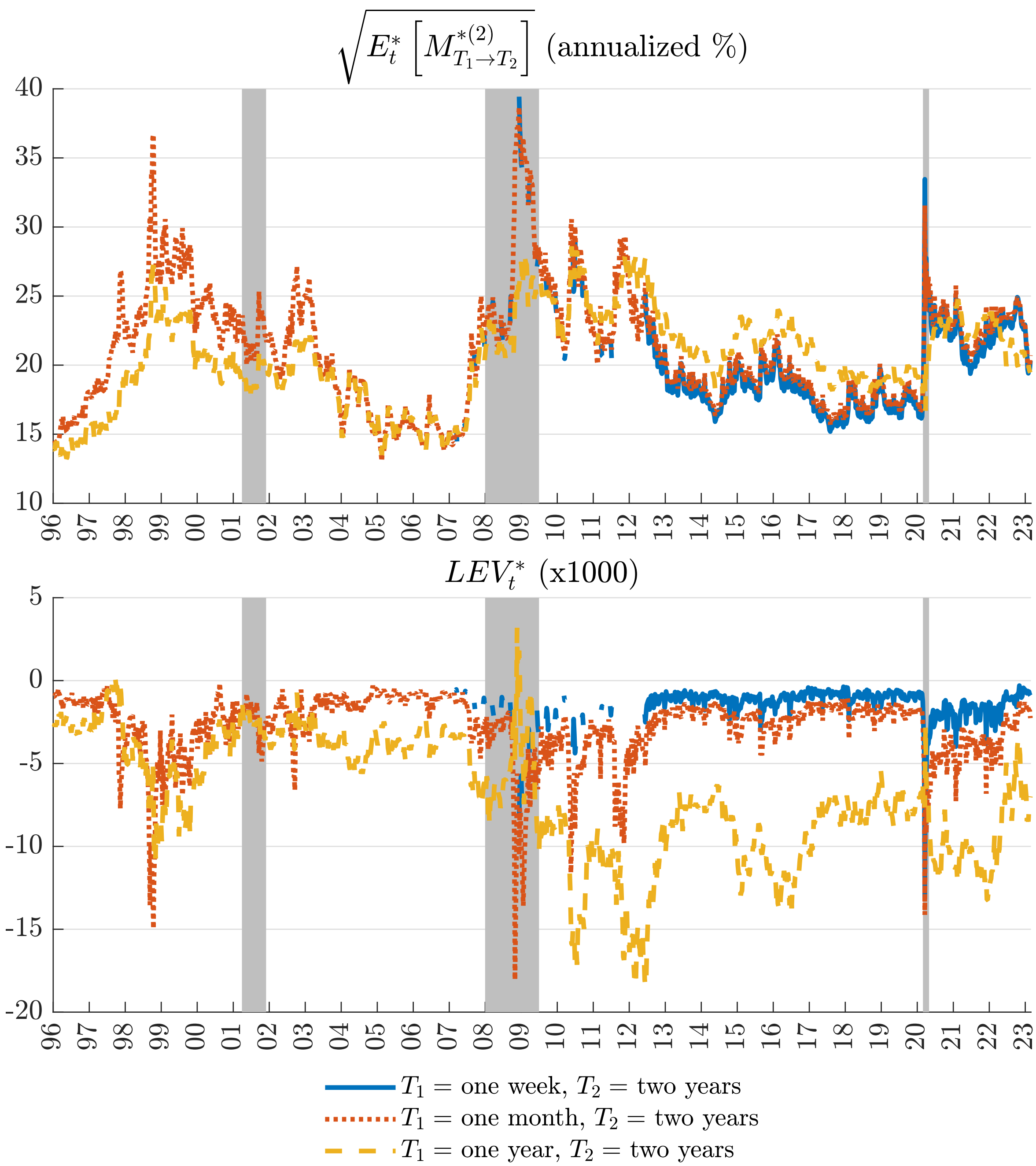

Figure 2: **Risk-neutral expected future variance and leverage.**
We report in the top graph the risk-neutral expected future volatility for the S&P 500 index. We report in the bottom graph the risk-neutral covariance between market returns and future variances in Equation (9). We use horizons T_1 of one week, one month, one quarter, and one year, and T_N = two years. We annualize each measure by multiplying by $\frac{365}{T_N - T_1}$. Data are weekly from January 1996 to February 2023. Gray areas are NBER recessions.

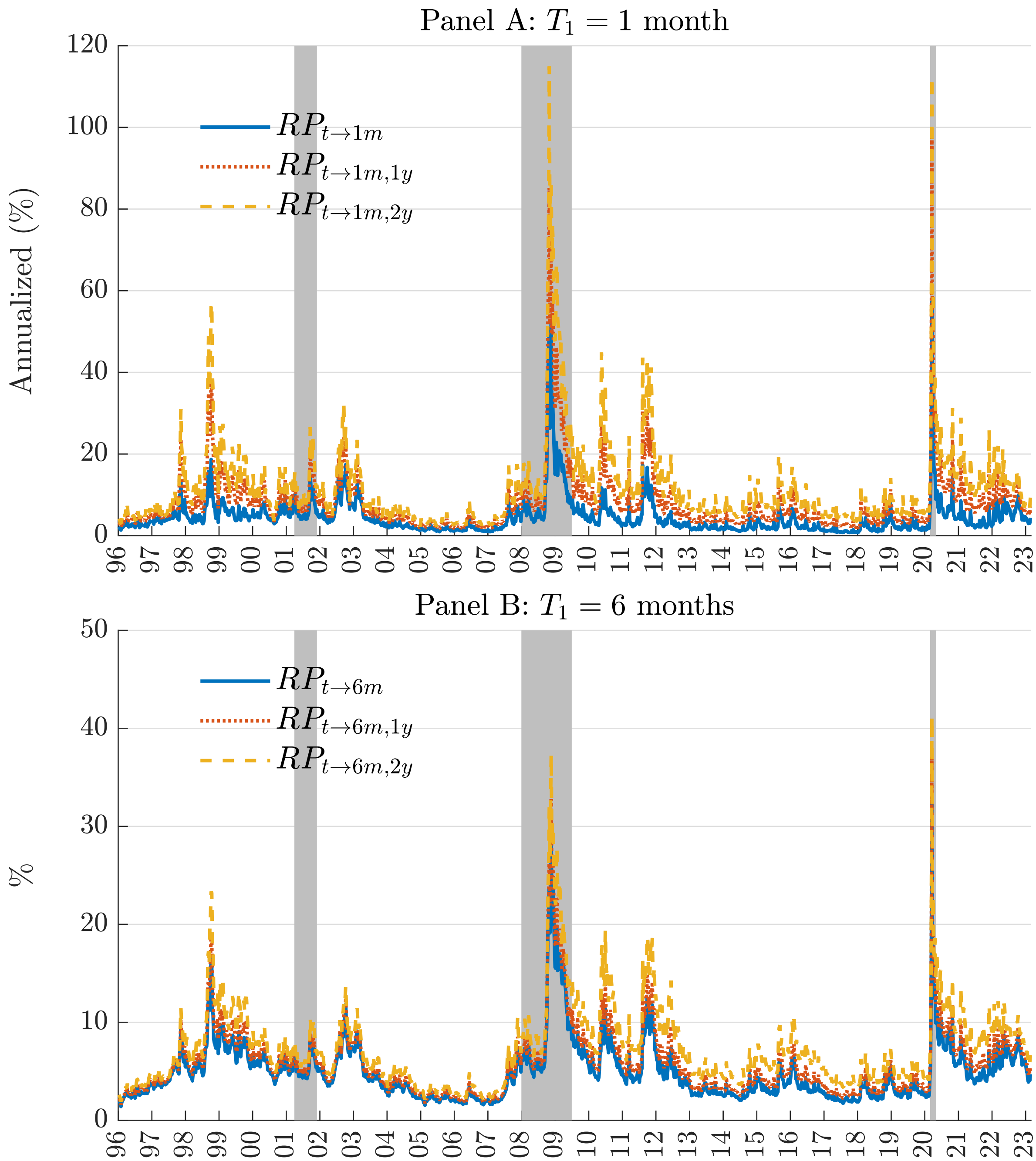

Figure 3: **Equity risk premium.**
This graph represents the different equity risk premium estimates, for a forecast horizon of 1 month (Panel A) and 6 months (Panel B). The following estimates are compared: the bound of Chabi-Yo and Loudis (2020), $RP_{t\to T_1}$, and our bound $RP_{t\to T_1,T_N}$ in Equation (10), for $T_N = 1$ year and 2 years.

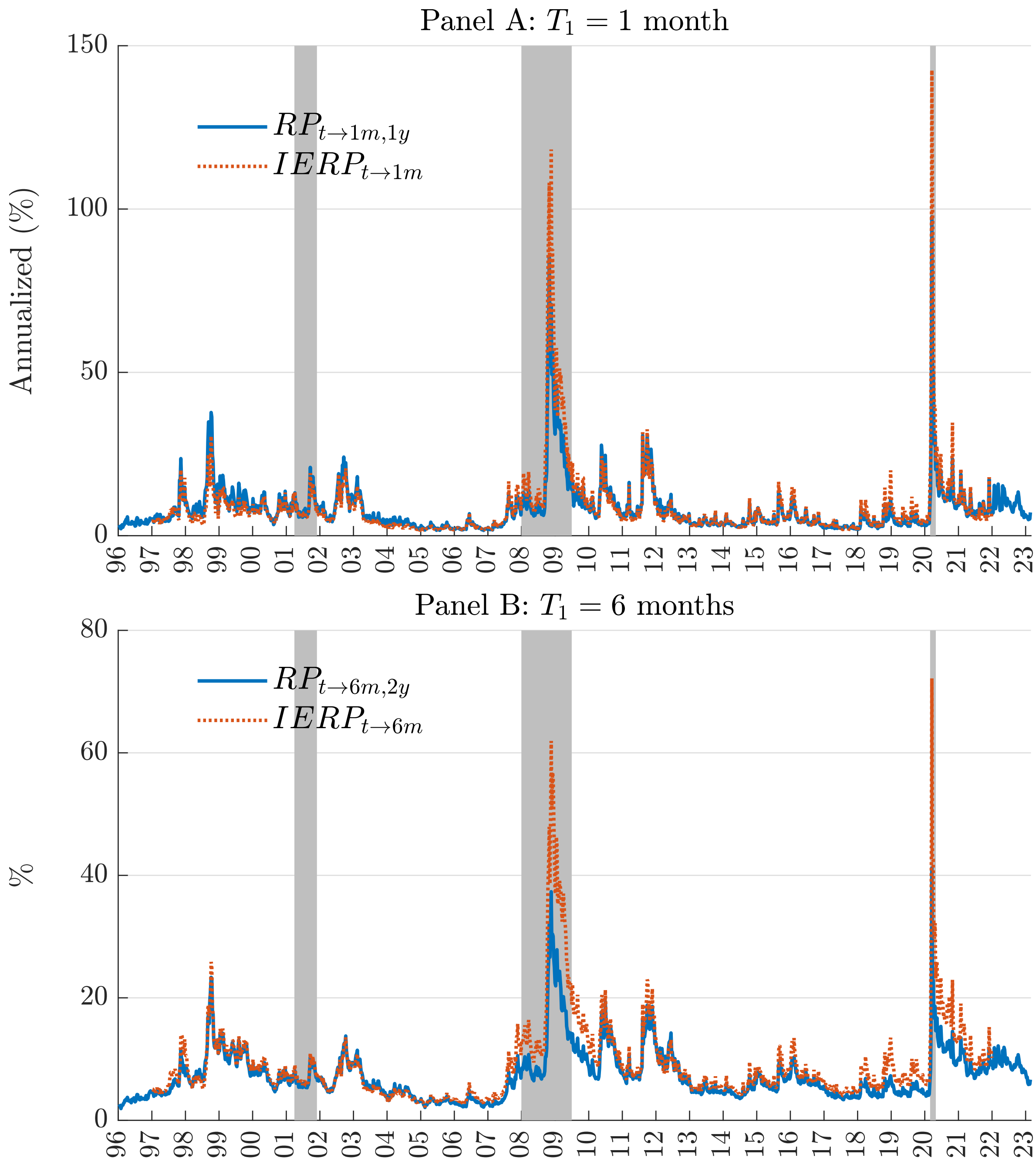

Figure 4: **Comparison with** $IERP_{t\to T_1}$
This graph compares our estimate of the equity risk premium, $RP_{t\to T_1,T_N}$, to the Implied Equity Risk Premium of Tetlock, McCoy, and Shah (2024), $IERP_{t\to T_1}$, for a forecast horizon of 1 month (Panel A) and 6 months (Panel B). In our bound, the investment horizon T_N is 1 year in Panel A, and 2 years in Panel B, chosen to match the IERP as closely as possible.

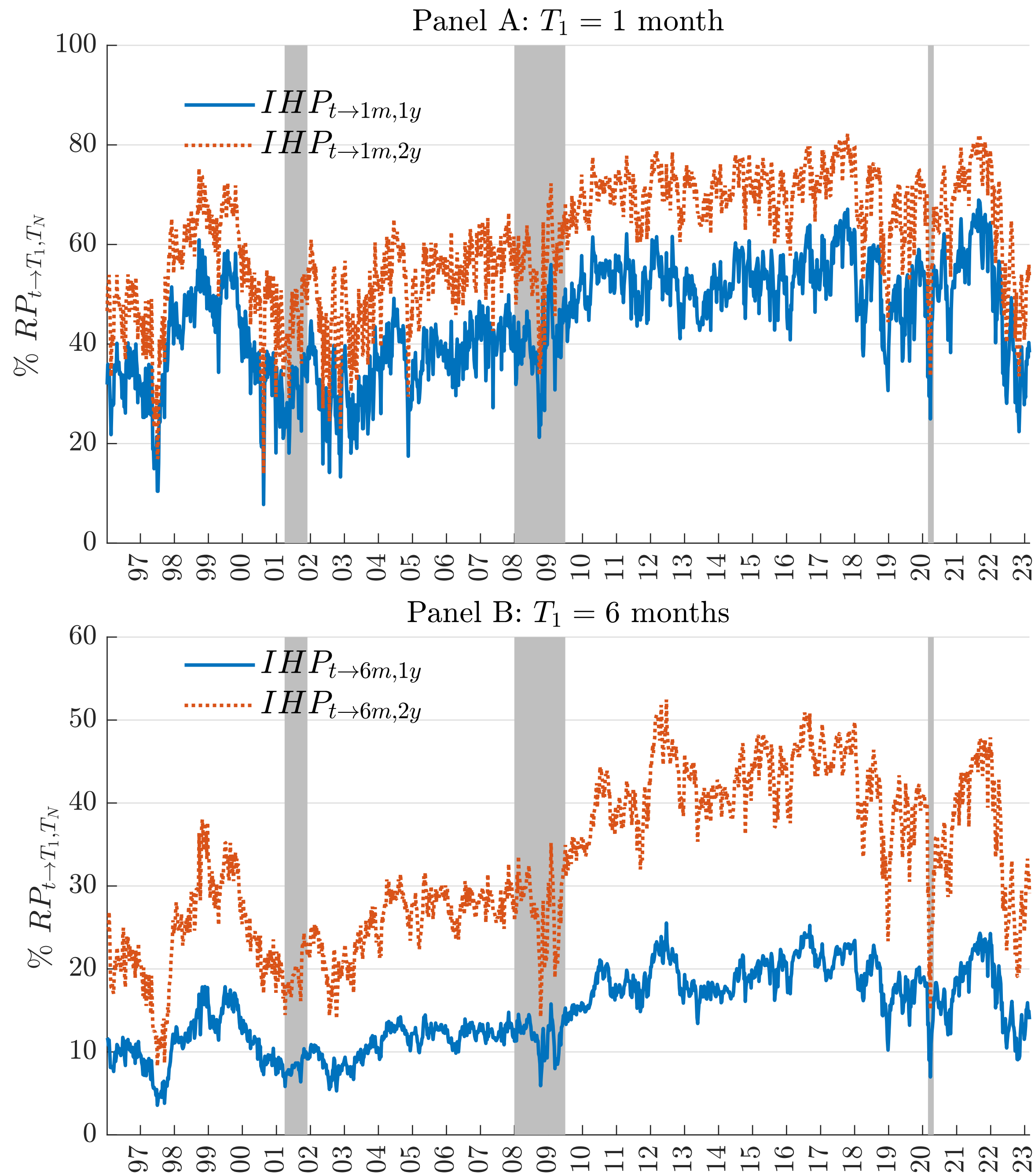

Figure 5: **Intertemporal hedging premium** $IHP_{t\to T_1,T_N}$

This graph represents the intertemporal hedging premium, $IHP_{t\to T_1,T_N}$, as defined in Equation (18), for different equity risk premium estimates $RP_{t\to T_1,T_N}$. $IHP_{t\to T_1,T_N}$ is displayed in percentages of $RP_{t\to T_1,T_N}$. The forecast horizon is of 1 month (Panel A) and 6 months (Panel B).

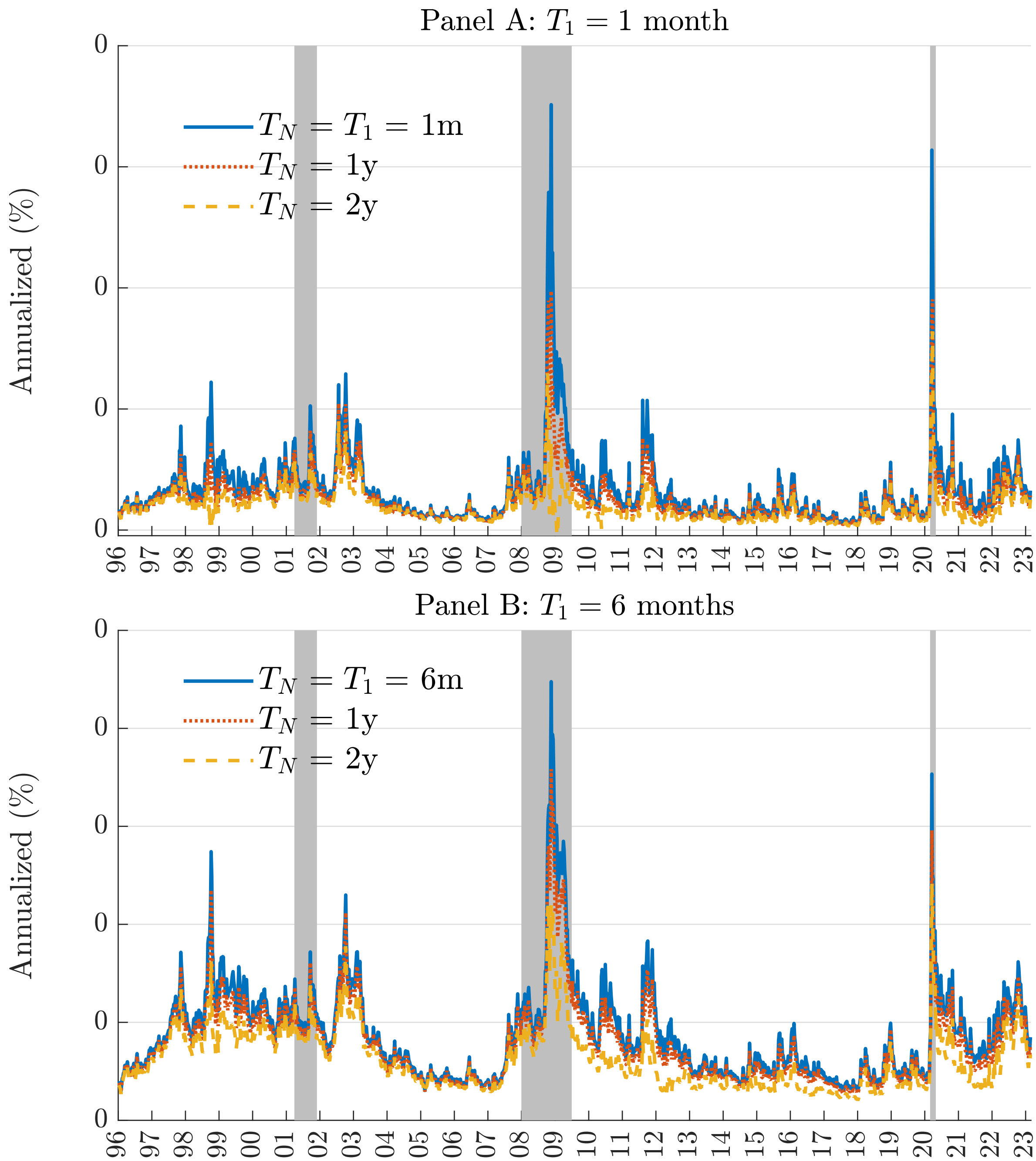

Figure 6: **Conditional variance under the physical measure.**
This graph represents the conditional physical variance as defined in Equations (20)-(22), for $T_1 = 1$ month (Panel A) and $T_1 = 6$ months (Panel B). The conditional variance without intertemporal hedging ($T_N = T_1$) is compared to the variance with intertemporal hedging, using $T_N = 1$ year and $T_N = 2$ years.

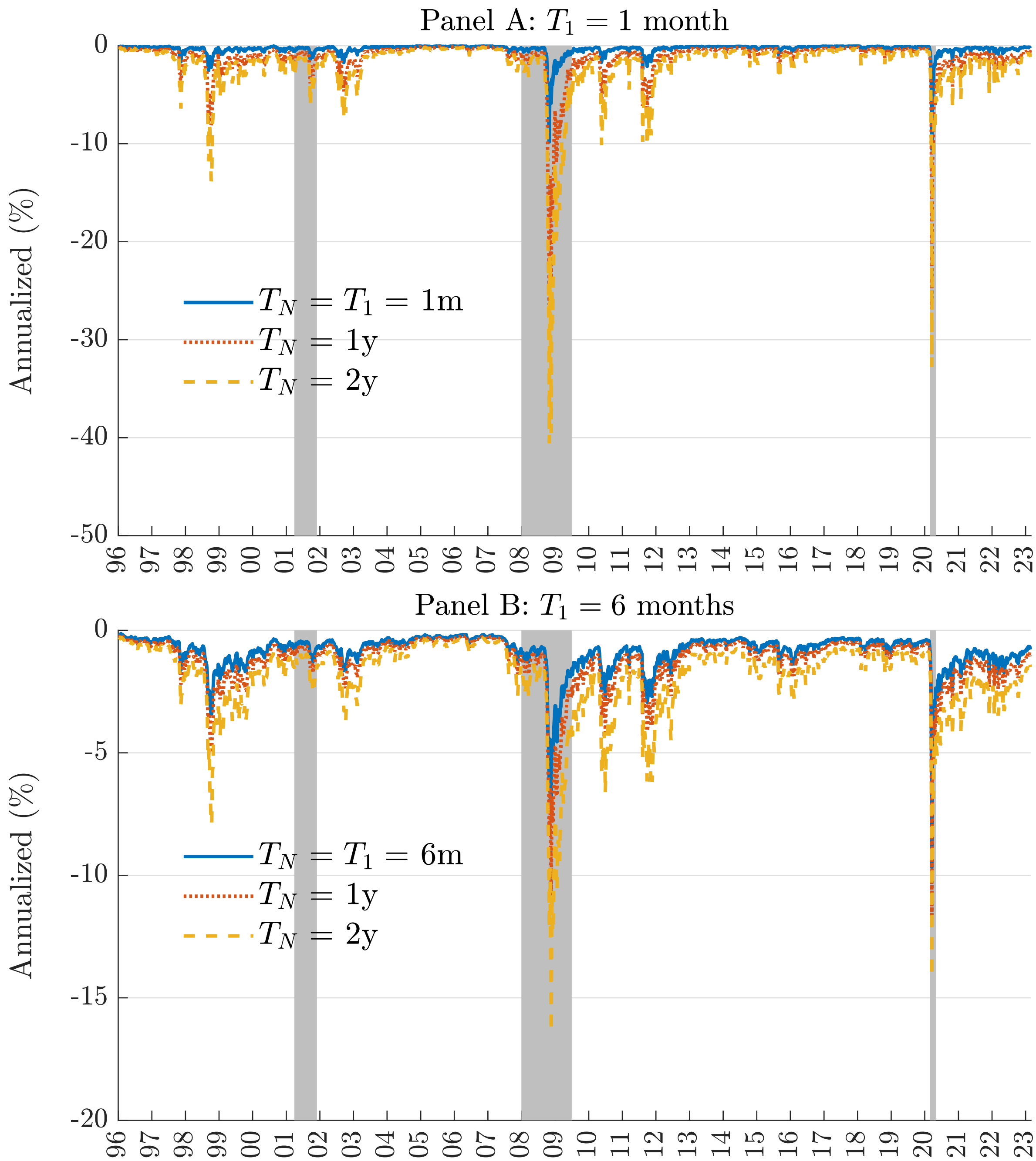

Figure 7: **Variance risk premium.**
This graph represents the variance risk premium for $T_1 = 1$ month (Panel A) and $T_1 = 6$ months (Panel B) without intertemporal hedging ($T_N = 1$ and 6 months, respectively) and with intertemporal hedging. The variance risk premium is defined as the difference between the conditional variance under the physical measure and under the risk-neutral measure.

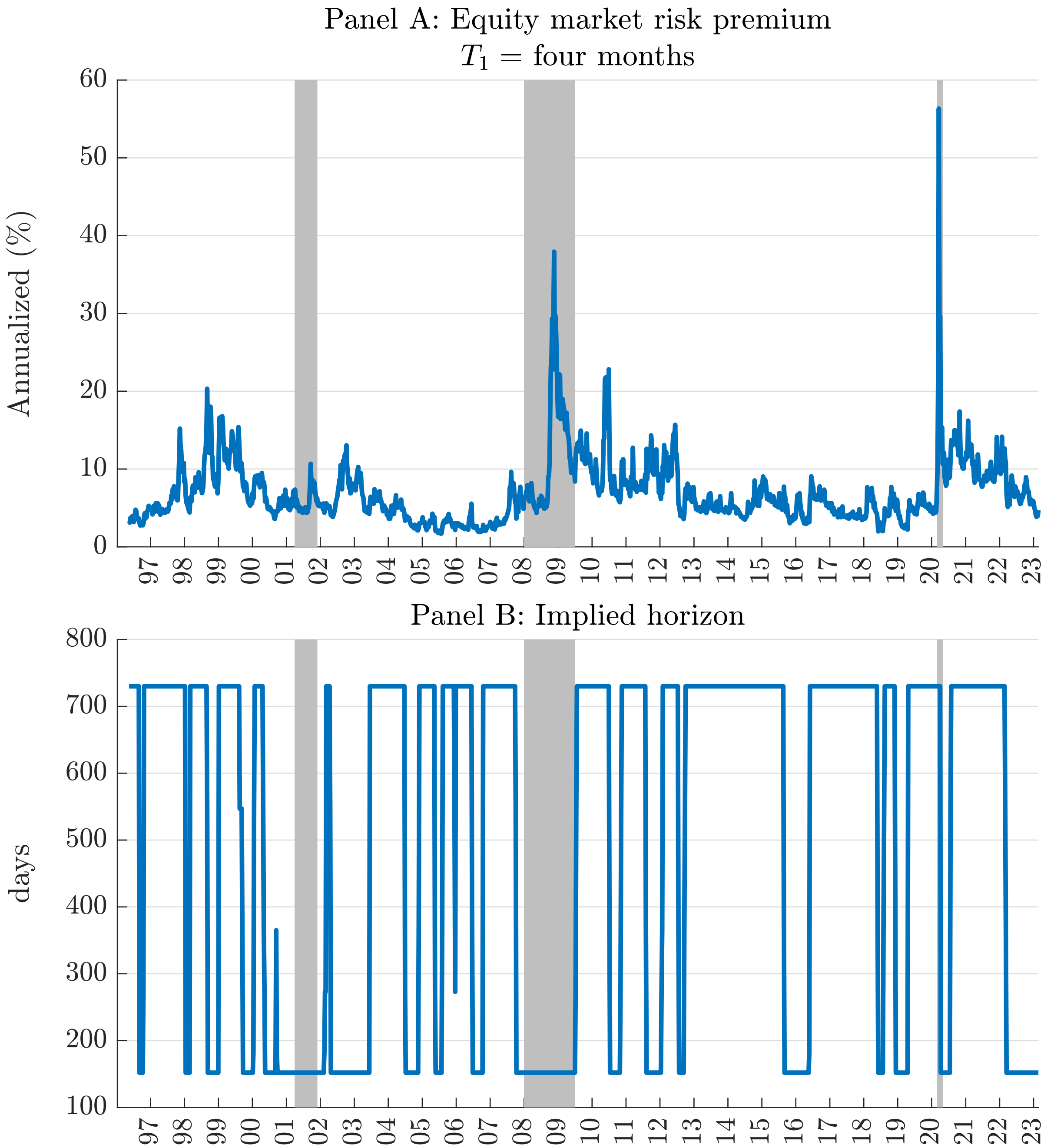

Figure 8: **Implied investors' horizon for T_1 = 4 months.**
This graph represents, in Panel A, the 4-month ERP obtained with an optimized investors' horizon. Panel B displays the implied investors' horizon $T^*_{N,t}$, which maximizes the in-sample fit of our bound to the realized returns, as measured by the R^2 over a window of 90 days.

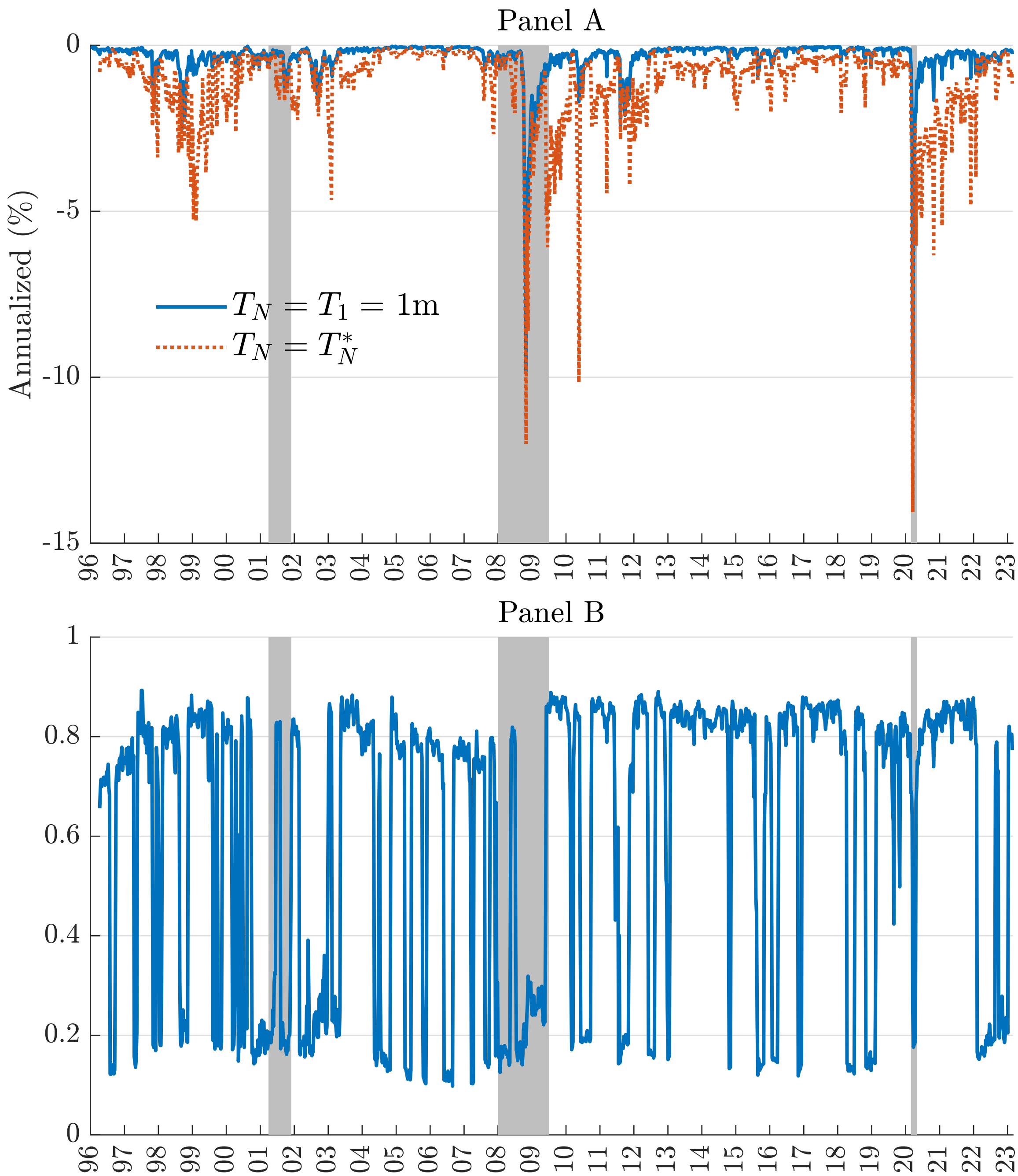

Figure 9: **Variance risk premium with the implied investors' horizon.**
This graph represents, in Panel A, the variance risk premium at $T_1 = 1$ month, obtained with an optimized investors' horizon. The variance risk premium is defined as the difference between the conditional variance under the physical measure and under the risk-neutral measure. Panel B displays the fraction of variance risk premium captured by intertemporal hedging, as a percentage of the total variance risk premium.

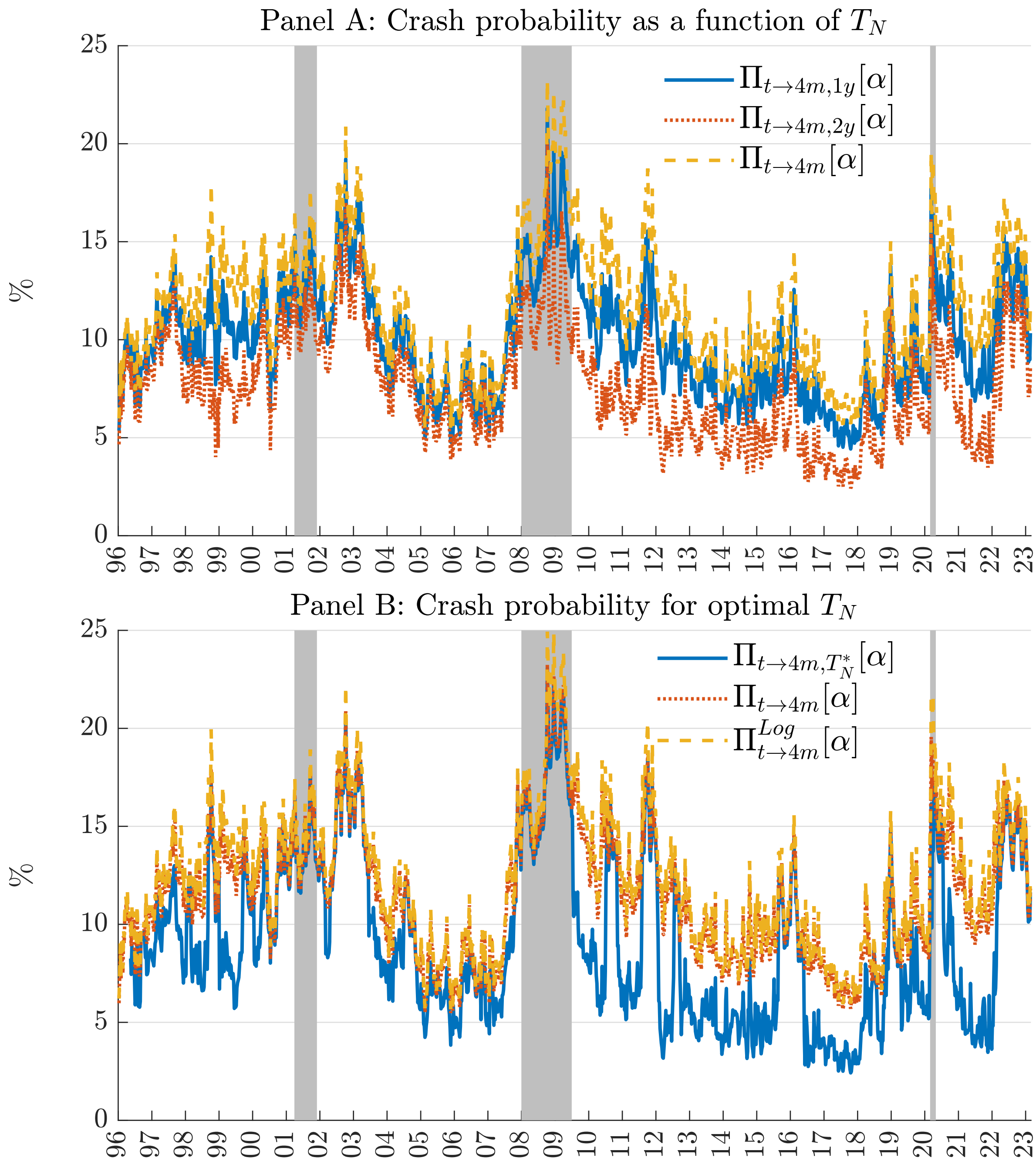

Figure 10: **Probability of a 10% market crash**
We report the time-varying probability of a 10% stock market crash from Proposition 5, for $T_1 = 4$ months. Panel A reports the crash probabilities for different values of T_N. Panel B compares our estimate of the crash probability with the optimal T_N, to the crash probabilities without intertemporal hedging and the one of Martin (2017a). Gray areas are NBER recessions.

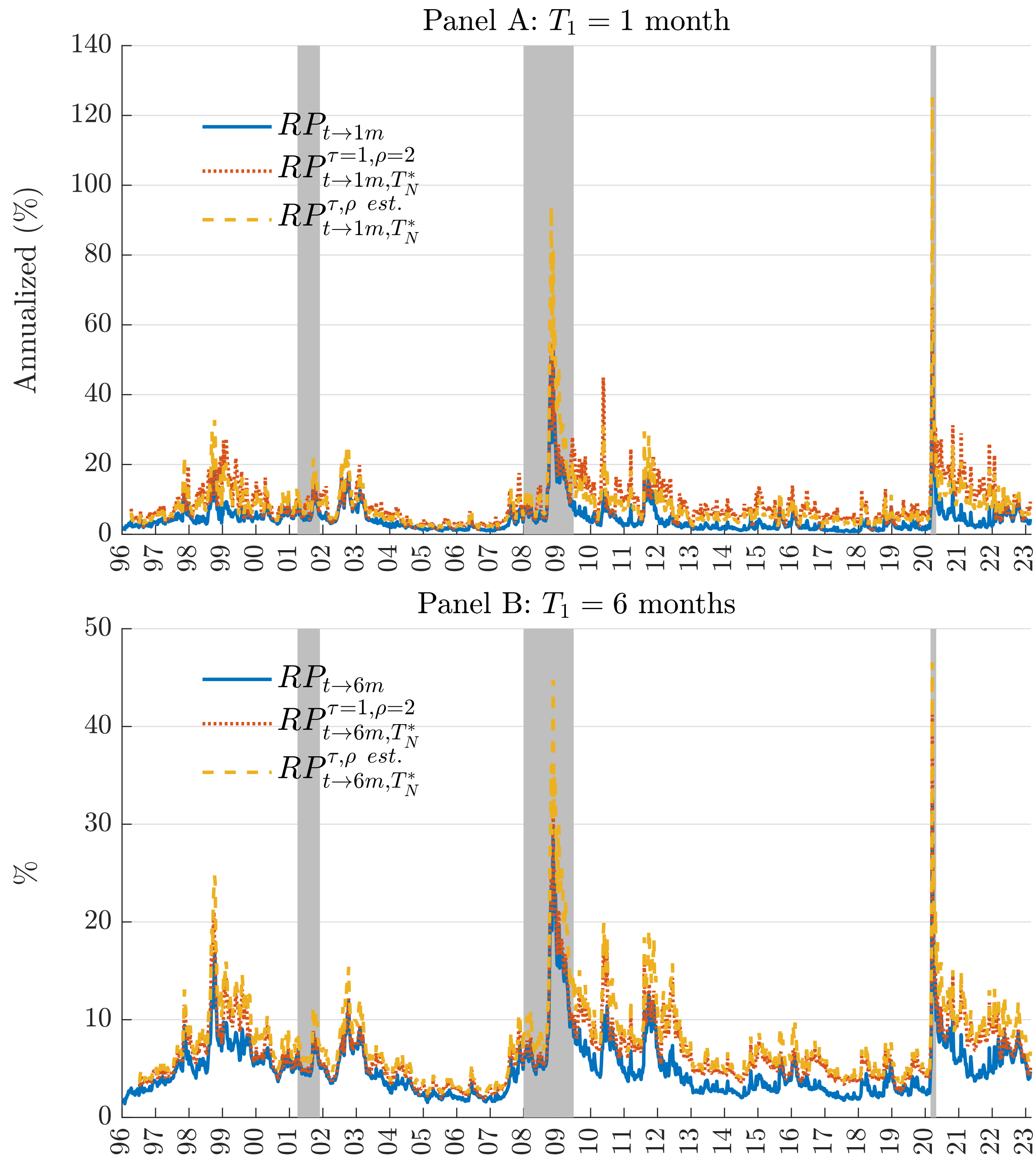

Figure 11: **Equity risk premium with estimated preference parameters.**
This graph compares the equity risk premium without intertemporal hedging $RP_{t\to T_1}$, to two estimates of the equity risk premium with intertemporal hedging. The dotted line, $RP^{\tau=1,\rho=2}_{t=T_1,T^*_N}$ has the preference parameters set to their default values. The dashed line, $RP^{\tau,\rho\ est.}_{t=T_1,T^*_N}$, has them estimated. In Panel A, the forecast horizon is $T_1 = 1$ month and in Panel B it is $T_1 = 6$ months.

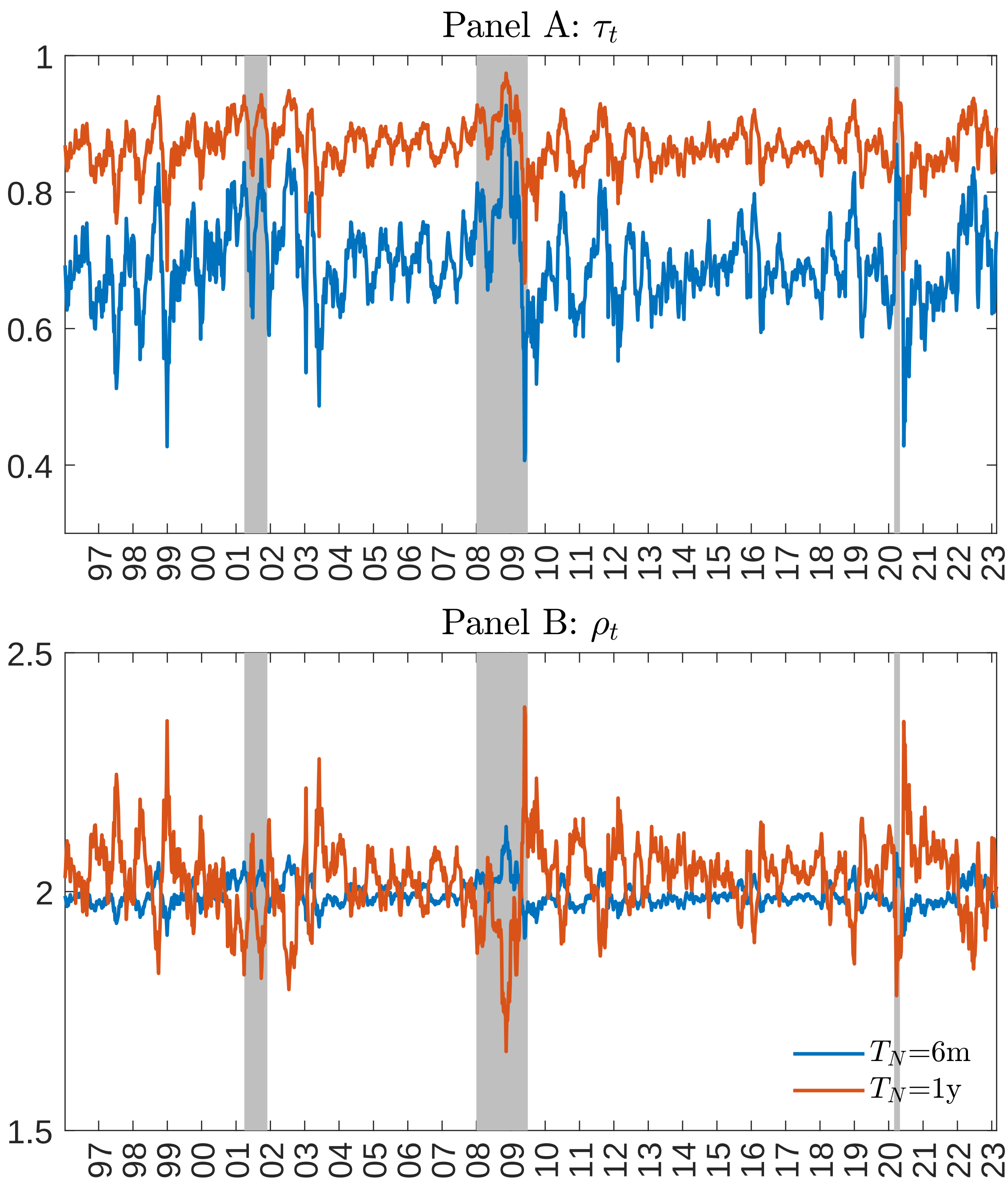

Figure 12: **Estimated preference parameters τ_t and ρ_t over the period 1996-2023.** This graph represents the estimated time series of risk aversion parameter τ_t and skewness tolerance parameter ρ_t, for $T_1 = 1$ month and varying T_N. Estimates are obtained by letting the preference parameters be linear functions of past 3-month returns, and applying the estimation methodology described in Section 5.1 on the whole dataset.

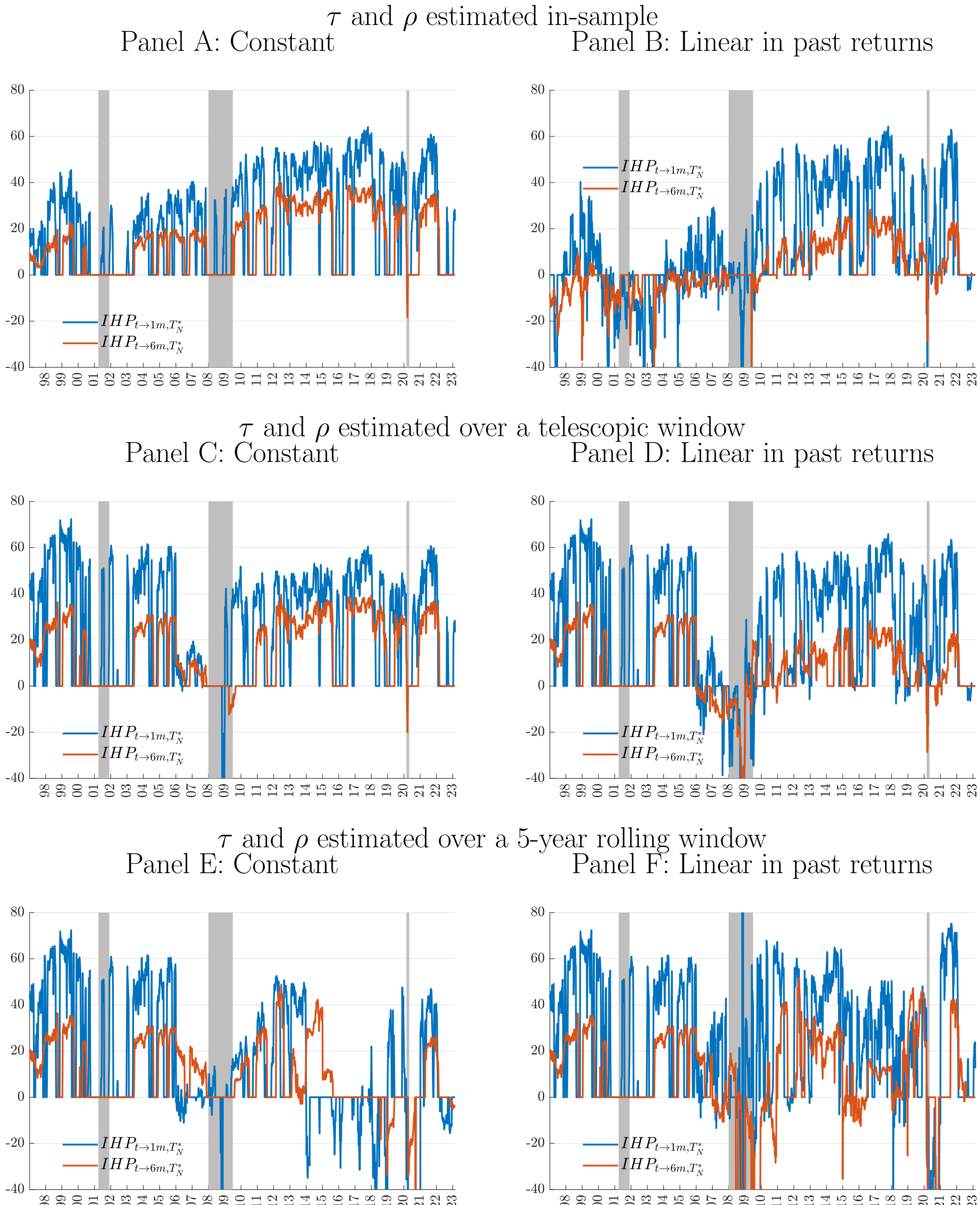

Figure 13: **Intertemporal hedging premium for all estimations** $IHP_{t\to T_1,T_N}$
This graph represents the intertemporal hedging premium, $IHP_{t\to T_1,T_N}$, as defined in Equation (18), for different equity risk premium estimates $RP_{t\to T_1,T_N}$. $IHP_{t\to T_1,T_N}$ is displayed in percentages of $RP_{t\to T_1,T_N}$. The forecast horizon is of 1 month (blue lines) and 6 months (red lines). In Panels A and B, the preferences parameters are estimated over the full sample from 1996 to 2023. In Panels C and D they are estimated over a telescopic window and in Panels E and F they are estimated over a rolling window of 5 years. In Panels A, C and E, preference parameters are estimated constant over the estimation period and in Panels B, D and F, they are assumed linear in past 3-month returns.

Table 1: **Summary statistics for risk premia**

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. All values are annualized and in percent. Data are monthly from January 1996 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the risk premium estimate of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premium measure in Equation (10), with $T_N = 1$ and 2 years.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
$RP^{Log}_{t\to T_1}$	3.43	4.87	6.29	59.08	1.01	1.36	2.05	3.55	6.19
$IERP_{t\to T_1}$	8.37	13.40	7.20	75.05	2.35	3.12	4.75	8.38	14.55
$RP_{t\to T_1}$	3.59	5.30	6.70	66.84	1.04	1.41	2.12	3.66	6.49
$RP_{t\to T_1,1y}$	9.44	11.98	6.89	76.40	3.35	4.16	6.24	10.22	16.36
$RP_{t\to T_1,2y}$	15.14	16.02	5.61	51.18	5.92	7.62	10.65	17.07	25.02
Panel B: One month									
$RP^{Log}_{t\to T_1}$	4.29	4.38	4.57	34.44	1.32	1.82	3.21	5.10	7.85
$IERP_{t\to T_1}$	8.74	11.11	5.69	47.26	2.46	3.66	5.88	9.82	15.32
$RP_{t\to T_1}$	4.60	4.89	4.84	38.09	1.37	1.92	3.43	5.40	8.48
$RP_{t\to T_1,1y}$	8.31	8.11	4.29	31.56	2.66	3.76	6.18	9.89	14.74
$RP_{t\to T_1,2y}$	11.96	10.87	3.74	24.10	4.19	5.82	8.93	14.13	20.76
Panel C: One quarter									
$RP^{Log}_{t\to T_1}$	4.35	3.39	3.39	21.22	1.68	2.15	3.51	5.32	7.52
$IERP_{t\to T_1}$	9.11	8.81	4.45	31.42	3.26	4.56	6.80	10.51	15.85
$RP_{t\to T_1}$	4.88	3.98	3.71	25.32	1.85	2.41	3.86	5.91	8.52
$RP_{t\to T_1,1y}$	6.74	5.18	3.42	22.36	2.55	3.50	5.44	8.11	11.76
$RP_{t\to T_1,2y}$	9.04	6.56	2.99	17.24	3.69	5.02	7.32	10.87	15.75
Panel D: Six months									
$RP^{Log}_{t\to T_1}$	4.32	2.68	2.61	14.02	1.94	2.45	3.70	5.33	7.02
$IERP_{t\to T_1}$	9.22	7.06	3.47	20.12	3.90	5.21	7.32	10.81	15.76
$RP_{t\to T_1}$	5.02	3.23	2.81	15.99	2.20	2.84	4.28	6.10	8.31
$RP_{t\to T_1,1y}$	5.87	3.66	2.67	14.82	2.61	3.42	5.04	7.06	9.78
$RP_{t\to T_1,2y}$	7.48	4.42	2.32	11.71	3.55	4.58	6.40	9.03	12.49

Table 2: **Out-of-sample prediction and allocation performance**

We report the out-of-sample performance of different risk premium prediction methods. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the Implied Equity Risk Premium of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). The results in the last column are based on predicted returns obtained by averaging $RP_{t\to T_1,T_N}$ across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices, using Equation (20). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. Negative certainty equivalents are not reported. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1997 to December 2021.

Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$IERP_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N}$ with $T_N =$ (in months): 1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2														
10d	−0.41	−0.60	−0.38	−0.33	−0.25	−0.16	−0.09	−0.02	0.06	0.15	0.11	−0.08	−0.69	0.13
1	0.92	1.39	1.08	-	1.25	1.39	1.51	1.62	1.71	1.84	1.84	1.62	1.00	1.80
2	1.51	2.18	1.96	-	-	2.23*	2.48*	2.70*	2.90*	3.36*	3.66	4.00	3.89	3.44
3	1.43	2.73	2.23	-	-	-	2.55**	2.86**	3.15**	3.85*	4.39*	5.21*	5.57	4.22*
4	2.17	5.22	3.35	-	-	-	-	3.69**	4.03**	4.88**	5.57**	6.65*	7.37*	5.62**
5	3.07	8.01	4.65	-	-	-	-	-	5.00***	5.92**	6.71**	7.98**	8.93**	7.12**
6	3.40	9.42	5.28	-	-	-	-	-	-	6.23***	7.05**	8.42**	9.54**	7.96**
12	2.67	10.38	5.58	-	-	-	-	-	-	-	-	6.90***	8.12***	7.53***
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$														
10d	3.66	-	3.74	3.92*	4.25*	4.60*	4.98*	5.34*	5.72*	6.25	5.62	-	-	6.28
1	4.71	-	4.89	-	5.11*	5.36*	5.61	5.85	6.10	6.59	6.37	-	11.83*	6.59
2	4.99	-	5.39	-	-	5.67**	5.96**	6.26**	6.56**	7.43**	8.31**	9.20	9.71	7.72**
3	5.53	-	6.13	-	-	-	6.41**	6.69**	6.98**	7.84**	8.75**	10.35*	-	8.50**
4	5.71	-	6.44	-	-	-	-	6.70***	6.96***	7.76**	8.69**	10.89**	6.56	8.81**
5	5.42	-	6.16	-	-	-	-	-	6.35**	6.97**	7.63**	9.15**	9.96*	8.11**
6	5.58	-	6.59	-	-	-	-	-	-	7.31***	8.13***	10.12***	9.99	9.34***
12	5.40	-	6.42	-	-	-	-	-	-	-	-	7.21	8.02	7.67

Table 3: **Out-of-sample prediction and allocation performance with T_N optimized**

We report the out-of-sample performance of different risk premium prediction methods, from January 1997 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the Implied Equity Risk Premium of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$IERP_{t\to T_1}$	$RP_{t\to T_1}$	$T_N = 1$y $RP_{t\to T_1,T_N}$	$T_N = 2$y $RP_{t\to T_1,T_N}$	Av. across T_N $RP_{t\to T_1,T_N}$	T_N opt. $RP_{t\to T_1,T_N}$
Panel A: Out-of-sample R^2							
10d	−0.40	−0.59	−0.37	0.12	−0.69	0.14	0.17
1	0.93	1.40	1.08	1.85	1.00	1.81	1.86
2	1.52	2.18	1.97	3.66	3.89	3.44	4.15**
3	1.43	2.73	2.23	4.39*	5.58	4.22*	5.39***
4	2.18	5.22	3.36	5.57**	7.38*	5.63**	6.54***
5	3.08	8.01	4.67	6.73**	8.94**	7.14**	7.70***
6	3.43	9.44	5.31	7.08**	9.56**	7.99**	8.40***
12	2.69	10.39	5.61	-	8.15***	7.56***	7.84***
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$							
10d	5.45	-	5.62	12.36*	-	11.18**	-
1	4.78	-	5.00	7.10	-	6.97*	3.44
2	4.90	-	5.30	7.95**	-	7.48**	3.00
3	5.25	-	5.79	8.29**	-	8.03***	10.22**
4	5.47	-	6.13	8.16**	-	8.28**	10.63***
5	5.19	-	5.87	7.38**	9.49*	7.85**	8.77**
6	5.21	-	5.99	7.16**	8.47	8.03**	8.71*
12	5.28	-	6.33	-	8.09*	7.67*	-

Table 4: **Out-of-sample crash prediction with T_N optimized**

We report the out-of-sample performance of different crash prediction methods. Each month, we use the crash probability from Martin (2017a) ($\Pi^{Log}_{t\to T_1}[\alpha]$), the one from Chabi-Yo and Loudis (2020) ($\Pi_{t\to T_1}[\alpha]$ in Equation (25)), and the one from our methodology, $\Pi_{t\to T_1,T_N}[\alpha]$, defined in Equation (24) of Proposition 5. T_N is set equal to the implied investors' horizon $T^*_{N,t}$ at each time t. We compute the loss function for $\Pi_{t\to T_1,T_N}[\alpha]$ as $l_{t\to T_1,T_N} = -(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\log(\Pi_{t\to T_1,T_N}[\alpha]) + (1-\mathbb{1}_{R_{M,t\to T_1}<\alpha})(1-\log(\Pi_{t\to T_1,T_N}[\alpha])))$. Similarly, we compute a loss function for other methods. For each method in rows, we test whether the average loss functions are significantly larger than those of the method in columns using the Diebold and Mariano (1995) test. A significantly positive test statistic indicates that the column-method outperforms the row-method. We estimate the variance of the difference in loss functions using a Newey-West correction with 12 lags. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. We report on a 90% ($\alpha = 0.10$), and 80% ($\alpha = 0.20$) crash size. Data are from January 1996 to February 2023.

	10% crash		20% crash	
	$\Pi_{t\to T_1}[\alpha]$	$\Pi_{t\to T_1,T_N}[\alpha]$	$\Pi_{t\to T_1}[\alpha]$	$\Pi_{t\to T_1,T_N}[\alpha]$
Panel A: One week				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.56*	1.92**	1.29*	−0.92
$\Pi_{t\to T_1}[\alpha]$	-	2.06**	-	−0.92
Panel B: One month				
$\Pi^{Log}_{t\to T_1}[\alpha]$	1.76**	−0.97	5.71***	6.58***
$\Pi_{t\to T_1}[\alpha]$	-	−0.98	-	6.42***
Panel C: One quarter				
$\Pi^{Log}_{t\to T_1}[\alpha]$	4.42***	7.14***	2.67***	2.58***
$\Pi_{t\to T_1}[\alpha]$	-	6.75***	-	2.40***
Panel D: Six months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	3.91***	8.21***	3.36***	3.71***
$\Pi_{t\to T_1}[\alpha]$	-	10.54***	-	3.45***
Panel E: Nine months				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.66***	5.10***	1.48*	2.18**
$\Pi_{t\to T_1}[\alpha]$	-	7.18***	-	2.36***
Panel F: One year				
$\Pi^{Log}_{t\to T_1}[\alpha]$	2.18**	2.79***	1.25	2.02**
$\Pi_{t\to T_1}[\alpha]$	-	3.34***	-	2.51***

Table 5: **Out-of-sample prediction and allocation performance with τ and ρ estimated in-sample**

We report the out-of-sample performance of different risk premium prediction methods, from January 1997 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (4), they are kept constant over the time series of data, but the constants are estimated. In column (5), they are modelled as linear functions of past 3-month returns. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices, using Equation (20). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ est. constant	ρ, τ est. linear in past returns
(months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$
	(1)	(2)	(3)	(4)	(5)
Panel A: Out-of-sample R^2					
10d	−0.40	−0.37	0.17	0.04	−0.08*
1	0.93	1.08	1.86	1.58	2.23
2	1.52	1.97	4.15**	3.79*	4.52
3	1.43	2.23	5.39***	4.67**	7.16**
4	2.18	3.36	6.54***	6.42**	8.39
5	3.08	4.67	7.70***	8.28**	10.26**
6	3.43	5.31	8.40***	9.48*	10.99
12	2.69	5.61	7.84***	8.36***	9.48
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	5.45	5.62	-	-	-
1	4.78	5.00	3.44	6.08	8.45**
2	4.90	5.30	3.00	8.80*	9.21
3	5.25	5.79	10.22**	10.23***	0.63*
4	5.47	6.13	10.63***	10.28**	-
5	5.19	5.87	8.77**	5.64	-
6	5.21	5.99	8.71*	-	-
12	5.28	6.33	-	-	-

Table 6: **Out-of-sample prediction and allocation performance with τ and ρ estimated without look-ahead bias**

We report the out-of-sample performance of different risk premium prediction methods, from January 2006 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the estimate of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). In columns (3) and (4), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (5), they are modelled constant and estimated on a telescopic window of time. In column (6), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1				$\tau = 1$ and $\rho = 2$	telescopic est.	rolling est.
	$RP^{Log}_{t\to T_1}$	$IERP_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$
	(1)	(2)	(3)	(4)	(5)	(6)
Panel A: Out-of-sample R^2						
10d	−0.41	−0.60	−0.38	0.16	−0.20	−0.59
1	0.39	0.64	0.56	2.21	1.03	0.90*
2	0.76	0.77	1.25	4.92**	2.10	4.19
3	−0.13	0.48	0.73	5.64***	3.70	5.98
4	1.11	4.63	2.52	7.05***	8.90	5.88
5	2.46	9.13	4.47	8.33***	10.96	7.34
6	2.76	11.46	5.26	9.12***	12.82	9.08
12	2.03	16.29	6.68	10.14***	11.66	9.54
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$						
10d	3.53	-	3.61	-	-	-
1	4.10	-	4.33	-	-	-
2	3.91	-	4.26	3.22	-	-
3	4.76	-	5.40	13.38***	0.20	14.13
4	4.19	-	4.54	6.48	1.13	-
5	4.24	-	4.96	8.61**	-	-
6	5.00	-	5.92	9.75*	-	-
12	4.83	-	6.22	7.22	0.95	5.58

Table 7: **Out-of-sample prediction and allocation performance with $\tau = 1$ and $\rho = 2$, with rebalancing**

We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023, setting the preference parameters to their default values $\tau = 1$ and $\rho = 2$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). In columns (2) and (3), results are reported using no rebalancing (benchmark). In columns (4) and (5), they are reported with rebalancing. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		No rebalancing		With rebalancing	
	$RP^{Log}_{t\to T_1}$ (1)	$RP_{t\to T_1}$ (2)	$RP_{t\to T_1,T^*_N}$ (3)	$RP_{t\to T_1}$ (4)	$RP_{t\to T_1,T^*_N}$ (5)
Panel A: Out-of-sample R^2					
10d	−0.09	−0.07	0.06	−0.07	0.16
1	1.09	1.18	1.73	1.18	1.65
2	1.34	1.59	3.84**	1.59	3.16
3	1.18	1.61	4.71***	1.61	3.76
4	2.16	2.86	5.47**	2.86	4.81
5	3.12	4.19	6.45**	4.19	5.94
6	3.61	4.97	7.26**	4.97	7.00
9	4.32	6.37	8.76**	6.37	8.75
12	4.00	6.54	8.44	6.54	8.89
18	2.29	6.17	7.66	6.17	7.66
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	4.56	4.69	5.75	4.69	5.34
1	3.55	3.68	3.52	3.68	2.78
2	3.69	3.96	6.40	3.96	6.51
3	4.14	4.54	9.50***	4.54	8.48
4	4.27	4.75	8.46**	4.75	7.96
5	4.01	4.50	6.85	4.50	6.69
6	4.26	4.89	7.24	4.89	7.23
9	4.18	4.88	6.19	4.88	6.18
12	4.52	5.45	6.85**	5.45	6.98
18	4.59	5.62	6.11**	5.62	6.11

Table 8: **Out-of-sample prediction and allocation performance of the third-order bound with $\tau = 1$, $\rho = 2$ and $\kappa = 4$**

We report the out-of-sample performance of different risk premium prediction methods, from January 2000 to February 2023, setting the preference parameters to their default values $\tau = 1$, $\rho = 2$ and $\kappa = 4$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). Results are reported setting the preference parameters to $\tau = 1$, $\rho = 2$ and $\kappa = 4$. Columns (2) and (3) report the second-order bounds while columns (4) and (5) report the thir-order bounds. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		2nd order		3rd order	
	$RP^{Log}_{t\to T_1}$ (1)	$RP_{t\to T_1}$ (2)	$RP_{t\to T_1,T_N^*}$ (3)	$RP_{t\to T_1}$ (4)	$RP_{t\to T_1,T_N^*}$ (5)
Panel A: Out-of-sample R^2					
10d	−0.40	−0.37	0.17	−0.37	−0.48
1	0.93	1.08	1.87	1.13	0.91
2	1.52	1.97	4.07**	2.07	3.51
3	1.43	2.23	5.32***	2.37	5.59**
4	2.18	3.36	6.49***	3.74	6.75**
5	3.08	4.67	7.70***	5.44	9.33**
6	3.43	5.31	8.40***	6.41	-
12	2.69	5.61	7.84***	8.00	-
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	6.00	6.17	13.00	6.17	-
1	4.50	4.69	-	4.75	-
2	4.59	4.88	3.91	4.97	-
3	5.01	5.52	8.66**	5.74	-
4	4.86	5.35	8.62*	5.53	-
5	5.09	5.80	8.93**	6.17	3.57
6	4.63	5.35	6.93	5.76	-
12	5.12	6.11	7.83**	6.37	-

A Proofs and derivations

This section contains the proofs and derivations of the main results presented in Section 2.

A.1 Proof of Equation (1)

Let $R_{k,t\to T_1}$ be the return of risky asset k from time t to time T_1 and $m_{t\to T_1}$ be the one-period SDF. We show that the conditional expected return of risky assets can be expressed as the risk-neutral covariance between the asset return and the inverse of the SDF $m_{t\to T_1}$. This result is not new, and was derived in Equation (2) of Chabi-Yo and Loudis (2019).

The conditional expected return of asset k can be expressed using the identity

$$\mathbb{E}_t\left(R_{k,t\to T_1}\right) = \mathbb{E}_t\left(R_{k,t\to T_1}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\right). \tag{A1}$$

The ratio $\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}$ defines the risk-neutral distribution. Hence, the Radon-Nykodym theorem allows us to express the conditional expected return of asset k as a function of moments under the risk-neutral measure:

$$\begin{aligned}
\mathbb{E}_t\left(R_{k,t\to T_1}\right) &= \mathbb{E}_t^*\left(R_{k,t\to T_1}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right)\\
&= \mathbb{COV}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}, R_{k,t\to T_1}\right) + \mathbb{E}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right)\mathbb{E}_t^*\left(R_{k,t\to T_1}\right)\\
&= \mathbb{COV}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}, R_{k,t\to T_1}\right) + R_{f,t\to T_1}.
\end{aligned} \tag{A2}$$

We use $\mathbb{E}_t^*\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right) = 1$ and $\mathbb{E}_t^*\left(R_{k,t\to T_1}\right) = R_{f,t\to T_1}$. This identity is reminiscent of the well-known asset pricing equation in which the expected excess return is negatively related to the covariance between the return and the SDF under the physical measure.

A.2 Proof of Equation (3)

We show that the inverse of the one-period SDF $m_{t\to T_1}$ can be expressed as a function of the marginal utility of wealth and expectations under the risk-neutral measure.

The representative agent's optimization problem can be re-written as

$$\max_{\omega_t} \mathbb{E}_t \left(u\left[W_{T_N}\right]\right) = \max_{\omega_t} \mathbb{E}_t \left(\max_{\omega_{T_1}} \mathbb{E}_{T_1} \left(u\left[W_{T_N}\right]\right)\right). \tag{A3}$$

Solving Problem (A3) backward, the first step is to solve

$$\max_{\omega_{T_1}} \mathbb{E}_{T_1} \left(u\left[W_{T_N}\right]\right). \tag{A4}$$

Equation (A4) produces an optimal weight $\omega^*_{T_1}$, and the terminal wealth achieved with this weight is $W^*_{T_N} = W_{T_1}\left(\omega^{*\intercal}_{T_1} R_{T_1\to T_N}\right)$. The corresponding one-period SDF from time T_1 to time T_N, $m_{T_1\to T_N}$, has the form

$$m_{T_1\to T_N} = \delta_{T_1} u^{'} \left[W^*_{T_N}\right]. \tag{A5}$$

Given the optimal value, $\omega^*_{T_1}$, the second step solves

$$\max_{\omega_t} \mathbb{E}_t \left(\mathbb{E}_{T_1} \left(u\left[W^*_{T_N}\right]\right)\right). \tag{A6}$$

This produces a one-period SDF from time t to time T_1 of the form

$$m_{t\to T_1} = \delta_t \mathbb{E}_{T_1} \left(u^{'} \left[W^*_{T_N}\right] \left(\omega^{*\intercal}_{T_1} R_{T_1\to T_N}\right)\right). \tag{A7}$$

From (A7), the constant δ_t can alternatively be written as

$$\delta_t = m_{t\to T_1} \left(\mathbb{E}_{T_1} \left(u^{'} \left[W^*_{T_N}\right] \left(\omega^{*\intercal}_{T_1} R_{T_1\to T_N}\right)\right)\right)^{-1}. \tag{A8}$$

Because parameter δ_t is a constant, we have $\delta_t = \mathbb{E}_t\delta_t$. We exploit the no-arbitrage conditions that allow us to move from the physical measure to the risk-neutral measure to obtain,

$$\begin{aligned}
\delta_t &= \mathbb{E}_t\left(m_{t\to T_1}\left(\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)\right)^{-1}\right) \\
&= \mathbb{E}_t\left(m_{t\to T_1}\right)\mathbb{E}_t\left(\frac{m_{t\to T_1}}{\mathbb{E}_t\left(m_{t\to T_1}\right)}\left(\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)\right)^{-1}\right) \\
&= \mathbb{E}_t\left(m_{t\to T_1}\right)\mathbb{E}^*_t\left(\left(\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)\right)^{-1}\right). \qquad \text{(A9)}
\end{aligned}$$

Next, we replace δ_t by its expression in (A7) and show that

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{1/\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)}{\mathbb{E}^*_t\left(1/\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)\right)}. \qquad \text{(A10)}$$

Similarly, we can use the SDF (A5) and show that

$$\frac{\mathbb{E}_{T_1} m_{T_1\to T_N}}{m_{T_1\to T_N}} = \frac{1/u^{'}\left[W^*_{T_N}\right]}{\mathbb{E}^*_{T_1}\left(1/u^{'}\left[W^*_{T_N}\right]\right)}. \qquad \text{(A11)}$$

Next, we write $\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right)$ in (A10) as a function of risk-neutral quantities:

$$\begin{aligned}
\mathbb{E}_{T_1}\left(u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right) &= \mathbb{E}_{T_1}\left(\frac{m_{T_1\to T_N}}{\mathbb{E}_{T_1}m_{T_1\to T_N}}\frac{\mathbb{E}_{T_1}m_{T_1\to T_N}}{m_{T_1\to T_N}}u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right) \\
&= \mathbb{E}^*_{T_1}\left(\frac{\mathbb{E}_{T_1}m_{T_1\to T_N}}{m_{T_1\to T_N}}u^{'}\left[W^*_{T_N}\right]\left(\omega^{*\intercal}_{T_1}R_{T_1\to T_N}\right)\right) \\
&= \frac{\omega^{*\intercal}_{T_1}\mathbb{E}^*_{T_1}R_{T_1\to T_N}}{\mathbb{E}^*_{T_1}\left(1/u^{'}\left[W^*_{T_N}\right]\right)}, \\
&= \frac{R_{f,T_1\to T_N}}{\mathbb{E}^*_{T_1}\left(1/u^{'}\left[W^*_{T_N}\right]\right)}, \qquad \text{(A12)}
\end{aligned}$$

where we have used the no-arbitrage conditions to move from the physical measure to the risk-neutral measure in the second equation, and Equation (A11) to obtain the third equation.

We replace (A12) in (A10) to obtain

$$
\begin{aligned}
\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} &= \frac{\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u'\left[W^*_{T_N}\right]}\right)}{R_{f,T_1\to T_N}}}{\mathbb{E}^*_t\left(\frac{\mathbb{E}^*_{T_1}\left(\frac{1}{u'\left[W^*_{T_N}\right]}\right)}{R_{f,T_1\to T_N}}\right)} \\
&= \frac{\left(\left(1/R_{f,T_1\to T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W^*_{T_N}\right]}\right)}{\mathbb{E}^*_t\left(\left(\left(1/R_{f,T_1\to T_N}\right)/\mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)\right)\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W^*_{T_N}\right]}\right)\right)}.
\end{aligned}
$$

Since there is no interest rate risk, $1/R_{f,T_1\to T_N} = \mathbb{E}_t\left(1/R_{f,T_1\to T_N}\right)$, this last expression simplifies to

$$
\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W^*_{T_N}\right]}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W^*_{T_N}\right]}\right)\right)}. \tag{A13}
$$

Assume that the gross return on the market can be used as proxy for the return on aggregate wealth:

$$
R_{M,t\to T_N} = \frac{W^*_{T_N}}{W_t} \quad \text{and} \quad R_{M,T_1\to T_N} = \frac{W^*_{T_N}}{W_{T_1}} \tag{A14}
$$

Equation (A13) can be rewritten as

$$
\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_t R_{M,t\to T_N}\right]}\right)}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left(\frac{u'\left[W_t R_{f,t\to T_N}\right]}{u'\left[W_t R_{M,t\to T_N}\right]}\right)\right)}. \tag{A15}
$$

This ends the proof.

A.3 Proof of Equation (4)

In this Section we detail the second-order expansion of the inverse of the marginal utility, which we use to derive Proposition 1.

Define the function

$$f[x,y] = \frac{u'[W_t x_0 y_0]}{u'[W_t xy]},$$

with $x = R_{M,t\to T_1}$, $y = R_{M,T_1\to T_N}$, $x_0 = R_{f,t\to T_1}$ and $y_0 = R_{f,T_1\to T_N}$. Since we assume no interest rate risk, $R_{M,t\to T_N} = xy$ and $R_{f,t\to T_N} = x_0 y_0$.

Let us write the inverse of the SDF as

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}(f[x,y])}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}(f[x,y])\right)}, \tag{A16}$$

where the function f is defined as

$$f[x,y] = \frac{u'[W_t x_0 y_0]}{u'[W_t xy]}$$

and $x = R_{M,t\to T_1}$, $x_0 = R_{f,t\to T_1}$, $y = R_{M,T_1\to T_N}$, and $y_0 = R_{f,t\to T_N}/R_{f,t\to T_1} = R_{f,T_1\to T_N}$. We adopt the following short notations. First, we use f_x and f_y to denote the first partial derivatives of the function f, f_{xx} and f_{yy} the second partial derivatives, and f_{xy} the cross-derivative, all evaluated at (x_0, y_0). Second, we denote as u', u'', and u''' the first, second, and third derivatives of $u[\cdot]$ evaluated at (x_0, y_0). We perform a second-order Taylor expansion series of $f[x,y]$ around $(x,y) = (x_0, y_0)$:

$$\begin{aligned} f[x,y] \approx\ & 1 + \frac{1}{1!}(x-x_0) f_x + \frac{1}{1!}(y-y_0) f_y + \frac{1}{2!}(x-x_0)^2 f_{xx} \\ & + \frac{1}{2!}(y-y_0)^2 f_{yy} + \frac{2}{2!}(x-x_0)(y-y_0) f_{xy}, \end{aligned}$$

where:

$$
\begin{aligned}
f_x &= \frac{y_0}{x_0} f_y = \frac{1}{x_0}\left(-\frac{\left(W_t x_0 y_0\right) u^{''}}{u^{'}}\right), \\
f_{xy} &= \frac{1}{x_0 y_0}\left(-\frac{W_t x_0 y_0 u^{''}}{u^{'}}\right) + \frac{1}{x_0 y_0}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{\left(u^{'}\right)^2}\left(2 - \frac{u^{'''} u^{'}}{\left(u^{''}\right)^2}\right), \\
f_{xx} &= \frac{y_0^2}{x_0^2} f_{yy} = \frac{1}{\left(x_0\right)^2}\frac{\left(W_t x_0 y_0 u^{''}\right)^2}{\left(u^{'}\right)^2}\left(2 - \frac{u^{'''} u^{'}}{\left(u^{''}\right)^2}\right).
\end{aligned}
$$

Note that $f_{xy} = f_{yx}$. Thus, we obtain,

$$
\begin{aligned}
f\left[x,y\right] \approx\ & 1 + \frac{1}{x_0}\frac{1}{\tau_t}\left(x - x_0\right) + \frac{1}{y_0}\frac{1}{\tau_t}\left(y - y_0\right) \\
& + \frac{1}{\left(x_0\right)^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(x - x_0\right)^2 + \frac{1}{\left(y_0\right)^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(y - y_0\right)^2 \\
& + \frac{1}{x_0 y_0}\left(\frac{1}{\tau_t} + \frac{2\left(1-\rho_t\right)}{\tau_t^2}\right)\left(x - x_0\right)\left(y - y_0\right), \qquad \text{(A17)}
\end{aligned}
$$

where τ_t and ρ_t are defined in equations (7) and (8). Replacing x, x_0, y, and y_0 by their expressions and using preference parameters $a_{1,t}$ and $a_{2,t}$ defined in Equations (7) and (8), we obtain,

$$
\begin{aligned}
\mathbb{E}_{T_1}^*\left(f\left[x,y\right]\right) =\ & 1 + \frac{a_{1,t}}{R_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) + \frac{a_{1,t}}{R_{f,T_1\to T_N}}\left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right) \\
& + \frac{a_{2,t}}{\left(R_{f,t\to T_1}\right)^2}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{\left(R_{f,T_1\to T_N}\right)^2}\mathbb{E}_{T_1}^*\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right) \\
& + \frac{a_{1,t} + 2a_{2,t}}{R_{f,t\to T_2}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\left(R_{f,T_1\to T_N} - R_{f,T_1\to T_N}\right). \qquad \text{(A18)}
\end{aligned}
$$

Thus, $\mathbb{E}_{T_1}^* f\left[x,y\right]$ simplifies to

$$
\mathbb{E}_{T_1}^* f\left[x,y\right] = \mathbb{E}_{T_1}^*\left(\frac{u^{'}\left[W_t R_{f,t\to T_1} R_{f,T_1\to T_N}\right]}{u^{'}\left[W_t R_{M,t\to T_1} R_{M,T_1\to T_N}\right]}\right) = 1 + z_{T_1}, \qquad \text{(A19)}
$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^{*(2)}_{T_1\to T_N} \tag{A20}$$

We then replace Equation (A19) in (A16) to obtain Equation (4).

A.4 Proof of Proposition 1

We use the expression for the SDF (4) derived in Section A.3, and plug it in the expected return expression identity (1). We obtain

$$\begin{aligned}\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \frac{1+z_{T_1}}{1+\mathbb{E}^*_t z_{T_1}}\right)\\ &= \mathbb{COV}^*_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}, \frac{1+z_{T_1}}{1+\mathbb{E}^*_t z_{T_1}}\right)\\ &= \mathbb{E}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\frac{1+z_{T_1}}{1+\mathbb{E}^*_t z_{T_1}}\right)\end{aligned} \tag{A21}$$

We then replace (A20) in (A21) and obtain the estimate for the market risk premium in Equation (10).

A.5 Restricted bound (13)

Let us assume that (i) odd market risk neutral moments are negative and (ii) conditions $1/\tau_t \geq 1$ and $\rho_t \geq 2$ hold. Under these conditions, $a_{1,t} \geq 1$ and $a_{2,t} \leq -1$. Hence,

$$\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} \geq \frac{1}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} \tag{A22}$$

$$\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} \geq \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} \tag{A23}$$

$$\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t \geq \frac{1}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t \tag{A24}$$

which shows that the numerator of (10) is larger than the numerator of (13). Furthermore,

$$\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} \leq \frac{-1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} \tag{A25}$$

$$\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N} \leq \frac{-1}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N} \tag{A26}$$

$$\tag{A27}$$

which shows that the denominator of (10) is smaller than the denominator of (13).

The inequality follows.

A.6 Proof of Corollary 2

The expected excess return can be decomposed into

$$\begin{aligned}\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right) &= \frac{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\left(\frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}\right)\\ &\quad+\frac{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\left(\frac{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}\right).\end{aligned}$$

Setting

$$\pi^*_t = \frac{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}}{1+\frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{t\to T_1}}$$

ends the proof.

Furthermore, provided that $\rho_t \geq 1$, i.e., $a_{2,t} \leq 0$, $\pi^*_t \geq 1$, i.e., $1-\pi^*_t \leq 0$. Assuming that $\mathbb{LEV}^*_t \leq 0$,

$$\mathbb{RP}^{\upsilon}_{t\to T_N} = \frac{\mathbb{LEV}^*_t}{\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}} \leq 0, \tag{A28}$$

which gives

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right) = \pi^*_t RP_{t\to T_1} + \left(1-\pi^*_t\right)\mathbb{RP}^{\upsilon}_{t\to T_N} \geq RP_{t\to T_1}. \tag{A29}$$

This shows that under minimal assumptions, our risk premium is always larger than the one of Chabi-Yo and Loudis (2020).

A.7 Comparison of the models of Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024)

In this subsection, we show that the SDF of Chabi-Yo and Loudis (2020) and Tetlock, McCoy, and Shah (2024) are equivalent.

In the model of Tetlock, McCoy, and Shah (2024), the representative agent can trade the market as well as $K-1$ derivatives securities, with payoffs the higher-order moments of the market returns. The excess return on the market is $\widetilde{R}_{M,t\to T_1} = R_{M,t\to T_1} - R_{f,t\to T_1}$ and the excess returns on the derivatives securities is $\widetilde{R}^k_{M,t\to T_1} - c_k$, with $c_k = \mathbb{E}^*_t\left(\widetilde{R}^k_{M,t\to T_1}\right)$.

Equation (9) of Tetlock, McCoy, and Shah (2024) gives the inverse SDF as follows:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = 1 + \sum_{k=1}^{K} \frac{1}{R_{f,t\to T_1}} \omega_{k,t} \left(\widetilde{R}^k_{M,t\to T_1} - c_k\right) \tag{A30}$$

where $\omega_{k,t}$ are the weights if the growth-optimal portfolio, held by logarithmic investors.

In comparison, the SDF of Chabi-Yo and Loudis (2020) is given by Equations (13) and (21) in their paper, as

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} \widetilde{R}^k_{M,t\to T_1}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} \tag{A31}$$

with

$$\widetilde{a}_{k,t} = \frac{a_{k,t}}{R^k_{f,t\to T_1}}$$

The SDF (A31) can be expanded as

$$\begin{aligned} \frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} &= \frac{1}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k} + \sum_{k=1}^{K} \left(\frac{\widetilde{a}_{k,t}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k}\right) \widetilde{R}^k_{M,t\to T_1} && \text{(A32)} \\ &= 1 + \sum_{k=1}^{K} \left(\frac{\widetilde{a}_{k,t}}{1 + \sum_{k=1}^{K} \widetilde{a}_{k,t} c_k}\right) \left(\widetilde{R}^k_{M,t\to T_1} - c_k\right) && \text{(A33)} \end{aligned}$$

Setting the weights as follows

$$\omega_{k,t} = R_{f,t\to T_1} \frac{\widetilde{a}_{k,t}}{1+\sum_{k=1}^{K} \widetilde{a}_{k,t} c_k},$$

we recover the SDF of Tetlock, McCoy, and Shah (2024).

The model of Tetlock, McCoy, and Shah (2024) allows for investors who are not logarithmic investors. The complexity of these investors portfolios determines the value of K. Consider $K = 3$, for illustration purposes. If the weights $\omega_{k,t}$ for $k = 1, ..., K$ are known (or computed) as it is the case in Tetlock, McCoy, and Shah (2024), then the coefficients $\widetilde{a}_{k,t}$ can be recovered and the preference parameters $1/\tau_t$, ρ_t and κ_t can be uniquely identified. In this case, the two SDFs are equivalent. Conversely, if the preference parameters $1/\tau_t$, ρ_t and κ_t are known, $\widetilde{a}_{k,t}$ can be recovered and the weights can be computed. The same reasoning extends to any finite $K > 3$.

A.8 Comparison of the models of Crescini, Trojani, and Vedolin (2025) and Tetlock, McCoy, and Shah (2024)

In this subsection, we show that the SDF in Tetlock, McCoy, and Shah (2024) can be rewritten in the form of the SDF used by Crescini, Trojani, and Vedolin (2025).

From the spanning formula of Carr and Madan (2001b), $\widetilde{R}^k_{M,t\to T_1}$, for $k > 1$ can be written as a collection of calls and puts:

$$\begin{aligned}
\widetilde{R}^k_{M,t\to T_1} &= \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^k \\
&= \frac{k(k-1)}{S_t^2} \left\{ \begin{array}{c} \int_{S_t R_{f,t\to T_1}}^{\infty} \left(\frac{K}{S_t} - R_{f,t\to T_1}\right)^k (S_{T_1} - K)^+ \, dK \\ + \int_0^{S_t R_{f,t\to T_1}} \left(\frac{K}{S_t} - R_{f,t\to T_1}\right)^k (K - S_{T_1})^+ \, dK \end{array} \right\}
\end{aligned}$$

where S_t is the spot market price at time t.

Substituting this expression in (A30), it follows that $\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}$ is a weighted sum of the excess market return, and payoffs of calls and puts (both tradable) with maturity T_1 and strike K. This SDF can thus be rewritten in form of the SDF of Crescini, Trojani, and Vedolin (2025):

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = 1 + \theta' \mathbf{R}^e$$

where $\mathbf{R}^e = \mathbf{R} - 1$ be the excess forward return of the index and options. $\theta' \mathbf{R}^e$ is a weighted sum of the excess market return and payoffs of tradable call and put options.

In the model of Crescini, Trojani, and Vedolin (2025), portfolio weights are allowed to vary across investors: $\theta = \theta_i$ to reflect varying demand of investors for the different assets. For logarithmic investors, the resulting SDF is equivalent to the one of Tetlock, McCoy, and Shah (2024). Other investors' SDF depend on their holdings, which allows Crescini, Trojani, and Vedolin (2025) to construct subjective measures of investor expected return and risk.

A.9 Comparison of our model to the model of Crescini, Trojani, and Vedolin (2025)

Unlike Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025), we do not model explicitely heterogenous investors' beliefs. It is therefore unclear whether our SDF incorporates information on option demand. In this subsection, we show that it does.

Our inverse SDF can be viewed as an extension of the inverse SDFs proposed by Chabi-Yo and Loudis (2020), augmented with the conditional risk-neutral variance of the market between time T_1 and time T_N (when a second-order expansion of the inverse marginal utility is used). We showed in Appendix A.7 that the SDF of Chabi-Yo and Loudis (2020) is equivalent to the one of Tetlock, McCoy, and Shah (2024), when the preference parameters are calibrated as functions of the weights in Tetlock, McCoy, and Shah (2024). Hence, our inverse SDF can be expressed as a weighted sum of the excess market return and a collection of call and put payoffs, augmented by the risk-neutral variance $\mathbb{M}^{*(2)}_{T_1\to T_N}$. As

$\mathbb{M}^{*(k)}_{T_1\to T_N} = \mathbb{E}^*_{T_1}\widetilde{R}^k_{M,T_1\to T_N}$ for $k > 1$ depends on future option prices, the spanning formula implies

$$\begin{aligned}\widetilde{R}^k_{M,T_1\to T_N} &= \left(\frac{S_{T_N}}{S_{T_1}} - R_{f,T_1\to T_N}\right)^k \\ &= \frac{k(k-1)}{S^2_{T_1}}\left\{\begin{array}{c} \int_{S_{T_1}R_{f,T_1\to T_N}}^{\infty}\left(\frac{K}{S_{T_1}} - R_{f,T_1\to T_N}\right)^k (S_{T_N}-K)^+\,dK \\ +\int_0^{S_{T_1}R_{f,T_1\to T_N}}\left(\frac{K}{S_{T_1}} - R_{f,T_1\to T_N}\right)^k (K-S_{T_N})^+\,dK \end{array}\right\}\end{aligned}$$

Thus

$$\mathbb{M}^{*(k)}_{T_1\to T_N} = \frac{k(k-1)R_{f,T_1\to T_N}}{S^2_{T_1}}\left\{\begin{array}{c} \int_{S_{T_1}R_{f,T_1\to T_N}}^{\infty}\left(\frac{K}{S_{T_1}} - R_{f,T_1\to T_N}\right)^k C_{T_N}[K]\,dK \\ +\int_0^{S_{T_1}R_{f,T_1\to T_N}}\left(\frac{K}{S_{T_1}} - R_{f,T_1\to T_N}\right)^k P_{T_N}[K]\,dK \end{array}\right\},$$

where $C_{T_N}[K]$ and $P_{T_N}[K]$ are the prices of call and put options with strike K and maturity T_N.

Therefore, our inverse SDF can be written as a weighted sum of the excess market return, a collection of payoffs on calls and puts and a set of payoffs from a strategy that buys call and put options expiring at T_N and sells them at T_1.

Our estimated parameters $a_{k,t}$ hence embed information on aggregate demand for options with maturity T_1 -as in Chabi-Yo and Loudis (2020), Tetlock, McCoy, and Shah (2024) and Crescini, Trojani, and Vedolin (2025)- as well as for strategies that involve buying options with maturity T_N and selling them at T_1. These strategies arise due to intertemporal hedging in a multi-period economy.

A.10 Physical variance

In this section, we provide expressions for the option-implied physical variance

$$\mathbb{E}_t\left(R_{M,t\to T_1} - \mathbb{E}_t R_{M,t\to T_1}\right)^2 = \mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 - \left(\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right)^2.$$

We already have an expression for $\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)$. Note that

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \mathbb{E}_t^*\left\{\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right\}.$$

Using the second-order approximation in Equation (4), we obtain

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(4)}_{t\to T_1} \\ + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\left(\mathbb{LEK}^*_t + \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}\right) \end{array}\right\}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}} \tag{A34}$$

where

$$\mathbb{LEK}^*_t = \mathbb{COV}^*_t\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2, \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right).$$

A.11 Proof of Proposition 5

Under no-arbitrage conditions, we use the Radon-Nikodym theorem. It allows us to move from the physical to the risk neutral measures and express the conditional crash probability as

$$\begin{aligned}\mathbb{P}_t\left(R_{M,t\to T_1} < \alpha\right) &= \mathbb{E}_t\left(\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) \\ &= \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right).\end{aligned} \tag{A35}$$

We then replace the inverse of the SDF by Equation (4) in the conditional crash probability to obtain,

$$\mathbb{P}_t\left(R_{M,t\to T_1} < \alpha\right) = \frac{\mathbb{E}^*_t\left(\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) + \mathbb{E}^*_t\left(z_{T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) + \mathbb{E}^*_t\left(z^{\upsilon}_{T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right)}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}}, \tag{A36}$$

where

$$
\begin{aligned}
\mathbb{E}_t^*\left(z_{T_1}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) &= \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) \\
&\quad + \frac{a_{2,t}}{R_{f,t\to T_1}^2}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right), \\
\mathbb{E}_t^*\left(z_{T_1}^v\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) &= \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\mathbb{E}_t^*\left(\mathbb{M}_{T_1\to T_N}^{*(2)}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right).
\end{aligned} \tag{A37}
$$

A.12 $RP_{t\to T_1,T_N}$ as a function of T_N

Using Equations (28) and (29), we can rewrite $RP_{t\to T_1,T_N}$ as a function of $\mathbb{M}_{t\to T_N}^{*(2)}$:

$$
RP_{t\to T_1,T_N} = \frac{A_{T_1} + B_{T_1}\mathbb{M}_{t\to T_N}^{*(2)}/R_{f,T_1\to T_N}^2}{C_{T_1} + D_{T_1}\mathbb{M}_{t\to T_N}^{*(2)}/R_{f,T_1\to T_N}^2} \tag{A38}
$$

with

$$
B_{T_1} = \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\frac{\mathbb{M}_{t\to T_1}^{*(3)}}{\mathbb{M}_{t\to T_1}^{*(4)} + 2R_{f,t\to T_1}\mathbb{M}_{t\to T_1}^{*(3)} + R_{f,t\to T_1}^2\mathbb{M}_{t\to T_1}^{*(2)}}, \tag{A39}
$$

$$
D_{T_1} = \frac{a_{2,t}}{R_{f,T_1\to T_N}^2}\frac{\mathbb{M}_{t\to T_1}^{*(2)}}{\mathbb{M}_{t\to T_1}^{*(4)} + 2R_{f,t\to T_1}\mathbb{M}_{t\to T_1}^{*(3)} + R_{f,t\to T_1}^2\mathbb{M}_{t\to T_1}^{*(2)}}, \tag{A40}
$$

and A_{T_1} and C_{T_1} gather all remaining terms in $RP_{t\to T_1,T_N}$.

Assuming $a_{2,t} \le 0$ and $\mathbb{LEV}_t^* \le 0$, we have $B_{T_1} \ge 0$ and $D_{T_1} \le 0$.

Compute the derivative of $RP_{t\to T_1,T_N}$ with respect to x to study the sign of this derivative.

$$
\frac{\partial RP_{t\to T_1,T_N}}{\partial x} = \frac{B_{T_1}(C_{T_1} + D_{T_1}x) - (A_{T_1} + B_{T_1}x)D_{T_1}}{(C_{T_1} + D_{T_1}x)^2} \tag{A41}
$$

We already know that $B_{T_1} \ge 0$ and $D_{T_1} \le 0$. Furthermore, for $x \ge 0$, the numerator of $RP_{t\to T_1,T_N}$ is positive, hence $A_{T_1} + B_{T_1}x \ge 0$. Assuming that $RP_{t\to T_1,T_N}$ is positive, the denominator $C_{T_1} + D_{T_1}x$ will also be positive: $C_{T_1} + D_{T_1}x \ge 0$. This gives

$$
\frac{\partial RP_{t\to T_1,T_N}}{\partial x} \ge 0, \tag{A42}
$$

hence, $RP_{t\to T_1,T_N}$ is increasing in x. As $R^2_{f,T_1\to T_N} \approx 1$, increasing T_N increases mechanically $x = \mathbb{M}^{*(2)}_{t\to T_N}/R^2_{f,T_1\to T_N}$, and hence $RP_{t\to T_1,T_N}$.

B Estimation of moments

We provide closed-form solutions to the risk-neutral and physical moments used in our analysis. In many cases, we use the spanning formula of Carr and Madan (2001a) and Bakshi and Madan (2000) to evaluate the risk-neutral expected value of a twice-differentiable function of the underlying asset price, $H(S_{T_1})$ as

$$\begin{aligned}\mathbb{E}_t^* H\left[S_{T_1}\right] &= H\left[S_t R_{f,t\to T_1}\right] + \mathbb{E}_t^* H_S\left[S_t R_{f,t\to T_1}\right] S_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\\ &\quad + R_{f,t\to T_1}\left[\int_{S_t R_{f,t\to T_1}}^{\infty} H_{SS}[K]\, C_t[K]\, dK + \int_0^{S_t R_{f,t\to T_1}} H_{SS}[K]\, P_t[K]\, dK\right],\end{aligned} \tag{B1}$$

where H_S and H_{SS} are the first and second derivative of function $H(\cdot)$, respectively. We evaluate the integral terms via numerical integration using the 1,000-point moneyness grid described in Section 3.2.

B.1 Closed-form expressions for $\mathbb{M}^{*(k)}_{t\to T_j}$ and $\mathbb{E}_t^*\left(R^k_{M,t\to T_j}\right)$

To evaluate the risk-neutral moments of order k, $\mathbb{M}^{*(k)}_{t\to T_j}$ and $\mathbb{E}_t^*\left(R^k_{M,t\to T_j}\right)$, we set $H\left(S_{T_j}\right) = \left(\frac{S_{T_j}}{S_t} - R_{f,t\to T_j}\right)^k$ and $H\left(S_{T_j}\right) = \left(\frac{S_{T_j}}{S_t}\right)^k$ in Equation (B1), respectively. Then, we use options with maturity T_j to evaluate Equation (B1).

B.2 Closed-form expression of $\mathbb{LEK}_t^*$

Notice that

$$\begin{aligned}\mathbb{LEK}_t^* &= \mathbb{COV}_t^*\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2, \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right)\\ &= \mathbb{E}_t^*\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\right)\\ &\quad - \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}_t^*\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2\\ &= \theta_t \mathbb{VAR}_t^*\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right)\end{aligned}$$

because

$$\mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2 = \theta_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2$$

Hence

$$\mathbb{LEK}^*_t = \theta_t\left(\mathbb{M}^{*(4)}_{t\to T_1} - \left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2\right)$$

B.3 Closed-form expression for $\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N}$ and $\mathbb{LES}^*_t$

We can write $\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N}$ and $\mathbb{LES}^*_t$ respectively as,

$$\begin{aligned}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N} &= \mathbb{E}^*_t\left(\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3\right)\\ &= \mathbb{E}^*_t\left(R^3_{M,T_1\to T_N} - R^3_{f,T_1\to T_N}\right) - 3R_{f,T_1\to T_N}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N},\end{aligned} \tag{B2}$$

and

$$\begin{aligned}\mathbb{LES}^*_t &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{M}^{*(3)}_{T_1\to T_N}\right)\\ &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N} - R^3_{f,T_1\to T_N}\right) - 3R_{f,T_1\to T_N}\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\\ &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N} - R^3_{f,T_1\to T_N}\right)\right) - 3R_{f,T_1\to T_N}\mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_1\to T_N}\right)\\ &= \mathbb{COV}^*_t\left(R_{M,t\to T_1}, \mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N} - R^3_{f,T_1\to T_N}\right)\right) - 3R_{f,T_1\to T_N}\mathbb{LEV}^*_t\end{aligned} \tag{B3}$$

Let us assume that the term $\mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N} - R^3_{f,T_1\to T_N}\right)$ is a nonlinear function g of $R_{M,t\to T_1} - R_{f,t\to T_1}$:

$$\mathbb{E}^*_{T_1}\left(R^3_{M,T_1\to T_N}\right) - R^3_{f,T_1\to T_N} = \gamma_t g[R_{M,t\to T_1} - R_{f,t\to T_1}] + \upsilon_t, \tag{B4}$$

with $\mathbb{E}_t^*\left(\upsilon_t|R_{M,t\to T_1}\right)=\mathbb{E}_t^*\left(\upsilon_t\right)=0$. Multiplying both sides of Equation (B4) by $R^3_{M,t\to T_1}$ and taking the time-t risk-neutral expectation, we obtain,

$$\gamma_t=\frac{\mathbb{M}^{*(3)}_{t\to T_N}+3R_{f,t\to T_N}\mathbb{M}^{*(2)}_{t\to T_N}-R^3_{f,T_1\to T_N}\left(\mathbb{M}^{*(3)}_{t\to T_1}+3R_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}\right)}{\mathbb{E}_t^*\left(R^3_{M,t\to T_1}g[R_{M,t\to T_1}-R_{f,t\to T_1}]\right)}. \tag{B5}$$

If we use $g[R_{M,t\to T_1}-R_{f,t\to T_1}]=R^3_{M,t\to T_1}$, we obtain

$$\gamma_t=\frac{\mathbb{M}^{*(3)}_{t\to T_N}+3R_{f,t\to T_N}\mathbb{M}^{*(2)}_{t\to T_N}-R^3_{f,T_1\to T_N}\left(\mathbb{M}^{*(3)}_{t\to T_1}+3R_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}\right)}{\mathbb{E}_t^*\left(R^6_{M,t\to T_1}\right)}, \tag{B6}$$

Taking the expectation of (B4) under the risk neutral measure,

$$\mathbb{E}_t^*\left(R^3_{M,T_1\to T_N}\right)-R^3_{f,T_1\to T_N}=\gamma_t\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right), \tag{B7}$$

Multiplying both sides of Equation (B4) by $R_{M,t\to T_1}$ and taking the time-t risk-neutral expectation

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}R^3_{M,T_1\to T_N}\right)=R_{f,t\to T_1}R^3_{f,T_1\to T_N}+\gamma_t\mathbb{E}_t^*\left(R^4_{M,t\to T_1}\right). \tag{B8}$$

Therefore, using Equations (B2) and (B3) we obtain $\mathbb{E}_t^*\mathbb{M}^{*(3)}_{T_1\to T_N}$ and $\mathbb{LES}_t^*$ as,

$$\mathbb{E}_t^*\mathbb{M}^{*(3)}_{T_1\to T_N}=\gamma_t\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right)-3R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_1\to T_N},$$

and

$$\mathbb{LES}_t^* \;=\; \gamma_t\mathbb{E}_t^*\left(R^4_{M,t\to T_1}\right)-R_{f,t\to T_1}\mathbb{E}_t^*\mathbb{M}^{*(3)}_{T_1\to T_N}-3R_{f,t\to T_1}R_{f,T_1\to T_N}\mathbb{E}_t^*\mathbb{M}^{*(2)}_{T_1\to T_N}-3R_{f,T_1\to T_N}\mathbb{LEV}_t^*$$

To compute the physical variance, we also need the following moments which we obtain using a similar approach:

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2=\mathbb{E}_t^*\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3\mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2$$

Using expression (26)

$$\mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2=\theta_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2+\epsilon_{T_1},$$

it follows that

$$\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2=\theta_t\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^5$$

In addition, let us provide a closed-form expression of another risk neutral quantity:

$$\begin{aligned}
& \mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^3 \\
= \; & \mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 R^3_{M,T_1\to T_N} \\
& -R^3_{f,T_1\to T_N}\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 \\
& +3R^2_{f,T_1\to T_N}\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 R_{M,T_1\to T_N} \\
& -3R_{f,T_1\to T_N}\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 R^2_{M,T_1\to T_N}
\end{aligned}$$

This expression simplifies to

$$\begin{aligned}
& \mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^3 \\
= \; & \mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 \mathbb{E}^*_{T_1}R^3_{M,T_1\to T_N} \\
& +R^3_{f,T_1\to T_N}\mathbb{M}^{*(2)}_{t\to T_1} \\
& -3R_{f,T_1\to T_N}\mathbb{E}^*_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}
\end{aligned}$$

Since

$$\mathbb{E}^*_{T_1}R^3_{M,T_1\to T_N}=\gamma_t R^3_{M,t\to T_1}$$

It follows that

$$\begin{aligned}
& \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^3 \\
= \; & \gamma_t \mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R^3_{M,t\to T_1}\right) \\
& + R^3_{f,T_1\to T_N} \mathbb{M}^{*(2)}_{t\to T_1} \\
& - 3R_{f,T_1\to T_N} \mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}\right)
\end{aligned}$$

where expression $\mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}\right)$ can be derived as follows:

$$\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N} = \theta_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^4 + \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \varepsilon_{T_1}$$

and

$$\begin{aligned}
\mathbb{E}_t^* \left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \mathbb{M}^{*(2)}_{T_1\to T_N}\right) &= \theta_t \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^4 + \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \varepsilon_{T_1} \\
&= \theta_t \mathbb{E}_t^* \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^4
\end{aligned}$$

B.4 Closed-form expression of $\mathbb{M}^{*(k)}_{t\to T_1}[\alpha]$

Recall that $\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] = \mathbb{E}_t^* \left\{\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^k \mathbb{1}_{S_{T_1} < \alpha S_t}\right\}$. Therefore, we set $H[x] = \left(\frac{x}{S_t} - R_{f,t\to T_1}\right)^k$ in Equation (B1) and obtain,

$$\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] = H[\alpha S_t] \mathbb{P}_t^* [S_{T_1} < \alpha S_t] - H_S[\alpha S_t] R_{f,t\to T_1} P_t[\alpha S_t] + R_{f,t\to T_1} \int_0^{\alpha S_t} H_{SS}[K] P_t[K] \, dK.$$

B.5 Closed-form expression of $\mathbb{E}_t^* \left(r^j_{M,t\to T_1} \mathbb{M}^{*(k)}_{T_1\to T_N} \mathbb{1}_{R_{M,t\to T_1} < \alpha}\right)$

Let us consider the case $k = 2$. First, we have

$$\mathbb{M}^*_{t,\upsilon}[\alpha] \equiv \mathbb{E}_t^* \left(\mathbb{M}^{*(2)}_{T_1\to T_N} \mathbb{1}_{R_{M,t\to T_1} < \alpha}\right) = \theta_t \mathbb{M}^{*(2)}_{t\to T_1}[\alpha], \tag{B9}$$

and

$$\mathbb{M}^{*}_{t,sv}\left[\alpha\right] \equiv \mathbb{E}^{*}_{t}\left(\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right) \quad = \quad \theta_t \mathbb{M}^{*(3)}_{t\to T_1}\left[\alpha\right]. \tag{B10}$$

Next, we can write the future third central moment as,

$$\mathbb{M}^{*(3)}_{T_1\to T_N} \quad = \quad \mathbb{E}^{*}_{T_1}\left(R^3_{M,T_1\to T_N}\right) - R^3_{f,T_1\to T_N} - 3R_{f,T_1\to T_N}\mathbb{E}^{*}_{T_1}\left(R^2_{M,T_1\to T_N}\right) + 3R^3_{f,T_1\to T_N} \tag{B11}$$

$$\begin{aligned}\mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(3)}_{T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \quad = \quad & \mathbb{E}^{*}_{t}\left(R^3_{M,T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \\ & -3R_{f,T_1\to T_N}\mathbb{E}^{*}_{t}\left(\mathbb{E}^{*}_{T_1}R^2_{M,T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \\ & +2R^3_{f,T_1\to T_N}\mathbb{E}^{*}_{t}1_{R_{M,t\to T_1}<\alpha}\end{aligned}$$

which simplifies to

$$\begin{aligned}\mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(3)}_{T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \quad = \quad & \mathbb{E}^{*}_{t}\left(R^3_{M,T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \\ & -3R_{f,T_1\to T_N}\mathbb{E}^{*}_{t}\left(\left(\mathbb{M}^{*(2)}_{T_1\to T_N} + R^2_{f,T_1\to T_N}\right)1_{R_{M,t\to T_1}<\alpha}\right) \\ & +2R^3_{f,T_1\to T_N}\mathbb{E}^{*}_{t}1_{R_{M,t\to T_1}<\alpha}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(3)}_{T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \quad = \quad & \mathbb{E}^{*}_{t}\left(1_{R_{M,t\to T_1}<\alpha}\mathbb{E}^{*}_{T_1}R^3_{M,T_1\to T_N}\right) \\ & -3R_{f,T_1\to T_N}\mathbb{E}^{*}_{t}\left(\mathbb{M}^{*(2)}_{T_1\to T_N}1_{R_{M,t\to T_1}<\alpha}\right) \\ & -R^3_{f,T_1\to T_N}\mathbb{E}^{*}_{t}1_{R_{M,t\to T_1}<\alpha}\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}^*_{T_1} R^3_{M,T_1\to T_N} &= \gamma_t R^3_{M,t\to T_1} + \varepsilon_{T_1} \\ \mathbb{M}^{*(2)}_{T_1\to T_N} &= \theta_t \left(R_{M,t\to T_1} - R_{M,t\to T_1}\right)^2 + \eta_{T_1}\end{aligned}$$

with

$$\mathbb{E}^*_t\left[\varepsilon_{T_1} | R_{M,t\to T_1}\right] = 0 \text{ and } \mathbb{E}^*_t\left[\eta_{T_1} | R_{M,t\to T_1}\right] = 0$$

Hence

$$\begin{aligned}\mathbb{E}^*_t\left(\mathbb{M}^{*(3)}_{T_1\to T_N} 1_{R_{M,t\to T_1}<\alpha}\right) &= \gamma_t \mathbb{E}^*_t\left(1_{R_{M,t\to T_1}<\alpha} R^3_{M,t\to T_1}\right) \\ &\quad -3R_{f,T_1\to T_N}\theta_t \mathbb{E}^*_t\left(\left(R_{M,t\to T_1} - R_{M,t\to T_1}\right)^2 1_{R_{M,t\to T_1}<\alpha}\right) \\ &\quad -R^3_{f,T_1\to T_N}\mathbb{E}^*_t 1_{R_{M,t\to T_1}<\alpha} \qquad \text{(B12)}\end{aligned}$$

Recall that

$$\begin{aligned}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3 &= R^3_{M,t\to T_1} - R^3_{f,t\to T_1} - 3R^2_{M,t\to T_1}R_{f,t\to T_1} + 3R^2_{f,t\to T_1}R_{M,t\to T_1} \\ &= R^3_{M,t\to T_1} - R^3_{f,t\to T_1} - 3\begin{pmatrix}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 \\ +2R_{M,t\to T_1}R_{f,t\to T_1} - R^2_{f,t\to T_1}\end{pmatrix} R_{f,t\to T_1} + 3R^2_{f,t\to T_1}R_{M,} \\ &= R^3_{M,t\to T_1} - R^3_{f,t\to T_1} - 3\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R_{f,t\to T_1} \\ &\quad -6R_{M,t\to T_1}R^2_{f,t\to T_1} + 3R^3_{f,t\to T_1} + 3R^2_{f,t\to T_1}R_{M,t\to T_1}\end{aligned}$$

and

$$\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3 = R^3_{M,t\to T_1} - 3\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R_{f,t\to T_1} - 3R_{M,t\to T_1}R^2_{f,t\to T_1} + 2R^3_{f,t\to T_1}$$

That is

$$R^3_{M,t\to T_1} = \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^3 + 3\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 R_{f,t\to T_1} + 3R_{M,t\to T_1}R^2_{f,t\to T_1} - 2R^3_{f,t\to T_1}$$

We can then simplify (B12) as

$$\mathbb{E}_t^*\left(\mathbb{M}_{T_1\to T_N}^{*(3)}1_{R_{M,t\to T_1}<\alpha}\right) = \gamma_t\mathbb{E}_t^*\left(1_{R_{M,t\to T_1}<\alpha}\left(\begin{array}{c}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3\\ +3\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 R_{f,t\to T_1}\\ +3R_{M,t\to T_1}R_{f,t\to T_1}^2-2R_{f,t\to T_1}^3\end{array}\right)\right)$$
$$-3R_{f,T_1\to T_N}\theta_t\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{M,t\to T_1}\right)^2 1_{R_{M,t\to T_1}<\alpha}\right)$$
$$-R_{f,T_1\to T_N}^3\mathbb{E}_t^*1_{R_{M,t\to T_1}<\alpha}$$

Finally

$$\mathbb{E}_t^*\left(\mathbb{M}_{T_1\to T_N}^{*(3)}1_{R_{M,t\to T_1}<\alpha}\right) = \gamma_t\left\{\begin{array}{c}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3 1_{R_{M,t\to T_1}<\alpha}\right)\\ +3R_{f,t\to T_1}\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2 1_{R_{M,t\to T_1}<\alpha}\right)\\ +3R_{f,t\to T_1}^2\mathbb{E}_t^*\left(R_{M,t\to T_1}1_{R_{M,t\to T_1}<\alpha}\right)\\ -2R_{f,t\to T_1}^3\mathbb{E}_t^*\left(1_{R_{M,t\to T_1}<\alpha}\right)\end{array}\right\}$$
$$-3R_{f,T_1\to T_N}\theta_t\mathbb{E}_t^*\left(\left(R_{M,t\to T_1}-R_{M,t\to T_1}\right)^2 1_{R_{M,t\to T_1}<\alpha}\right)$$
$$-R_{f,T_1\to T_N}^3\mathbb{E}_t^*1_{R_{M,t\to T_1}<\alpha}$$

and

$$\mathbb{E}_t^*\left(\mathbb{M}_{T_1\to T_N}^{*(3)}1_{R_{M,t\to T_1}<\alpha}\right) = \gamma_t\left\{\begin{array}{c}\mathbb{M}_{t\to T_1}^{*(3)}[\alpha]+3R_{f,t\to T_1}\mathbb{M}_{t\to T_1}^{*(2)}[\alpha]\\ +3R_{f,t\to T_1}^2\mathbb{M}_{t\to T_1}^{*(1)}[\alpha]+R_{f,t\to T_1}^3\mathbb{M}_{t\to T_1}^{*(0)}[\alpha]\end{array}\right\}$$
$$-3R_{f,T_1\to T_N}\mathbb{M}_{t,v}^{*}[\alpha]-R_{f,T_1\to T_N}^3\mathbb{M}_{t\to T_1}^{*(0)}[\alpha]$$

C Validity and tightness tests

We follow the methodology of Back, Crotty, and Kazempour (2022) and test for the validity and tightness of these bounds. We recall below the methodology.

The test is based on the set of inequalities

$$\mathbb{E}_t[((R_{M,t\to T_1} - R_{f,t\to T_1}) - RP_{t\to T_1})z_t] \geq 0$$

for a vector z_t of nonnegative conditioning variables in the time-t information set. Back, Crotty, and Kazempour (2022) refer to $((R_{M,t\to T_1} - R_{f,t\to T_1}) - RP_{t\to T_1})$ as realized slackness. We consider $RP_{t\to T_1} \in \{RP^{Log}_{t\to T_1}, RP_{t\to T_1}, RP_{t\to T_1,T_N^*}\}$. The vector z_t contains variables from Welch and Goyal (2008): Dividend Price Ratio (defined as the difference between the log of dividends and the log of the S&P 500 index price level, plus 5 to ensure a positive conditioning variable); Earnings Price Ratio (defined as the difference between the log of earnings and the log of the S&P 500 index price level, plus 5 to ensure a positive conditioning variable); Book-to-Market Ratio (the ratio of book value to market value for the Dow Jones Industrial Average); T-bill Rate (the 3-month Treasury bill rate); 1 + Term Spread (defined as the difference between the long-term yield from Ibbotson's and the 3-month T-bill rate, plus 1 to ensure a positive conditioning variable); Credit Spread (defined as difference between BAA and AAA-rated corporate bond yields); Stock Variance (defined as the sum of squared daily returns on the S&P 500); 1 + Net Equity Issuance (the ratio of 12-month moving sums of net issues by NYSE-listed stocks to the total end-of-year market capitalizations of NYSE stocks, plus 1 to ensure a positive conditioning variable); 1 + Inflation (the Consumer Price Index, plus 1 to ensure a positive conditioning variable).

Denote by λ_0 the population mean of $\mathbb{E}_t[((R_{M,t\to T_1} - R_{f,t\to T_1}) - RP_{t\to T_1})z_t]$. To evaluate a bound's validity, we test the null hypothesis $\lambda_0 \geq 0$ against the alternative that λ_0 is unrestricted. To test tightness, we test the null hypothesis $\lambda_0 = 0$ against the alternative that $\lambda_0 \geq 0$.

Let $\bar{\lambda}$ denote the sample mean of $((R_{M,t\to T_1} - R_{f,t\to T_1}) - RP_{t\to T_1})z_t$, and Σ its sample covariance. Validity is tested using the statistic D_1, defined as

$$D_1 = \min_{\lambda \geq 0} (\lambda - \bar{\lambda})\Sigma^{-1}(\lambda - \bar{\lambda}) \tag{C13}$$

Under the null that $\lambda_0 \geq 0$, Back, Crotty, and Kazempour (2022) show that D_1 is asymptotically distributed as a mixture of chi- square distributions.

Under validity, tightness is tested using the statistic

$$D_2 = \hat{\lambda}' \Sigma^{-1} \hat{\lambda}, \tag{C14}$$

where $\hat{\lambda}$ is the vector of λ which reaches the minimum in equation (C13). under the null that $\lambda_0 = 0$, D_2 is also asymptotically distributed as a mixture of chi-square distributions.

Following Back, Crotty, and Kazempour (2022), we calculate finite sample $p-$ values using Monte-Carlo simulations.

Table A1 reports the results, for investors' horizons $T_2 = 1$ and 2 years. The statistic D_1 is zero for $RP^{Log}_{t\to T_1}$ and for $RP_{t\to T_1}$ for most forecast horizons T_1. It is positive for $RP_{t\to T_1,1y}$, but always below the 10% critical value. It is positive but larger for $RP_{t\to T_1,2y}$, such that validity is rejected for half of the forecast horizons. The statistic D_2 is the largest for $RP^{Log}_{t\to T_1}$, slightly smaller for $RP_{t\to T_1}$, and smaller for $RP_{t\to T_1,1y}$ and $RP_{t\to T_1,2y}$. Tightness is rejected for all bounds, except for $RP_{t\to T_1,1y}$ and $RP_{t\to T_1,2y}$ at $T_1 = 10$ days.

D Portfolio Rebalancing: Implementation

To compute the risk neutral quantities, we use an approach similar to (26) by considering the decomposition:

$$\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} = \theta_{T_{Q_{j-1}}\to T_{Q_j}} \left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}} \right)^2 + \eta_{T_{Q_{j-1}}} \tag{D1}$$

Table A1: **Validity and tightness tests**

We report test statistics for bound validity and bound tightness for different equity risk premium bounds. The test statistics D_1 and D_2 are defined in equation (C13) and (C14). The sample moments are means of weekly realized slackness and weekly realized slackness interacted with each of the variables from Welch and Goyal (2008) in z_t. The p-values (in percent notation) are based on 100 Monte-Carlo simulations. Data are weekly from January 1996 to February 2023.

	$RP_{t\to T_1}^{Log}$	$RP_{t\to T_1}$	$RP_{t\to T_1,1y}$	$RP_{t\to T_1,2y}$
Panel A: One week				
D_1	0.00	0.00	0.39	1.91
p-value for validity	0.91	0.91	0.76	0.43
D_2	15.19	15.00	7.94	4.11
p-value for tightness	0.05	0.06	0.42	0.81
Panel B: One month				
D_1	0.00	0.07	3.06	10.94
p-value for validity	0.93	0.76	0.24	0.01
D_2	82.60	80.94	69.15	61.51
p-value for tightness	0.00	0.00	0.00	0.00
Panel C: Two months				
D_1	0.00	0.00	2.30	10.46
p-value for validity	0.88	0.83	0.27	0.01
D_2	225.41	221.85	193.71	168.46
p-value for tightness	0.00	0.00	0.00	0.00
Panel D: One quarter				
D_1	0.00	0.00	1.71	8.02
p-value for validity	0.83	0.82	0.32	0.02
D_2	405.06	395.76	352.33	304.26
p-value for tightness	0.00	0.00	0.00	0.00
Panel E: Six months				
D_1	0.00	0.00	0.64	4.33
p-value for validity	0.79	0.83	0.46	0.11
D_2	707.61	680.64	636.71	548.28
p-value for tightness	0.00	0.00	0.00	0.00

with $\mathbb{E}^*\left(\eta_{T_{Q_{j-1}}}|R_{M,t\to T_{Q_{-1}}}\right)=0$. We then show:

$$\theta_{T_{Q_{j-1}}\to T_{Q_j}}=\frac{\mathbb{M}^{*(2)}_{t\to T_{Q_j}}-R^2_{f,T_{Q_{j-1}}\to T_{Q_j}}\mathbb{M}^{*(2)}_{t\to T_{Q_{j-1}}}}{\mathbb{E}^*_t\left(R^2_{M,t\to T_{Q_{j-1}}}\left(R_{M,t\to T_{Q_{j-1}}}-R_{f,t\to T_{Q_{j-1}}}\right)^2\right)}.$$

and

$$\mathcal{LEV}^*_t=\sum_{j>1}^{J}\frac{1}{R^2_{f,T_{Q_{j-1}}\to T_{Q_j}}}\mathbb{COV}^*_t\left(R_{M,t\to T_1},\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right). \tag{D2}$$

with

$$\mathbb{COV}^*_t\left(R_{M,t\to T_1},\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) = \mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right)-R_{f,t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} \tag{D3}$$

Taking the expectation, under the risk neutral measure, of (D1) at time t leads to

$$\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}} = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\mathbb{M}^{*(2)}_{t\to T_{Q_{j-1}}}$$

If $T_{Q_{j-1}}=T_1$, (D3) simplifies to

$$\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_{Q_j}}\right) = \theta_{T_1\to T_{Q_j}}\left(\mathbb{M}^{*(3)}_{t\to T_1}+R_{f,t\to T_1}\mathbb{M}^{*(2)}_{t\to T_1}\right)$$

Now, assume that $T_{Q_{j-1}}>T_1$. We then replace $\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}$ by its decomposition and show

$$\begin{aligned}\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) &= \mathbb{E}^*_t\left(R_{M,t\to T_1}\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left(R_{M,t\to T_{Q_{j-1}}}-R_{f,t\to T_{Q_{j-1}}}\right)^2\right)\\ &\quad+\mathbb{E}^*_t\left(R_{M,t\to T_1}\eta_{T_{Q_{j-1}}}\right)\end{aligned}$$

Since $T_{Q_{j-1}}>T_1$, it follows that

$$\mathbb{E}^*_t\left(R_{M,t\to T_1}\eta_{T_{Q_{j-1}}}\right)=\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{E}^*_{T_1}\eta_{T_{Q_{j-1}}}\right)$$

Given that $\mathbb{E}^*_{T_1}\eta_{T_{Q_{j-1}}} = 0$, it follows that

$$\begin{aligned}\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) &= \mathbb{E}^*_t\left(R_{M,t\to T_1}\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}}\right)^2\right) \\ &= \theta_{T_{Q_{j-1}}\to T_{Q_j}}\mathbb{E}^*_t\left(R_{M,t\to T_1}\left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}}\right)^2\right)\end{aligned}$$

Observe that

$$\left(R_{M,t\to T_{Q_{j-1}}} - R_{f,t\to T_{Q_{j-1}}}\right)^2 = R^2_{M,t\to T_{Q_{j-1}}} - 2R_{M,t\to T_{Q_{j-1}}}R_{f,t\to T_{Q_{j-1}}} + R^2_{f,t\to T_{Q_{j-1}}}$$

Thus

$$\begin{aligned}\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) &= \theta_{T_{Q_{j-1}}\to T_{Q_j}}\mathbb{E}^*_t\left(R_{M,t\to T_1}\left(R^2_{M,t\to T_{Q_{j-1}}} - 2R_{M,t\to T_{Q_{j-1}}}R_{f,t\to T_{Q_{j-1}}} + R^2_{f,t\to T_{Q_{j-1}}}\right)\right) \\ &= \theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\mathbb{E}^*_t\left(R_{M,t\to T_1}R^2_{M,t\to T_{Q_{j-1}}}\right) \\ -2R_{f,t\to T_{Q_{j-1}}}\mathbb{E}^*_t\left(R_{M,t\to T_1}R_{M,t\to T_{Q_{j-1}}}\right) \\ +\mathbb{E}^*_t\left(R_{M,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\right)\end{array}\right\}\end{aligned}$$

Since $R_{M,t\to T_{Q_{j-1}}} = R_{M,t\to T_1}R_{M,T_1\to T_{Q_{j-1}}}$, the above expression simplifies to

$$\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\mathbb{E}^*_t\left(R^3_{M,t\to T_1}R^2_{M,T_1\to T_{Q_{j-1}}}\right) \\ -2R_{f,t\to T_{Q_{j-1}}}\mathbb{E}^*_t\left(R^2_{M,t\to T_1}R_{M,T_1\to T_{Q_{j-1}}}\right) \\ +R_{f,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\end{array}\right\}$$

and

$$\mathbb{E}^*_t\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right) = \theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\mathbb{E}^*_t\left(R^3_{M,t\to T_1}R^2_{M,T_1\to T_{Q_{j-1}}}\right) \\ -2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}^*_t\left(R^2_{M,t\to T_1}\right) \\ +R_{f,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\end{array}\right\}$$

We further expand this expression to

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right)=\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\left(R^2_{M,T_1\to T_{Q_{j-1}}}-R^2_{f,T_1\to T_{Q_{j-1}}}+R^2_{f,T_1\to T_{Q_{j-1}}}\right)\right)\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^2_{M,t\to T_1}\right)\\+R_{f,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\end{array}\right\}$$

which simplifies to

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right)=\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_{Q_{j-1}}}\right)\\+R^2_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right)\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{M}^{*(2)}_{t\to T_1}\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}R^2_{f,t\to T_1}+R_{f,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\end{array}\right\} \tag{D4}$$

Recall that

$$\mathbb{M}^{*(2)}_{T_1\to T_{Q_{j-1}}}=\theta_{T_1\to T_{Q_{j-1}}}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2+\eta_{T_1}\text{ with }\mathbb{E}_t^*\left(\eta_{T_1}|R_{M,t\to T_1}\right)$$

Hence, (D4)

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right)=\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\theta_{T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\right)\\+R^2_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right)\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{M}^{*(2)}_{t\to T_1}\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}R^2_{f,t\to T_1}+R_{f,t\to T_1}R^2_{f,t\to T_{Q_{j-1}}}\end{array}\right\} \tag{D5}$$

Thus

$$\mathbb{E}_t^*\left(R_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}\right)=\theta_{T_{Q_{j-1}}\to T_{Q_j}}\left\{\begin{array}{c}\theta_{T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\right)\\+R^2_{f,T_1\to T_{Q_{j-1}}}\mathbb{E}_t^*\left(R^3_{M,t\to T_1}\right)-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}\mathbb{M}^{*(2)}_{t\to T_1}\\-2R_{f,t\to T_{Q_{j-1}}}R_{f,T_1\to T_{Q_{j-1}}}R^2_{f,t\to T_1}+R^2_{f,t\to T_{Q_{j-1}}}R_{f,t\to T_1}\end{array}\right\}$$

Provided that odd market risk neutral moments and the risk neutral leverage $\mathcal{LEV}_t^*$ are negative and conditions $1/\tau_t \geq 1$and $\rho_t - 1 \geq 1$ hold, we can further bound (34) as follows:

$$RP_{t\to T_1,T_N} \geq \frac{\frac{1}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} - \mathcal{LEV}_t^*}{1 - \frac{1}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} - \mathbb{E}_t^*\mathcal{M}^{*(2)}_{T_{Q_{j-1}}\to T_{Q_j}}}.$$

We then use option prices to recover the expected excess market return.

E Higher-order expansion

In this section, we investigate how higher-order moments contribute to the equity risk premium. We show that increasing the order of the approximation, therefore allowing for kurtosis preference, generates additional terms that contribute to the equity risk premium.

E.1 One-period SDF

Under no-arbitrage assumptions, a third-order Taylor expansion-series produces a one-period SDF in a three-date (two-period) economy of the form

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} \approx \frac{1 + z_{T_1} + z^{\upsilon}_{T_1}}{\mathbb{E}_t^*\left(1 + z_{T_1} + z^{\upsilon}_{T_1}\right)}, \tag{E1}$$

where

$$z_{T_1} = \frac{a_{1,t}}{R_{f,t\to T_1}}(R_{M,t\to T_1} - R_{f,t\to T_1}) + \frac{a_{2,t}}{R^2_{f,t\to T_1}}(R_{M,t\to T_1} - R_{f,t\to T_1})^2 + \frac{a_{3,t}}{R^3_{f,t\to T_1}}(R_{M,t\to T_1} - R_{f,t\to T_1})^3,$$

$$z^{\upsilon}_{T_1} = \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^{*(2)}_{T_1\to T_N} + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{M}^{*(3)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}(R_{M,t\to T_1} - R_{f,t\to T_1})\mathbb{M}^{*(2)}_{T_1\to T_N}, \tag{E2}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$.

Proof. Consider the partial derivatives

$$
\begin{aligned}
f_{xxy} &= \frac{2}{(x_0)^2 y_0}\frac{\left(W_t x_0 y_0 u''\right)^2}{(u')^2}\left(2-\frac{u''' u'}{(u'')^2}\right) \\
&\quad +\frac{1}{(x_0)^2 y_0}\left\{6\frac{(W_t x_0 y_0)^3 u'' u' u'''}{(u')^3}-(W_t x_0 y_0)^3\frac{u''''}{u'}-6\frac{\left(W_t x_0 y_0 u''\right)^3}{(u')^3}\right\}, \\
f_{xxx} &= \frac{y_0^3}{x_0^3} f_{yyy} = \frac{1}{(x_0)^3}\left(6\frac{(W_t x_0 y_0)^3 u'' u'''}{(u')^2}-\frac{(W_t x_0 y_0)^3 u''''}{u'}-6\frac{(W_t x_0 y_0)^3 (u'')^3}{(u')^3}\right).
\end{aligned}
$$

Thus, a third order Taylor expansion-series yields

$$
\begin{aligned}
f[x,y] &= f[x,y]^{2nd} \\
&\quad +\frac{1}{(x_0)^3}\frac{(\kappa_t+1-2\rho_t)}{\tau_t^3}(x-x_0)^3+\frac{1}{(y_0)^3}\frac{(\kappa_t+1-2\rho_t)}{\tau_t^3}(y-y_0)^3 \\
&\quad +\frac{1}{(x_0)^2 y_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}+\frac{3(\kappa_t+1-2\rho_t)}{\tau_t^3}\right)(x-x_0)^2(y-y_0) \\
&\quad +\frac{1}{x_0 (y_0)^2}\left(\frac{2(1-\rho_t)}{\tau_t^2}+\frac{3(\kappa_t+1-2\rho_t)}{\tau_t^3}\right)(y-y_0)^2(x-x_0), \qquad \text{(E3)}
\end{aligned}
$$

where $f[x,y]^{2nd}$ is the second order Taylor expansion-series in Equation (A17).

Replacing x, x_0, y, and y_0 by their expressions and using preference parameters a_1, a_2, and a_3 defined in Equation (6), we obtain,

$$
\begin{aligned}
\mathbb{E}^*_{T_1}(f[x,y]) &= 1+\frac{a_{1,t}}{R_{f,t\to T_1}}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)+\frac{a_{1,t}}{R_{f,T_1\to T_N}}\left(R_{f,T_1\to T_N}-R_{f,T_1\to T_N}\right) \\
&\quad +\frac{a_{2,t}}{\left(R_{f,t\to T_1}\right)^2}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2+\frac{a_{2,t}}{\left(R_{f,T_1\to T_N}\right)^2}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2\right) \\
&\quad +\frac{a_{1,t}+2a_{2,t}}{R_{f,t\to T_2}}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\left(R_{f,T_1\to T_N}-R_{f,T_1\to T_N}\right) \\
&\quad +\frac{a_{3,t}}{\left(R_{f,t\to T_1}\right)^3}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^3+\frac{a_{3,t}}{\left(R_{f,T_1\to T_N}\right)^3}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^3\right) \\
&\quad +\frac{2a_{2,t}+3a_{3,t}}{\left(R_{f,t\to T_1}\right)^2 R_{f,T_1\to T_N}}\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)^2\left(R_{f,T_1\to T_N}-R_{f,T_1\to T_N}\right) \\
&\quad +\frac{2a_{2,t}+3a_{3,t}}{R_{f,t\to T_1}\left(R_{f,T_1\to T_N}\right)^2}\mathbb{E}^*_{T_1}\left(\left(R_{M,T_1\to T_N}-R_{f,T_1\to T_N}\right)^2\right)\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right) \qquad \text{(E4)}
\end{aligned}
$$

which gives the desired result when interest rates are deterministic.

■

When (E2) is removed from the SDF specification (E1), which corresponds to a static SDF in a one-period economy, the equity risk premium reduces to the expected excess return in Chabi-Yo and Loudis. We refer to the Chabi-Yo and Loudis bounds to as $RP^{3rd}_{t\to T_1}$.

Using this third-order expansion, we next derive the equity risk premium.

E.2 Equity risk premium

With the third-order Taylor expansion-series approach, Equation (E1) depends on, in addition to risk-neutral variance, new terms such as risk-neutral skewness and cross-term between risk-neutral volatility and market excess return. These additional terms, as shown below, introduce additional high-order leverage effects in the expected excess return decomposition. To find a closed-form expression for the equity risk premium in terms of risk-neutral moments and high-order leverages, we first define high-order leverage effects under the risk-neutral measure as:

$$\mathbb{LES}^*_t = \mathbb{COV}^*_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}, \mathbb{M}^{*(3)}_{T_1\to T_N}\right), \tag{E5}$$

$$\mathbb{LEK}^*_t = \mathbb{COV}^*_t\left((R_{M,t\to T_1} - R_{f,t\to T_1})^2, \mathbb{M}^{*(2)}_{T_1\to T_N}\right). \tag{E6}$$

We then show how the equity risk premium depends on these terms in the following Proposition.

Proposition 6 *Up to the third-order Taylor expansion-series of the inverse marginal utility, the one-period expected excess market return obeys the following decomposition*

$$\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) = \pi^o_t RP^{3rd}_{t\to T_1} + (1-\pi^o_t)\,\mathbb{RP}^{v,s}_t, \tag{E7}$$

with

$$RP^{3rd}_{t\to T_1} = \frac{\frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T_1}}\mathbb{M}^{*(4)}_{t\to T_1}}{1 + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_3}{R^3_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}}, \tag{E8}$$

$$\mathbb{RP}^{\upsilon,s}_t = \frac{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{LES}^*_t + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\left(\mathbb{LEK}^*_t + \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}\right)}{\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N} + \frac{a_3}{R^3_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}, \tag{E9}$$

and

$$\pi^o_t = \frac{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}}{1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}. \tag{E10}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ *and the risk-neutral quantities* $\mathbb{LEV}^*_t$, $\mathbb{M}^{*(k)}_{T_i\to T_j}$, $\mathbb{LES}^*_t$ *and* $\mathbb{LEK}^*_t$ *are defined in Equations* (11), (12), (E5), *and* (E6), *respectively.*

Proof. This proposition results from Proposition 7 below. ■

Proposition 7 *Up to a third-order expansion-series, the one-period expected excess market return is*

$$RP^{3rd}_{t\to T_1,T_N} = \frac{\mathcal{D}_{1,t} + \mathcal{D}_{2,t}}{\mathcal{D}_{3,t} + \mathcal{D}_{4,t}} \tag{E11}$$

with

$$\begin{aligned}
\mathcal{D}_{1,t} &= \sum_{k=1}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k+1)}_{t\to T_1}\\
\mathcal{D}_{2,t} &= \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{LES}^*_t + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\left(\mathbb{LEK}^*_t + \mathbb{M}^{*(2)}_{t\to T_1}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\\
\mathcal{D}_{3,t} &= 1 + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}\\
\mathcal{D}_{4,t} &= \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t
\end{aligned}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ *and the risk-neutral quantities* $\mathbb{LEV}^*_t$, $\mathbb{M}^{*(k)}_{T_i \to T_j}$, $\mathbb{LES}^*_t$ *and* $\mathbb{LEK}^*_t$ *are defined in Equations* (11), (12), (E5), *and* (E6), *respectively.*

Proof. The expected excess market return is

$$\mathbb{E}_t \left(R_{t\to T_1} - R_{f,t\to T_1} \right) = \mathbb{COV}^*_t \left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}, \left(R_{t\to T_1} - R_{f,t\to T_1} \right) \right).$$

We then replace the inverse SDF by its expression and obtain

$$\begin{aligned}
\mathbb{E}_t \left(R_{t\to T_1} - R_{f,t\to T_1} \right) &= \mathbb{COV}^*_t \left(\frac{1 + z_{T_1} + z^{\upsilon}_{T_1}}{1 + \mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}}, \left(R_{M,t\to T_1} - R_{f,t\to T_1} \right) \right) \\
&= \frac{\mathbb{COV}^*_t \left(z_{T_1}, r_{M,t\to T_1} \right) + \mathbb{COV}^*_t \left(z^{\upsilon}_{T_1}, r_{M,t\to T_1} \right)}{1 + \mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}}
\end{aligned}$$

Setting $r_{M,t\to T_1} = R_{M,t\to T_1} - R_{f,t\to T_1}$ and using the definitions of z_{T_1} and $z^{\upsilon}_{T_1}$, it follows that

$$\begin{aligned}
\mathbb{E}^*_t z_{T_1} &= \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{E}^*_t r^2_{M,t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T_1}} \mathbb{E}^*_t r^3_{M,t\to T_1} \\
\mathbb{E}^*_t z^{\upsilon}_{T_1} &= \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N} + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}} \mathbb{E}^*_t \mathbb{M}^{*(3)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1} R^2_{f,T_1\to T_N}} \mathbb{E}^*_t r_{M,t\to T_1} \mathbb{M}^{*(2)}_{T_1\to T_N}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^*_t z_{T_1} \left(R_{M,t\to T_1} - R_{f,t\to T_1} \right) &= \frac{a_{1,t}}{R_{f,t\to T_1}} \mathbb{E}^*_t r^2_{M,t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T}} \mathbb{E}^*_t r^3_{M,t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T}} \mathbb{E}^*_t r^4_{M,t\to T_1} \\
&= \frac{a_{1,t}}{R_{f,t\to T}} \mathbb{M}^{*(2)}_{t\to T_1} + \frac{a_{2,t}}{R^2_{f,t\to T_1}} \mathbb{M}^{*(3)}_{t\to T_1} + \frac{a_{3,t}}{R^3_{f,t\to T}} \mathbb{M}^{*(4)}_{t\to T_1}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^*_t z^{\upsilon}_{T_1} \left(R_{M,t\to T} - R_{f,t\to T_1} \right) = &\frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{COV}^*_t \left(r_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_1\to T_N} \right) + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}} \mathbb{COV}^*_t \left(r_{M,t\to T_1}, \mathbb{M}^{*(3)}_{T_1\to T_N} \right) \\
&+ \frac{a_{2,3,t}}{R_{f,t\to T_1} R^2_{f,T_1\to T_N}} \left(\mathbb{COV}^*_t \left(r^2_{M,t\to T_1}, \mathbb{M}^{*(2)}_{T_1\to T_N} \right) + \mathbb{M}^{*(2)}_{t\to T_1} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N} \right)
\end{aligned}$$

This ends the proof. ■

E.3 Conditional crash probability

We next express the conditional probability of a crash using a third-order Taylor expansion series for the inverse marginal utility. To derive this probability, we define additional truncated moments as

$$\mathbb{M}^*_{t,s}[\alpha] = \mathbb{E}^*_t\left(\mathbb{M}^{*(3)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right), \tag{E12}$$

$$\mathbb{M}^*_{t,sv}[\alpha] = \mathbb{E}^*_t\left(r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}\mathbb{1}_{R_{M,t\to T_1}<\alpha}\right). \tag{E13}$$

Proposition 8 *Up to the third-order expansion-series of the inverse marginal utility, the conditional crash probability in a two-period (three-date) economy is a weighted average of two probabilities:*

$$\mathbb{P}_t\left(R_{M,t\to T_1}<\alpha\right) = \pi^o_t\Pi^{3rd}_{t\to T_1}[\alpha] + (1-\pi^o_t)\,\Pi^{\upsilon,s}_{t\to T_1}[\alpha], \tag{E14}$$

with

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \frac{a_{1,t}}{R_{f,t\to T_1}}\mathbb{M}^{*(1)}_{t\to T_1}[\alpha] + \frac{a_{2,t}}{R^2_{f,t\to T_1}}\mathbb{M}^{*(2)}_{t\to T_1}[\alpha] + \frac{a_{3,t}}{R^3_{f,t\to T_1}}\mathbb{M}^{*(3)}_{t\to T_1}[\alpha]}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}}, \tag{E15}$$

$$\Pi^{\upsilon,s}_{t\to T_1}[\alpha] = \frac{\frac{a_{2,t}}{\overline{R}^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,\upsilon}[\alpha] + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{M}^*_{t,s}[\alpha] + \frac{a_{2,3,t}}{R_{f,t\to T_1}\overline{R}^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,s\upsilon}[\alpha]}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{LEV}^*_t}, \tag{E16}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$ *and* π^o_t *is defined in Equation* (E10)

Proof. The proof results from Proposition 9 below. ■

Proposition 9 *Up to a third-order approximation, the conditional probability of a crash,* $\Pi^{3rd}_{t\to T_1}[\alpha] = P_t\left(R_{M,t\to T} < \alpha\right)$, *is*

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] \\ +\frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,\upsilon}[\alpha] + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{M}^*_{t,s}[\alpha] + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,s\upsilon}[\alpha] \end{array}\right\}}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{E}^*_t r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}} \tag{E17}$$

where $a_{2,3,t} = 2a_{2,t} + 3a_{3,t}$.

Proof. The probability of crash is

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \mathbb{E}^*_t\left(\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}1_{R_{M,t\to T}<\alpha}\right)$$

We then replace the inverse SDF by its expression and obtain

$$\begin{aligned}
\Pi^{3rd}_{t\to T_1}[\alpha] &= \frac{\mathbb{E}^*_t\left(\left(1+z_{T_1}+z^{\upsilon}_{T_1}\right)1_{R_{M,t\to T}<\alpha}\right)}{1+\mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}} \\
&= \frac{\mathbb{E}^*_t\left(1_{R_{M,t\to T}<\alpha}\right) + \mathbb{E}^*_t\left(z_{T_1}1_{R_{M,t\to T}<\alpha}\right) + \mathbb{E}^*_t\left(z^{\upsilon}_{T_1}1_{R_{M,t\to T}<\alpha}\right)}{1+\mathbb{E}^*_t z_{T_1} + \mathbb{E}^*_t z^{\upsilon}_{T_1}} \\
&= \frac{\left\{\begin{array}{c} \mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1}[\alpha] + \\ \frac{a_{2,t}}{R^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,\upsilon}[\alpha] + \frac{a_{3,t}}{R^3_{f,T_1\to T_N}}\mathbb{M}^*_{t,s}[\alpha] + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{M}^*_{t,s\upsilon}[\alpha] \end{array}\right\}}{1+\sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,t\to T_1}}\mathbb{M}^{*(k)}_{t\to T_1} + \sum_{k=2}^{3}\frac{a_{k,t}}{R^k_{f,T_1\to T_N}}\mathbb{E}^*_t\mathbb{M}^{*(k)}_{T_1\to T_N} + \frac{a_{2,3,t}}{R_{f,t\to T_1}R^2_{f,T_1\to T_N}}\mathbb{E}^*_t r_{M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}}
\end{aligned}$$

This ends the proof. ■

When $z^{v}_{T_1}$ is absent in the SDF expression (E1), the SDF corresponds to the SDF in a one-period static economy. Under this scenario, the probability of crash reduces to

$$\Pi^{3rd}_{t\to T_1}[\alpha] = \frac{\mathbb{M}^{*(0)}_{t\to T_1}[\alpha] + \sum_{k=2}^{3} \frac{a_{k,t}}{R^k_{f,t\to T_1}} \mathbb{M}^{*(k)}_{t\to T_1}[\alpha]}{1 + \sum_{k=2}^{3} \frac{a_{k,t}}{R^k_{f,t\to T_1}} \mathbb{M}^{*(k)}_{t\to T_1}}$$

We refer to our crash probability in (E14) as $\Pi^{3rd}_{t\to T_1,T_N}[\alpha]$.

E.4 Empirical results with fixed preference parameters

We detail in Appendix B.3 our calculation of $\mathbb{E}^*_t \mathbb{M}^{*(3)}_{T_1\to T_N}$ and $\mathbb{LES}^*_t$.

Table A2 reports the out-of-sample performance of our bound using the third-order Taylor expansion-series for the inverse SDF. We find that the predictions are overall not better than those of the second-order case. They are slightly worse for long investment horizons T_N, illustrating the challenge of accurately estimating higher order moments for long maturities, and slightly better for short maturities. While these results are in favor of our simpler second-order bounds, they are likely to improve should the liquidity of longer-maturity options improve with time, yielding better estimations of risk-neutral moments.

F Out-of-sample performance over the full sample

In this section, we reproduce the out-of-sample performance results but on the full sample up to 2023.

Table A2: **Out-of-sample prediction and allocation performance of $RP^{3rd}_{t\to T_1,T_N}$, with fixed preference parameters**
We report the out-of-sample performance of different risk premium prediction methods. The preference parameters are set to $\tau = 1$ and $\rho = 2$. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (E11). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1996 to February 2023.

			$RP_{t\to T_1,T_N}$ with $T_N =$ (in months)										
Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.08	−0.08	−0.53	−1.23	−2.31	−3.30	−4.35	−9.35	−17.77	−37.19	−62.73	−7.76
1	0.98	1.16	-	1.64	1.64	1.27	0.62	−0.24	−5.24	−14.28	−39.93	−80.07	−6.44
2	1.50	1.98	-	-	2.70	3.01	2.98	2.68	−1.09	−9.49	−36.27	−83.30	−4.51
3	1.34	2.15	-	-	-	3.09	3.70	4.03	2.18	−4.17	−27.41	−71.73	−2.41
4	1.91	3.25	-	-	-	-	4.30	5.16	5.16	1.14	−17.31	−54.74	0.37
5	2.66	4.76	-	-	-	-	-	5.97**	7.58	5.86	−7.12	−36.29	3.38
6	2.84	5.56	-	-	-	-	-	-	8.23*	8.10	−0.62	−22.98	4.15
9	2.40	6.67	-	-	-	-	-	-	-	9.08*	7.73	−2.58	7.98
12	1.05	6.14	-	-	-	-	-	-	-	-	8.44	4.67	7.50
18	−2.21	3.87	-	-	-	-	-	-	-	-	-	7.26	7.26
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	4.56	4.71	5.80	−2.88	−25.18	−51.02	-	-	-	-	-	-	-
1	4.71	4.95	-	5.62	−70.31	−132.18	-	-	-	-	-	-	-
2	4.83	5.30	-	-	6.05*	-	-	-	-	-	-	-	-
3	5.03	5.66	-	-	-	−12.35	-	-	-	-	-	-	-
4	5.21	5.87	-	-	-	-	3.61	-	-	-	-	-	-
5	5.26	6.14	-	-	-	-	-	6.99**	−84.35	-	-	-	-
6	5.21	4.90	-	-	-	-	-	-	-	-	-	-	-
9	5.27	6.55	-	-	-	-	-	-	-	−24.40	-	-	-
12	5.51	-	-	-	-	-	-	-	-	-	-	-	-
18	-	-	-	-	-	-	-	-	-	-	-	-	-

Table A3: **Summary statistics for risk premia**

We report summary statistics for times series of risk premia predictions. We use weekly time series of overlapping values for horizons longer than 10 days. All values are annualized and in percent. Data are monthly from January 1996 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the risk premium estimate of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premium measure in Equation (10), with $T_N = 1$ and 2 years.

Prediction	Mean	Standard deviation	Skew.	Kurt.	10%	25%	50%	75%	90%
Panel A: One week									
$RP^{Log}_{t\to T_1}$	3.58	4.71	6.25	60.48	1.03	1.41	2.27	3.96	6.71
$RP_{t\to T_1}$	3.74	5.12	6.70	69.05	1.06	1.45	2.33	4.03	7.00
$RP_{t\to T_1,1y}$	9.53	11.49	7.09	81.93	3.41	4.38	6.55	10.56	16.30
$RP_{t\to T_1,2y}$	15.09	15.38	5.78	54.88	6.09	7.81	10.93	17.12	24.82
Panel B: One month									
$RP^{Log}_{t\to T_1}$	4.27	4.24	4.66	36.24	1.34	1.89	3.25	5.11	7.72
$RP_{t\to T_1}$	4.57	4.73	4.94	40.19	1.41	1.98	3.45	5.41	8.28
$RP_{t\to T_1,1y}$	8.19	7.85	4.39	33.33	2.70	3.79	6.17	9.82	14.50
$RP_{t\to T_1,2y}$	11.74	10.55	3.82	25.36	4.22	5.71	8.87	13.95	20.53
Panel C: One quarter									
$RP^{Log}_{t\to T_1}$	4.32	3.29	3.43	22.09	1.70	2.21	3.53	5.34	7.34
$RP_{t\to T_1}$	4.84	3.86	3.76	26.43	1.87	2.43	3.86	5.93	8.25
$RP_{t\to T_1,1y}$	6.67	5.04	3.45	23.21	2.58	3.46	5.44	8.14	11.41
$RP_{t\to T_1,2y}$	8.92	6.40	3.01	17.81	3.66	4.91	7.33	10.87	15.27
Panel D: Six months									
$RP^{Log}_{t\to T_1}$	4.30	2.62	2.60	14.30	1.95	2.45	3.71	5.35	6.94
$RP_{t\to T_1}$	4.99	3.16	2.80	16.34	2.21	2.82	4.29	6.12	8.22
$RP_{t\to T_1,1y}$	5.83	3.58	2.65	15.07	2.61	3.36	5.06	7.11	9.49
$RP_{t\to T_1,2y}$	7.40	4.33	2.30	11.85	3.38	4.49	6.41	9.07	12.20

Table A4: **Out-of-sample prediction and allocation performance**

We report the out-of-sample performance of different risk premium prediction methods. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). The results in the last column are based on predicted returns obtained by averaging $RP_{t\to T_1,T_N}$ across T_N. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices, using Equation (20). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. Negative certainty equivalents are not reported. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively. Data are from January 1997 to February 2023.

Horizon T_1 (in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N}$ with $T_N =$ (in months) 1	2	3	4	5	6	9	12	18	24	Average across T_N
Panel A: Out-of-sample R^2													
10d	−0.10	−0.08	−0.04	0.00	0.05	0.08	0.12	0.16	0.15	0.03	−0.28	−0.93	0.15
1	0.98	1.11	-	1.26	1.38	1.47	1.55	1.62	1.69	1.64	1.32	0.65	1.65
2	1.50	1.89	-	-	2.11	2.31	2.49	2.66	2.99	3.18	3.32	3.03	3.04
3	1.34	2.03	-	-	-	2.31*	2.57*	2.81*	3.37*	3.77	4.34	4.47	3.65*
4	1.91	2.94	-	-	-	-	3.22**	3.51**	4.20**	4.74*	5.55*	6.00	4.79*
5	2.66	4.07	-	-	-	-	-	4.38**	5.16**	5.82**	6.83*	7.54*	6.16**
6	2.84	4.56	-	-	-	-	-	-	5.39**	6.10**	7.25**	8.14**	6.87**
12	1.05	3.87	-	-	-	-	-	-	-	-	5.12	6.23	5.70
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$													
10d	2.80	2.85	2.94	3.08	3.23	3.38	3.51	3.63	3.31	1.83	-	-	3.55
1	4.39	4.56	-	4.78*	4.97*	5.16*	5.35*	5.50*	5.78	5.57	0.07	-	5.77
2	5.04	5.50	-	-	5.76**	6.04***	6.29***	6.53***	7.23**	7.71**	5.80	-	7.46**
3	5.49	6.10	-	-	-	6.33***	6.55***	6.76***	7.36**	7.12	-	-	7.72**
4	5.46	6.17	-	-	-	-	6.36**	6.55**	7.00**	6.54	-	-	7.25
5	5.45	6.07	-	-	-	-	-	6.25**	6.81**	7.35**	8.39*	6.84	7.72**
6	5.41	6.17	-	-	-	-	-	-	6.43	5.53	-	-	4.27
12	5.92	2.77	-	-	-	-	-	-	-	-	-	-	-

Table A5: **Out-of-sample prediction and allocation performance with T_N optimized**

We report the out-of-sample performance of different risk premium prediction methods, from January 1997 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

Horizon T_1			$T_N = 1$y	$T_N = 2$y	Av. across T_N	T_N opt.
(in months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N}$	$RP_{t\to T_1,T_N}$	$RP_{t\to T_1,T_N}$	$RP_{t\to T_1,T^*_N}$
Panel A: Out-of-sample R^2						
10d	−0.09	−0.07	0.04	−0.93	0.16	0.08
1	1.11	1.25	1.72	0.66	1.75	1.73
2	1.66	2.04	3.28	3.07	3.16	3.95**
3	1.65	2.33	4.02	4.66	3.91	5.09**
4	2.36	3.36	5.13*	6.34	5.18*	5.96**
5	3.18	4.57	6.29**	7.95*	6.62**	7.08**
6	3.53	5.22	6.73**	8.71**	7.48**	7.86**
12	2.86	5.66	-	8.00	7.46	7.68
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$						
10d	4.56	4.69	8.18	-	8.19*	-
1	4.42	4.60	5.85	-	5.99	2.69
2	4.60	4.94	7.03*	-	6.71**	4.59
3	4.95	5.43	7.50**	-	7.30**	9.48**
4	5.17	5.75	7.44**	-	7.55**	9.17**
5	5.03	5.64	6.97**	8.39	7.38**	8.01*
6	5.07	5.78	6.83**	7.61	7.58**	8.12
12	5.24	6.27	-	7.91**	7.54**	-

Table A6: **Out-of-sample prediction and allocation performance with τ and ρ estimated in-sample**

We report the out-of-sample performance of different risk premium prediction methods, from January 1997 to December 2021. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). In columns (2) and (3), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (4), they are kept constant over the time series of data, but the constants are estimated. In column (5), they are modelled as linear functions of past 3-month returns. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices, using Equation (20). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		ρ, τ est. constant	ρ, τ est. linear in past returns
(months)	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1,T_N^*}$	$RP_{t\to T_1,T_N^*}$
	(1)	(2)	(3)	(4)	(5)
Panel A: Out-of-sample R^2					
10d	−0.10	−0.08	0.07	0.01	−0.02
1	0.73	0.87	1.96	0.83	1.58
2	1.03	1.41	4.56**	2.96	4.08
3	0.30	0.98	5.16***	3.27*	6.29
4	1.43	2.57	6.14**	5.44	9.00
5	2.64	4.35	7.38**	7.92	11.82
6	2.94	5.13	8.29**	9.86	12.90
12	2.30	6.73	9.84	11.18	12.90
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	2.67	2.73	-	2.79	3.64
1	3.75	3.94	-	3.26	6.00
2	3.42	3.69	2.56	-	4.94
3	4.26	4.79	11.44**	8.25*	8.17
4	3.94	4.21	5.14	-	-
5	4.05	4.67	8.01*	7.63	-
6	4.55	5.29	8.70*	9.06	2.44
12	5.01	6.43	7.45	8.06	5.51

Table A7: **Out-of-sample prediction and allocation performance with τ and ρ estimated without look-ahead bias**

We report the out-of-sample performance of different risk premium prediction methods, from January 2006 to February 2023. $RP^{Log}_{t\to T_1}$ is the lower bound of Martin (2017a). $IERP_{t\to T_1}$ is the estimate of Tetlock, McCoy, and Shah (2024). $RP_{t\to T_1}$ is the second-order lower bound of Chabi-Yo and Loudis (2020) in Equation (15). $RP_{t\to T_1,T_N}$ is the risk premia measure in Equation (10). In columns (3) and (4), results are reported setting the preference parameters to $\tau = 1$ and $\rho = 2$ (benchmark). In column (5), they are modelled constant and estimated on a telescopic window of time. In column (6), they are modelled constant and estimated on a rolling window of five years. We report in Panel A the out-of-sample prediction R^2_{OOS} in percent (see Equation (30)). Values smaller than -1 are not reported and left blank. For each prediction method, we test for the significance of the R^2_{OOS} difference relative to $RP_{t\to T_1}$ using a Diebold and Mariano (1995) test. We estimate the variance of the differences using a Newey-West correction with 12 lags. We report in Panel B the realized mean-variance certainty equivalents using each period the predicted risk premium and physical variance to obtain the optimal allocation (see Equation (32)). The physical variances are computed using option prices (see Appendix A.10). For each prediction method, we test for the significance of the realized certainty equivalent difference relative to $RP_{t\to T_1}$ using a block-bootstrap with average block length of three years and 10,000 bootstraps. Realized certainty equivalents are computed from non-overlapping returns. $*$, $**$, and $***$ denote significance at the 10%, 5%, and 1% level, respectively.

T_1		$\tau = 1$ and $\rho = 2$		telescopic est.	rolling est.
	$RP^{Log}_{t\to T_1}$	$RP_{t\to T_1}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$	$RP_{t\to T_1,T^*_N}$
	(1)	(2)	(3)	(4)	(5)
Panel A: Out-of-sample R^2					
10d	−0.10	−0.08	0.07	−0.21	−0.56
1	0.73	0.87	1.96	0.81	0.50
2	1.03	1.41	4.56**	1.36	3.07
3	0.30	0.98	5.16***	2.79	4.59
4	1.43	2.57	6.14**	7.14	4.04
5	2.64	4.35	7.38**	9.07	5.16
6	2.94	5.13	8.29**	11.08	7.10
12	2.30	6.73	9.84	11.25	8.66
Panel B: Out-of-sample mean-variance certainty equivalent with $\gamma = 3$					
10d	2.67	2.73	-	-	-
1	3.75	3.94	-	-	-
2	3.42	3.69	2.56	-	-
3	4.26	4.79	11.44**	-	9.94
4	3.94	4.21	5.14	0.18	-
5	4.05	4.67	8.01*	-	-
6	4.55	5.29	8.70*	-	-
12	5.01	6.43	7.45	1.67	6.57

G Online Appendix

G.1 Volatility Dynamic Implied by (26)

To further show that our formulation (26) is different from the GARCH (1,1), we use the closed-form expression of θ_t displayed in (27) and show that

$$\mathbb{M}^{*(2)}_{t\to T_N} = \theta_t \mathbb{E}^*_t R^2_{M,t\to T_1} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + R^2_{f,T_1\to T_2} \mathbb{M}^{*(2)}_{t\to T_1}. \tag{G1}$$

Since $R^2_{M,t\to T_1} = \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 + 2R_{M,t\to T_1} R_{f,t\to T_1} - R^2_{f,t\to T_1}$, it follows that

$$\mathbb{E}^*_t R^2_{M,t\to T_1} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2 = \mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1} \mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,t\to T_1} \mathbb{M}^{*(2)}_{t\to T_1}.$$

We then replace this expression in the RHS of (G1) and obtain

$$\mathbb{M}^{*(2)}_{t\to T_N} = \theta_t \mathbb{M}^{*(4)}_{t\to T_1} + 2R_{f,t\to T_1} \theta_t \mathbb{M}^{*(3)}_{t\to T_1} + R^2_{f,T_1\to T_N} \left(\theta_t + 1\right) \mathbb{M}^{*(2)}_{t\to T_1}.$$

This shows that the process of $\mathbb{M}^{*(2)}_{t\to T_N}$ is different from a GARCH dynamic. To check similarities with the GARCH process, let's assume for illustration purpose that $\mathbb{M}^{*(3)}_{t\to T_1} = 0$ and $\mathbb{M}^{*(4)}_{t\to T_1} = 3\left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2$ then

$$\mathbb{M}^{*(2)}_{t\to T_N} = 3\theta_t \left(\mathbb{M}^{*(2)}_{t\to T_1}\right)^2 + R^2_{f,T_1\to T_N} \left(\theta_t + 1\right) \mathbb{M}^{*(2)}_{t\to T_1}. \tag{G2}$$

Expression (G2) is reminiscent but distinct from the GARCH process.

G.2 The case with consumption

In this section, we introduce consumption in the representative agent problem. Under the minimal assumption that (i) odd risk neutral moments are negative, (ii) preference parameters satisfy the restrictions $a_{2,t} > 0$, $a_{2,t} \leq 0$, $a_{3,t} \geq 0$, $a_{2,3,t} \geq 0$ (see Eq), (iii) consumption-wealth ratio is positively related to the market return and (iv) the correlation of the square of the consumption wealth ratio and market return is negative (condition reminiscent of market coskewness), our measure of expected excess return remains a lower bound to the true measure of market expected excess return.

To proceed, we start by having the representative agent solve the problem

$$\max_{\omega_t, c_t} \mathbb{E}_t \left\{ \max_{\omega_{T_1}, c_{T_1}} \left\{ \mathbb{E}_{T_1} u \left[W_{t \to T_N} \right] \right\} \right\},$$

where the terminal wealth is

$$W_{t \to T_N} = (1 - c_{T_1}) W_{T_1} \left(\omega_{T_1}^{\intercal} R_{T_1 \to T_N} \right) \text{ with } W_{T_1} = (1 - c_t) W_t \left(\omega_t^{\intercal} R_{t \to T_1} \right)$$

and c_t is the consumption wealth ratio. The terminal wealth can alternatively be written as

$$W_{t \to T_N} = (1 - c_{T_1}) (1 - c_t) W_t \left(\omega_t^{\intercal} R_{t \to T_1} \right) \left(\omega_{T_1}^{\intercal} R_{T_1 \to T_N} \right).$$

For simplicity, we assume no interest rate risk. Notice that the SDF is given by the identity:

$$\frac{\mathbb{E}_t m_{t \to T_1}}{m_{t \to T_1}} = \frac{\upsilon_{T_1}}{\mathbb{E}_t^* \left(\upsilon_{T_1} \right)},$$

where

$$\upsilon_{T_1} = \mathbb{E}_{T_1}^* \left(\frac{u' \left[\overline{W}_{t \to T_N} \right]}{u' \left[W_{t \to T_N} \right]} \right) \text{ with } \overline{W}_{t \to T_N} = W_t R_{f, t \to T_1} R_{f, T_1 \to T_N}. \tag{G3}$$

We set

$$R_{M, t \to T_1} = \omega_t^{\intercal} R_{t \to T_1}, \; R_{M, T_1 \to T_N} = \omega_{T_1}^{\intercal} R_{T_1 \to T_N}, \; cc_{tT_1} = (1 - c_{T_1}) (1 - c_t). \tag{G4}$$

Next, we define

$$\mathbf{x} = cc_{tT_1},\, \mathbf{y} = \omega_t^{\intercal} R_{t\to T_1},\, \mathbf{z} = \omega_{T_1}^{\intercal} R_{T_1\to T_N} \tag{G5}$$

$$\mathbf{x}_0 = 1,\, \mathbf{y}_0 = R_{f,t\to T_1},\, \mathbf{z}_0 = R_{f,T_1\to T_N} \tag{G6}$$

and set

$$\mathbf{X} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \text{ and } \mathbf{X}_0 = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0).$$

Notice that $0 < cc_{tT_1} \leq 1$ since $0 < c_{T_1} \leq 1$ and $0 < c_t \leq 1$. Now, assume that the utility function is well-behaved and admits high-order derivatives that exist. Denote

$$\mathbf{G} = \frac{u'\left[\overline{W}_{t\to T_N}\right]}{u'\left[W_{t\to T_N}\right]}$$

G.2.1 Second-order Taylor expansion-series

A second-order Taylor expansion of $\mathbf{G}$ around $\mathbf{X} = \mathbf{X}_0$ gives

$$\begin{aligned}
\mathbf{G} = {} & 1 - (\mathbf{x}-\mathbf{x}_0)\frac{W_t\mathbf{y}_0\mathbf{z}_0 u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} - (\mathbf{y}-\mathbf{y}_0)\frac{W_t\mathbf{z}_0 u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} \\
& - (\mathbf{z}-\mathbf{z}_0)\frac{W_t\mathbf{y}_0 u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} + \frac{1}{2}W_t^2\mathbf{y}_0^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)(\mathbf{x}-\mathbf{x}_0)^2 \\
& + \frac{1}{2}W_t^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)(\mathbf{y}-\mathbf{y}_0)^2 \\
& + \frac{1}{2}W_t^2\mathbf{y}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)(\mathbf{z}-\mathbf{z}_0)^2 \\
& + W_t^2\mathbf{y}_0\mathbf{x}_0\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]} + \frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
& + \left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{x}-\mathbf{x}_0)(\mathbf{z}-\mathbf{z}_0) + \left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{X}=\mathbf{X}_0}(\mathbf{z}-\mathbf{z}_0)(\mathbf{y}-\mathbf{y}_0).
\end{aligned}$$

Notice that

$$\mathbb{E}_{T_1}^{*}(\mathbf{z}-\mathbf{z}_0) = 0$$

and

$$\begin{aligned}
\mathbb{E}^*_{T_1}\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{z}-\mathbf{z}_0\right) &= \left(\mathbf{x}-\mathbf{x}_0\right)\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)=0,\\
\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right) &= \left(\mathbf{y}-\mathbf{y}_0\right)\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)=0.
\end{aligned}$$

We use these expressions to simplify (G3) as

$$\begin{aligned}
\upsilon_{T_1} &= 1-\frac{W_t\mathbf{y}_0\mathbf{z}_0u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}\left(\mathbf{x}-\mathbf{x}_0\right)-\left(\mathbf{y}-\mathbf{y}_0\right)\frac{W_t\mathbf{z}_0u''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}\\
&+\frac{1}{2}W_t^2\mathbf{y}_0^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)^2\\
&+\frac{1}{2}W_t^2\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{y}-\mathbf{y}_0\right)^2\\
&+\frac{1}{2}W_t^2\mathbf{y}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^2\\
&+W_t^2\mathbf{y}_0\mathbf{x}_0\mathbf{z}_0^2\left(-\frac{u'''\left[\overline{W}_{t\to T_N}\right]}{u'\left[\overline{W}_{t\to T_N}\right]}+\frac{2\left(u''\left[\overline{W}_{t\to T_N}\right]\right)^2}{\left(u'\left[\overline{W}_{t\to T_N}\right]\right)^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\upsilon_{T_1} &= 1+\frac{1}{\tau_t}\mathbb{E}^*_{T_1}\left(cc_{tT_1}-1\right)+\frac{1}{\tau_tR_{f,t\to T_1}}\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right)+\frac{(1-\rho_t)}{\tau_t^2}\left(cc_{tT_1}-1\right)^2\\
&+\frac{(1-\rho_t)}{\tau_t^2R_{f,t\to T_1}^2}\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right)^2+\frac{(1-\rho_t)}{\tau_t^2R_{f,T_1\to T_2}^2}\mathbb{E}^*_{T_1}\left(\omega_{T_1}^\intercal R_{T_1\to T_2}-R_{f,T_1\to T_N}\right)^2\\
&+\frac{2\left(1-\rho_t\right)}{\tau_t^2R_{f,t\to T_1}}\mathbb{E}^*_{T_1}\left(cc_{tT_1}-1\right)\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right).
\end{aligned}$$

We then exploit the notation $R_{M,t\to T_1} = \omega_t^\intercal R_{t\to T_1}$, $R_{M,T_1\to T_N} = \omega_{T_1}^\intercal R_{T_1\to T_N}$ and express the expected value of υ_{T_1} under the risk neutral measure as

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} &= 1 + \frac{1}{\tau_t}\mathbb{E}_t^*\left(cc_{tT_1} - 1\right) + \frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}_t^*\left(cc_{tT_1} - 1\right)^2 \\
&\quad + \frac{(1-\rho_t)}{\tau_t^2 R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{\tau_t^2 R_{f,T_1\to T_N}^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)} \\
&\quad + \frac{2(1-\rho_t)}{\tau_t^2 R_{f,t\to T_1}}\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t\to T_1}\right). \qquad \text{(G7)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{M}_{t\to T_1}^{*(n)} &= \mathbb{E}_t^*\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^n \\
\mathbb{M}_{T_1\to T_N}^{*(2)} &= \mathbb{E}_{T_1}^*\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^2
\end{aligned}$$

The expected excess market return is

$$\begin{aligned}
\mathbb{E}_t\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right) &= \mathbb{E}_t\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right] \\
&= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)\right] \\
&= \frac{\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t\to T_1}\right]}{\mathbb{E}_t^*\upsilon_{T_1}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbb{COV}_t^*\left[\upsilon_{T_1}, R_{M,t\to T_1}\right] &= 1 + \frac{1}{\tau_t}\mathbb{COV}_t^*\left(cc_{tT_1}, R_{M,t\to T_1}\right) + \frac{1}{\tau_t R_{f,t\to T_1}}\mathbb{M}_{t\to T_1}^{*(2)} \\
&\quad + \frac{(1-\rho_t)}{\tau_t^2}\mathbb{COV}_t^*\left(\left(cc_{tT_1} - 1\right)^2, R_{M,t\to T_1}\right) \\
&\quad + \frac{(1-\rho_t)}{\tau_t^2 R_{f,t\to T_1}^2}\mathbb{M}_{t\to T_1}^{*(3)} + \frac{(1-\rho_t)}{\tau_t^2 R_{f,T_1\to T_N}^2}\mathbb{LEV}_t^* \\
&\quad + \frac{2(1-\rho_t)}{\tau_t^2 R_{f,t\to T_1}^2}\mathbb{E}_t^*\left(\left(cc_{tT_1} - 1\right)\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^2\right). \qquad \text{(G8)}
\end{aligned}$$

Notice that $\mathbb{E}_t^*\left((cc_{tT_1}-1)\left(\omega_t^\intercal R_{t\to T_1}-R_{f,t\to T_1}\right)^2\right)<0$ because $cc_{tT_1}-1<0$. In addition, $\mathbb{M}_{t\to T_1}^{*(3)}\leq 0$, $\mathbb{LEV}_t^*\leq 0$, and $\mathbb{COV}_t^*\left(R_{M,t\to T_1},\mathbb{LEV}_t^*\right)\leq 0$. Recall that

$$\frac{1}{\tau_t}>0 \text{ and } 1-\rho_t\leq 0. \tag{G9}$$

In theory, each factor risk factor in υ_{T_1} positively contributes to the risk premium. Thus each term in (G8) is positive. Assuming (G9) is satisfied, one should expect

$$\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t\to T_1}\right)>0 \text{ and } \mathbb{COV}_t^*\left((cc_{tT_1}-1)^2,R_{M,t\to T_1}\right)\leq 0. \tag{G10}$$

Since $1-c_{T_1}=\frac{W_{T_1}-C_{T_1}}{W_{T_1}}$ is the fraction of wealth W_{T_1} invested at T_1, it follows that

$$\begin{aligned}\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t\to T_1}\right) &= (1-c_t)\,\mathbb{COV}_t^*\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}},R_{M,t\to T_1}\right) \text{ and}\\ \mathbb{COV}_t^*\left((cc_{tT_1}-1)^2,R_{M,t\to T_1}\right) &= (1-c_t)^2\,\mathbb{COV}_t^*\left(\left(\frac{W_{T_1}-C_{T_1}}{W_{T_1}}\right)^2,R_{M,t\to T_1}\right).\end{aligned}$$

The positive sign of $\mathbb{COV}_t^*\left(cc_{tT_1},R_{M,t\to T_1}\right)$ is motivated by the positive impact of wealth-consumption ratio on the market expected excess return. Conditions (G10) are reminiscent of the dependence between the wealth-consumption ratio and the return on the market under the physical measure. Under the physical measure, the wealth-consumption ratio is positively correlated to the market. Under conditions (G9) and (G10), the covariance $\mathbb{COV}_t^*\left[\upsilon_{T_1},R_{M,t\to T_1}\right]$ is bounded:

$$\mathbb{COV}_t^*\left[\upsilon_{T_1},R_{M,t\to T_1}\right]\geq\frac{1}{R_{f,t\to T_1}}\frac{1}{\tau_t}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(3)}+\frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{LEV}_t^*. \tag{G11}$$

Next, since $cc_{tT_1}\leq 1$, we use (G7) and exploit (G9) and (G10) to obtain

$$\mathbb{E}_t^*\upsilon_{T_1}\leq 1+\frac{(1-\rho_t)}{R_{f,t\to T_1}^2\tau_t^2}\mathbb{M}_{t\to T_1}^{*(2)}+\frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2\tau_t^2}\mathbb{E}_t^*\mathbb{M}_{T_1\to T_N}^{*(2)}.$$

Therefore,

$$\frac{1}{\mathbb{E}_t^* \upsilon_{T_1}} \geq \frac{1}{1 + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2 \tau_t^2} \mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2 \tau_t^2} \mathbb{E}_t^* \mathbb{M}_{T_1\to T_N}^{*(2)}}. \tag{G12}$$

Combining (G11) and (G12), the expected excess return is bounded

$$\mathbb{E}_t \left[R_{M,t\to T_1} - R_{f,t\to T_1}\right] \geq \underbrace{\frac{\frac{1}{R_{f,t\to T_1}} \frac{1}{\tau_t} \mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2 \tau_t^2} \mathbb{M}_{t\to T_1}^{*(3)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2 \tau_t^2} \mathbb{LEV}_t^*}{1 + \frac{(1-\rho_t)}{R_{f,t\to T_1}^2 \tau_t^2} \mathbb{M}_{t\to T_1}^{*(2)} + \frac{(1-\rho_t)}{R_{f,T_1\to T_N}^2 \tau_t^2} \mathbb{E}_t^* \mathbb{M}_{T_1\to T_N}^{*(2)}}}_{\text{This is our measure of expected excess return}}$$

This shows that under minimal conditions, our measure of expected excess return is a bound on the true expected excess return when consumption is taken into account.

Next, we focus on the third-order Taylor expansion-series of the inverse marginal utility function.

G.2.2 Third-order Taylor expansion-series

A Third-order Taylor expansion of $\frac{u'\left[\overline{W}_{t\to T_2}\right]}{u'\left[W_{t\to T_2}\right]}$ arround $\mathbf{X} = \mathbf{X}_0$ gives

$$
\begin{aligned}
\mathbf{G} \quad = \quad & 1+\left(\mathbf{x}-\mathbf{x}_0\right)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t}+\left(\mathbf{y}-\mathbf{y}_0\right)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}+\left(\mathbf{z}-\mathbf{z}_0\right)\frac{1}{\mathbf{z}_0}\frac{1}{\tau_t}+\frac{1}{\mathbf{x}_0^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(\mathbf{x}-\mathbf{x}_0\right)^2 \\
& +\frac{1}{\mathbf{y}_0^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(\mathbf{y}-\mathbf{y}_0\right)^2+\frac{1}{\mathbf{z}_0^2}\frac{\left(1-\rho_t\right)}{\tau_t^2}\left(\mathbf{z}-\mathbf{z}_0\right)^2+\frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2\left(1-\rho_t\right)}{\tau_t^2}\right)\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right) \\
& +\left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{z}-\mathbf{z}_0\right)+\left(\frac{\partial^2\mathbf{G}}{\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}\left(\mathbf{z}-\mathbf{z}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right). \\
& +\frac{1}{\mathbf{x}_0^3}\frac{\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\left(\mathbf{x}-\mathbf{x}_0\right)^3+\frac{1}{\mathbf{z}_0^3}\frac{\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\left(\mathbf{z}-\mathbf{z}_0\right)^3+\frac{1}{\mathbf{y}_0^3}\frac{\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\left(\mathbf{y}-\mathbf{y}_0\right)^3 \\
& +\frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4\left(1-\rho_t\right)}{\tau_t^2}+\frac{6\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\right)\left(\mathbf{x}-\mathbf{x}_0\right)^2\left(\mathbf{y}-\mathbf{y}_0\right) \\
& +\frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4\left(1-\rho_t\right)}{\tau_t^2}+\frac{6\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\right)\left(\mathbf{y}-\mathbf{y}_0\right)^2\left(\mathbf{x}-\mathbf{x}_0\right) \\
& +\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4\left(1-\rho_t\right)}{\tau_t^2}+\frac{6\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\right)\left(\mathbf{z}-\mathbf{z}_0\right)^2\left(\mathbf{x}-\mathbf{x}_0\right) \\
& +\frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4\left(1-\rho_t\right)}{\tau_t^2}+\frac{6\left(\kappa_t-2\rho_t+1\right)}{\tau_t^3}\right)\left(\mathbf{z}-\mathbf{z}_0\right)^2\left(\mathbf{y}-\mathbf{y}_0\right) \\
& +6\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial\mathbf{x}\partial\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}\left(\mathbf{z}-\mathbf{z}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right)\left(\mathbf{x}-\mathbf{x}_0\right) \\
& +3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{x}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}\left(\mathbf{x}-\mathbf{x}_0\right)^2\left(\mathbf{z}-\mathbf{z}_0\right) \\
& +3\frac{1}{3!}\left(\frac{\partial^3\mathbf{G}}{\partial^2\mathbf{y}\partial\mathbf{z}}\right)_{\mathbf{x}=\mathbf{x}_0}\left(\mathbf{y}-\mathbf{y}_0\right)^2\left(\mathbf{z}-\mathbf{z}_0\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1 + (\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}\frac{1}{\tau_t} + (\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t} + \frac{1}{\mathbf{x}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{x}-\mathbf{x}_0)^2 \\
&+ \frac{1}{\mathbf{y}_0^2}\frac{(1-\rho_t)}{\tau_t^2}(\mathbf{y}-\mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2}\frac{(1-\rho_t)}{\tau_t^2}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 + \frac{1}{\mathbf{x}_0\mathbf{y}_0}\left(\frac{2(1-\rho_t)}{\tau_t^2}\right)(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&+ \frac{1}{\mathbf{x}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{x}-\mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3}\frac{(\kappa_t-2\rho_t+1)}{\tau_t^3}(\mathbf{y}-\mathbf{y}_0)^3 \\
&+ \frac{3}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) \\
&+ \frac{3}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&+ \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&+ \frac{3}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}\left(\frac{4(1-\rho_t)}{\tau_t^2} + \frac{6(\kappa_t-2\rho_t+1)}{\tau_t^3}\right)(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2
\end{aligned}
$$

Using Eq (6) in the main text of the paper, it follows that

$$
\begin{aligned}
\upsilon_{T_1} &= \mathbb{E}^*_{T_1}\mathbf{G} \\
&= 1 + (\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0}a_{1,t} + (\mathbf{y}-\mathbf{y}_0)\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t} + \frac{1}{\mathbf{x}_0^2}a_{2,t}(\mathbf{x}-\mathbf{x}_0)^2 \\
&+ \frac{1}{\mathbf{y}_0^2}a_{2,t}(\mathbf{y}-\mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 + \frac{2}{\mathbf{x}_0\mathbf{y}_0}a_{2,t}(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&+ \frac{1}{\mathbf{x}_0^3}a_{3,t}(\mathbf{x}-\mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3}a_{3,t}(\mathbf{y}-\mathbf{y}_0)^3 \\
&+ \frac{6}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)^2(\mathbf{y}-\mathbf{y}_0) + \frac{6}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) \\
&+ \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}a_{2,3,t}(\mathbf{x}-\mathbf{x}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \\
&+ \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}(\mathbf{y}-\mathbf{y}_0)\mathbb{E}^*_{T_1}(\mathbf{z}-\mathbf{z}_0)^2 \qquad \text{(G13)}
\end{aligned}
$$

We then compute the expected value $\mathbb{E}_t^* \upsilon_{T_1}$ to obtain

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} &= 1 + (\mathbf{x}-\mathbf{x}_0)\frac{1}{\mathbf{x}_0} a_{1,t} + \frac{1}{\mathbf{x}_0^2} a_{2,t}\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^2 \\
&+ \frac{1}{\mathbf{y}_0^2} a_{2,t}\mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2 + \frac{2}{\mathbf{x}_0\mathbf{y}_0} a_{2,t}\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) \\
&+ \frac{1}{\mathbf{x}_0^3} a_{3,t}\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^3 + \frac{1}{\mathbf{z}_0^3} a_{3,t}\mathbb{E}_t^*\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t}\mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^3 \\
&+ \frac{1}{\mathbf{y}_0^2\mathbf{x}_0} a_{2,3,t}\mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) + \frac{1}{\mathbf{z}_0^2\mathbf{x}_0} a_{2,3,t}\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2 \\
&+ \frac{1}{\mathbf{z}_0^2\mathbf{y}_0} a_{2,3,t}\mathbb{COV}_t^*\left(\mathbf{y}, \mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2\right)
\end{aligned}$$

Notice that

$$\mathbf{x}-\mathbf{x}_0 \leq 0,$$

and the following inequalities hold:

$$a_{1,t} > 0,\ a_{2,t} \leq 0,\ \ a_{3,t} \geq 0,\ a_{2,3,t} \geq 0, \tag{G14}$$

and

$$\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^3 \leq 0,\ \ \mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^3 \leq 0,\ \mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)^3 \leq 0,$$

and

$$\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)(\mathbf{y}-\mathbf{y}_0) = \mathbb{COV}_t^*(\mathbf{x}-\mathbf{x}_0, \mathbf{y}) \geq 0$$

and

$$\begin{aligned}
\mathbb{E}_t^*(\mathbf{y}-\mathbf{y}_0)^2(\mathbf{x}-\mathbf{x}_0) &\leq 0 \text{ (because } (\mathbf{x}-\mathbf{x}_0) \leq 0) \\
\mathbb{E}_t^*(\mathbf{x}-\mathbf{x}_0)\mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2 &\leq 0 \text{ (because } (\mathbf{x}-\mathbf{x}_0) \leq 0) \\
\mathbb{COV}_t^*\left(\mathbf{y}, \mathbb{E}_{T_1}^*(\mathbf{z}-\mathbf{z}_0)^2\right) &= \mathbb{LEV}_t^* \leq 0.
\end{aligned}$$

This allows us to bound $\mathbb{E}_t^* \upsilon_{T_1}$ as

$$\begin{aligned}
\mathbb{E}_t^* \upsilon_{T_1} \quad \leq \quad & 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2 \\
& + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^3 \\
& + \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2\right).
\end{aligned}$$

As a result,

$$\frac{1}{\mathbb{E}_t^* \upsilon_{T_1}} \geq \frac{1}{\left\{ \begin{array}{c} 1 + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^2 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2 \\ + \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{E}_t^* \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^3 + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{E}_t^* \left(\mathbf{y} - \mathbf{y}_0\right)^3 \\ + \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left(\mathbf{y}, \mathbb{E}_{T_1}^* \left(\mathbf{z} - \mathbf{z}_0\right)^2\right) \end{array} \right\}} \tag{G15}$$

Next, our goal is to bound $\mathbb{COV}_t^*\left(\upsilon_{T_1}, \mathbf{y}-\mathbf{y}_0\right) = \mathbb{COV}_t^*\left(\upsilon_{T_1}, R_{M,t\to T_1} - R_{f,t\to T_1}\right)$. We then use (G13) to compute this covariance as

$$
\begin{aligned}
& \mathbb{COV}_t^*\left(\upsilon_{T_1}, \mathbf{y}-\mathbf{y}_0\right) \\
= \; & \frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}_t^*\left(\mathbf{y}\right) + \frac{1}{\mathbf{x}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{2}{\mathbf{x}_0\mathbf{y}_0}a_{2,t}\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{x}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) + \frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^3, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{x}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)^2\left(\mathbf{y}-\mathbf{y}_0\right), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{y}_0^2\mathbf{x}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)^2\left(\mathbf{x}-\mathbf{x}_0\right), \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{x}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \\
& + \frac{6}{3!}\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}_t^*\left(\left(\mathbf{y}-\mathbf{y}_0\right)\mathbb{E}_{T_1}^*\left(\mathbf{z}-\mathbf{z}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right).
\end{aligned}
$$

Notice that

$$\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right), \mathbf{y}-\mathbf{y}_0\right) = \mathbb{E}_t^*\left(\mathbf{x}-\mathbf{x}_0\right)\left(\mathbf{y}-\mathbf{y}_0\right)^2 \leq 0 \text{ (since } \mathbf{x} \leq \mathbf{x}_0\text{)},$$

and

$$\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)^2\left(\mathbf{y}-\mathbf{y}_0\right), \mathbf{y}-\mathbf{y}_0\right) = \mathbb{E}_t^*\left(\mathbf{x}-\mathbf{x}_0\right)^2\left(\mathbf{y}-\mathbf{y}_0\right)^2 \geq 0.$$

We assume

$$\mathbb{COV}_t^*\left(\left(\mathbf{x}-\mathbf{x}_0\right)^2, \mathbf{y}-\mathbf{y}_0\right) \leq 0 \tag{G16}$$

and

$$\mathbb{COV}_t^* \left((\mathbf{x} - \mathbf{x}_0)^3, \mathbf{y} - \mathbf{y}_0 \right) \geq 0, \tag{G17}$$

$$\mathbb{COV}_t^* \left((\mathbf{y} - \mathbf{y}_0)^2 (\mathbf{x} - \mathbf{x}_0), \mathbf{y} - \mathbf{y}_0 \right) \geq 0, \tag{G18}$$

$$\mathbb{COV}_t^* \left((\mathbf{x} - \mathbf{x}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2, \mathbf{y} - \mathbf{y}_0 \right) \geq 0. \tag{G19}$$

These conditions are reminiscent of the sign of coskewness and cokurtosis when random variables of interest are return. While $\mathbf{y} - \mathbf{y}_0$ and $\mathbf{z} - \mathbf{z}_0$ are realized excess returns, $\mathbf{x} - \mathbf{x}_0$ is a function of wealth-consumption ratio (See (G4)-(G6)). Because coskewness is negative (see Harvey and Siddique (2000)) and cokurtosis is positive (Dittmar (2002)) and the wealth-consumption ratio is positively correlated to the market return, one should expect (G17)-(G19) to hold.

Under conditions (G16)-(G19), it follows that

$$\begin{aligned} \mathbb{COV}_t^* (\upsilon_{T_1}, \mathbf{y} - \mathbf{y}_0) \geq{}& \frac{1}{\mathbf{y}_0} \frac{1}{\tau_t} \mathbb{VAR}_t^* (\mathbf{y}) + \frac{1}{\mathbf{y}_0^2} a_{2,t} \mathbb{E}_t^* (\mathbf{y} - \mathbf{y}_0)^3 + \frac{1}{\mathbf{z}_0^2} a_{2,t} \mathbb{COV}_t^* \left(\mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2, \mathbf{y} - \mathbf{y}_0 \right) \\ &+ \frac{1}{\mathbf{z}_0^3} a_{3,t} \mathbb{COV}_t^* \left(\mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^3, \mathbf{y} - \mathbf{y}_0 \right) + \frac{1}{\mathbf{y}_0^3} a_{3,t} \mathbb{COV}_t^* \left((\mathbf{y} - \mathbf{y}_0)^3, \mathbf{y} - \mathbf{y}_0 \right) \\ &+ \frac{1}{\mathbf{z}_0^2 \mathbf{y}_0} a_{2,3,t} \mathbb{COV}_t^* \left((\mathbf{y} - \mathbf{y}_0) \mathbb{E}_{T_1}^* (\mathbf{z} - \mathbf{z}_0)^2, \mathbf{y} - \mathbf{y}_0 \right) \end{aligned} \tag{G20}$$

Combining (G15) and (G20) leads to

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right) = \frac{\mathbb{COV}^*_t\left[\upsilon_{T_1},R_{M,t\to T_1}\right]}{\mathbb{E}^*_t\upsilon_{T_1}}$$

$$\geq \frac{\left\{\begin{array}{c}\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{VAR}^*_t\left(\mathbf{y}\right)+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}^*_t\left(\mathbf{y}-\mathbf{y}_0\right)^3\\ \frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{COV}^*_t\left(\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^2,\mathbf{y}-\mathbf{y}_0\right)\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{COV}^*_t\left(\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^3,\mathbf{y}-\mathbf{y}_0\right)\\ +\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{COV}^*_t\left(\left(\mathbf{y}-\mathbf{y}_0\right)^3,\mathbf{y}-\mathbf{y}_0\right)\end{array}\right\}}{\left\{\begin{array}{c}1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{E}^*_t\left(\mathbf{y}-\mathbf{y}_0\right)^2+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_t\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^2\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_t\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^3+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{E}^*_t\left(\mathbf{y}-\mathbf{y}_0\right)^3\\ +\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{COV}^*_t\left(\mathbf{y},\mathbb{E}^*_{T_1}\left(\mathbf{z}-\mathbf{z}_0\right)^2\right)\end{array}\right\}}$$

which simplifies to

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\geq\frac{\left(\begin{array}{c}\frac{1}{\mathbf{y}_0}\frac{1}{\tau_t}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}^{*(3)}_{t\to T_1}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}^{*(4)}_{t\to T_1}\\ +\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{LEV}^*_t+\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{LES}^*_t\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{\mathbf{y}_0^2}a_{2,t}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{1}{\mathbf{z}_0^2}a_{2,t}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}+\frac{1}{\mathbf{y}_0^3}a_{3,t}\mathbb{M}^{*(3)}_{t\to T_1}\\ +\frac{1}{\mathbf{z}_0^3}a_{3,t}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N}+\frac{1}{\mathbf{z}_0^2\mathbf{y}_0}a_{2,3,t}\mathbb{LEV}^*_t\end{array}\right)}$$

We, thereafter, replace $\mathbf{y}_0$ and $\mathbf{z}_0$ by their expressions

$$\mathbb{E}_t\left(R_{M,t\to T_1}-R_{f,t\to T_1}\right)\geq\underbrace{\frac{\left(\begin{array}{c}\frac{1}{R_{f,t\to T_1}}\frac{1}{\tau_t}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{1}{R^2_{f,t\to T_1}}a_{2,t}\mathbb{M}^{*(3)}_{t\to T_1}+\frac{1}{R^3_{f,t\to T_1}}a_{3,t}\mathbb{M}^{*(4)}_{t\to T_1}\\ +\frac{1}{R^2_{f,T_1\to T_N}}a_{2,t}\mathbb{LEV}^*_t+\frac{1}{R^3_{f,T_1\to T_N}}a_{3,t}\mathbb{LES}^*_t\end{array}\right)}{\left(\begin{array}{c}1+\frac{1}{R^2_{f,t\to T_1}}a_{2,t}\mathbb{M}^{*(2)}_{t\to T_1}+\frac{1}{R^2_{f,T_1\to T_2}}a_{2,t}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}+\frac{1}{R^3_{f,t\to T_1}}a_{3,t}\mathbb{M}^{*(3)}_{t\to T_1}\\ +\frac{1}{R^3_{f,T_1\to T_N}}a_{3,t}\mathbb{E}^*_t\mathbb{M}^{*(3)}_{T_1\to T_N}+\frac{1}{R^2_{f,T_1\to T_N}R_{f,t\to T_1}}a_{2,3,t}\mathbb{LEV}^*_t\end{array}\right)}}$$

G.3 Is our Market Expected Return a Lower Bound to the Expected Return?

Setting consumption-wealth ratio to 1 in Section G.2 and using reasonable minimal assumptions that (i) odd risk neutral moments are negative, (ii) preference parameters satisfy the restrictions (G9) proves that our measure of expected excess market return (10) remains a lower bound to the true expected excess market return.