Two Trees and a Gardener: Asset Pricing with Real Capital Flows

March 18, 2025

Abstract

We introduce a neoclassical model of a two-sector production economy featuring a venture capital firm that can transform and create capital. Our model extends the two-trees framework from Cochrane, Longstaff, and Santa-Clara (2008) by adding a gardener (the venture capital firm) that nurtures and prunes the trees. The model highlights a fundamental feature of capital investments: they change the economy's capital mix (e.g., office buildings versus factories, or plows versus tractors), which has several implications. A factor constructed from real capital flows captures risks associated with sector capital imbalance, prices the cross-section of stock returns, and generates positive CAPM alpha, consistent with the empirical success of the investment factor. In contrast to most existing q-theory models, the stochastic discount factor and fluctuations of risk premia and investments are all endogenous. We find empirical support for these predictions.

Keywords: Capital allocation, general equilibrium, factor-based asset pricing, investments.

The investment factor has become an essential pricing factor in widely used asset pricing models, such as "FF5" (Fama and French, 2015) and "Q4" (Hou et al., 2015). The empirical success of this factor, which takes a long position in stocks with low corporate investment and a short position in stocks with high corporate investment, raises important questions. Does the high average return of stocks with low corporate investment represent compensation for some type of aggregate investment risk? Or is it attributed to behavioral biases that cause stock prices to depart from their intrinsic value?

Our understanding of the investment factor primarily comes from a growing collection of q-theory models of the firm (see Carlson et al. (2004), Zhang (2009), Li et al. (2009), Novy-Marx (2013), Hou et al. (2015) Hou et al. (2018), among others). These models successfully relate a stock's risk premium to several of the firm's characteristics, including its level of investment. However, the models have limitations. Because they focus on individual firms and specify exogenous dynamics for the stochastic discount factor, they have little to say about the origins of the investment factor. What is more, because the specified stochastic discount factor generally has only one factor, which captures the returns of the market portfolio, a conditional capital asset pricing model (CAPM) holds. These models therefore do not explain why the investment factor empirically generates CAPM alpha.

Building on these theoretical advances, our paper develops a theory of the investment factor that is consistent with both standard q-theory models of the firm and risk-based models of investors. Our theory highlights a fundamental feature of capital investments: they change the economy's capital mix (e.g., office buildings versus factories, or plows versus tractors). Because the process of capital transformation requires effort to implement, investors require compensation for risk associated with the cost of sector capital imbalance. We show that a factor constructed from real capital flows captures this risk, prices the cross-section of stock returns, and generates positive CAPM alpha in the public equity market. These predictions are consistent with the empirical performance of the investment factor.

Our framework is parsimonious and delivers quasi-closed-form solutions. In contrast to most existing q-theory models, the stochastic discount factor is endogenous in our model, as are fluctuations of risk premia and investments. We build on the two-tree dynamic framework from Cochrane et al. (2008) where the two sectors (the "two trees") produce a risky amount of consumption good every period. The innovation relative to Cochrane et al. (2008) is that our economy features a venture capital firm (the "gardener") that can create and transform capital across sectors. The transformation of capital is costly because it involves potentially re-training workers or re-deploying capital equipment. Because of this friction, the gardener re-allocates capital across sectors slowly and predictably.

We obtain four key insights from the model. First, "sector capital imbalance" is a state variable that jointly drives capital flows and asset prices. At all times, the gardener seeks to maintain a balanced portfolio of the two sectors based on their riskiness and growth potential. So, when a sector incurs a negative shock and its capital level falls below its optimal capital share, its remaining capital stock becomes increasingly valued. The gardener is keen to supply more capital to that sector and bring it back to balance.

Second, sector capital imbalance is hard to measure empirically but can be directly recovered from capital flows. Measuring the direction and degree of sector capital imbalance is challenging because it requires knowing what is the optimal level of capital allocation to each sector. This in turn requires knowing what are the underlying risk and return parameters for each sector - parameters that are not directly observable by the econometrician. However, because the gardener systematically moves capital away from the sector where the capital share is too high and towards the sector where the capital share is too low, the econometrician can recover from capital flows information about the direction and the level of the sector capital imbalance.

Third, a portfolio constructed from real capital flows can be used as a priced risk factor. In our setting, households derive positive utility from aggregate consumption of the good and negative utility from the effort spent to transform capital. A two-factor pricing model arises where the first factor is captured by aggregate consumption, as in standard consumption-based asset pricing models, and the second factor is captured by the degree of sector capital imbalance. This second factor can be recovered from a portfolio that is long capital outflows and short capital inflows.

Fourth, this factor constructed from capital flows generates both a high average return when capital flows are large and a positive alpha with respect to a publicly traded market portfolio. In our model, even though a conditional CAPM holds with respect to the aggregate equity portfolio, it does not hold with respect to the publicly traded market benchmark when some of the newly created firms do not immediately become publicly listed. More specifically, when high-valuation firms go public fast and low-valuation firms stay private longer, consistent with the empirical evidence in the IPO literature, then the publicly traded market portfolio systematically over-weights the sector with capital inflows and under-weights the sector with capital outflows. Consequently, we show that firms in the sector with capital inflows have negative CAPM alphas whereas firms in the sector with capital outflows have positive CAPM alphas.

Altogether, the model produces a rich set of equilibrium outcomes linking real capital flows to asset prices. Capital flows endogenously and positively respond to asset prices. A portfolio that takes a long position in the sector with capital outflows and a short position in the sector with capital inflows serves as a wind vane that indicates the direction and the degree of sector capital imbalance. This portfolio generates a high average return, positive CAPM alpha, and can be used as a risk factor to explain the cross-section of risk premia in conjunction with the publicly traded market factor. Our general-equilibrium (GE) framework of real capital flows is therefore not only consistent with q-theory under adjustment costs but also provides a risk-based explanation of the investment factor centered on sector capital imbalance.

We emphasize that our results only arise in a production-GE setting with multiple sectors and finite adjustment costs. In the extreme case when capital adjustment costs are infinite, the model converges to the "two trees" endowment economy of Cochrane et al. (2008). Prices adjust to clear capital markets but capital flows do not respond to asset prices. At the extreme other end of the spectrum, when capital adjustment costs tend to zero the model converges to a Cox et al. (1985) economy (see also Merton (1971)) where capital flows immediately adjust to ensure the portfolio of sectors remains optimally balanced. Because capital shocks are fully "absorbed" by the ensuing capital flows, asset prices are unaffected. And because sectors always remain at their optimal capital share, the risk premium for bearing sector capital imbalance risk is zero in this case.

We next turn to the data and find empirical support for the model's key predictions. Using accounting data from Compustat on a sample of over 2,000 U.S. firms every year between 1976 and 2023, we propose a simple but efficient methodology to measure aggregate real capital flows. Starting at the firm level, we calculate the annual change in a firm's book value net of operating assets, i.e. net of cash holdings and other short-term financial assets. We categorize a positive change in a firm's book value as "in-capital," a negative change as "out-capital," and refer to the firm's book value from the previous year as "previous-capital." We then add up these capital values across firms and construct every year the aggregate portfolios of previous-, in-, and out-capital. Together, the three portfolios make up the total capital across listed firms. This decomposition of capital is consistent with the model, has clean addition properties, and allows us to bypass standard industry classifications which may not fully capture how the U.S. economy has transformed over the past 50 years (Bergsman et al., 1975; Fort and Klimek, 2016; Keil, 2017).

We uncover several empirical findings. First, capital flows are slow-moving and persistent. Focusing on the in-capital portfolio, we find that, conditional on investing in new capital in a given quarter, a firm has an above-average probability of investing in more capital in the following quarters. Over time, this probability converges to the unconditional average. The portfolio of out-capital exhibits similar persistency. These facts are consistent with the prediction from the model that capital re-allocation is predictable and takes time to implement.

Second, capital flows positively respond to asset prices. Every year, we compare the market-to-book ratio of the in-, out-, and previous-capital portfolios. We find the market-to-book ratio is always highest for the in-capital portfolio and always lowest for the out-capital portfolio, with the market-to-book ratio of the previous-capital portfolio always in between. These facts are consistent with the prediction from the model that capital flows away from where it is valued the least and towards where it is valued the most.

Third, capital flows negatively predict a stock's expected return. Using daily stock return data from CRSP, we examine the daily returns of each portfolio over one year after the portfolio is formed. We find that the out-capital portfolio systematically outperforms the in-capital portfolio. This fact is consistent with the model's prediction.

Fourth, capital flows negatively predict CAPM alpha. A zero-cost portfolio that is long the out-capital portfolio and short the in-capital portfolio has a CAPM-alpha of approximately 8% per year. This alpha is economically large, strongly statistically significant with a t-value of 3.9, and driven both by the long and short legs. Additionally, the return of this "out-minus-in" portfolio is highly correlated with the investment factor in Fama and French (2015) and Hou et al. (2015). These facts are consistent with the prediction from the model that a portfolio constructed from real capital flows can be used as a pricing factor in conjunction with the publicly traded market portfolio.

Our paper contributes to the general equilibrium literature with multiple production (Garleanu et al., 2012; Gomes et al., 2003; Kogan et al., 2020; Papanikolaou, 2011; Parlour and Walden, 2011; Betermier et al., 2023). Our model provides a link between real capital flows and the cross-section of equity premia in the presence of capital adjustment costs, and also provides a micro-foundation for the investment factor that is consistent with both q-theory models of the firm and consumption-based models of investors. A key difference from the q-theory literature is that in our model, the cost friction is associated with the transformation of capital from one sector to another, and not with the scaling up or down of aggregate capital investments.

Our study also contributes to the existing body of research on capital flows, which have been used extensively across the different finance research streams to reveal information about risk and return. In the mutual fund literature, investor flows reveal information about the underlying risk factors that investors care about (Barber et al., 2016; Berk and van Binsbergen, 2016). The capital flows of the mutual funds themselves are informative about fire-sale risk (Aragon and Kim, 2023). In the microstructure literature, order flow is a powerful predictor of exchange-rate dynamics (Evans and Lyons, 2002). In the macro-finance literature, studies have repeatedly shown how "flights" of capital towards safety and liquidity can help to explain large variations in the prices of real estate, gold, currencies, sovereign debt, international equities, and other assets (Beber et al., 2009; Baele et al., 2020). Our paper shows that real capital flows can be used to determine the cross-section of stock returns.

The rest of the paper is structured as follows. Section 1 introduces the model and highlights how capital flows reveal information about sector capital imbalance. Section 2 derives the main asset pricing results. Section 3 presents the data analysis and the empirical results. Section 4 concludes. An Internet Appendix provides the proofs and additional theoretical and empirical results.

1 Setup

We consider an economy with three types of agents: (i) firms, (ii) a venture capitalist firm, and (iii) households. Firms belong to one of two industries, they own industry-specific capital, and rent labor from households. The venture capitalist reallocates capital across sectors, which is the distinguishing feature of our model. Households receive equity as compensation for the effort they exert on behalf of the venture capitalist. Time is continuous and indexed by $t \in [0, +\infty)$, and there exists a unique consumption good.

1.1 Firms

There are two production sectors and a continuum of identical firms in each sector. We denote by $K_t^n \geq 0$ the amount of capital used in sector $n \in \{1, 2\}$ at time t. The firms in the sector own the n-capital, K_t^n, and each type of capital is distinct. We use the term "sector" but there are several possible interpretations for what the different types of capital represent. They could, for example, denote assets in place in two industries, one representing farming equipment and the other computer hardware. They could also represent tangible versus intangible capital in the economy, different types of human capital, etc. We are agnostic about the exact interpretation. The total capital stock is $K_t = K_t^1 + K_t^2$.

Capital generates a flow of the consumption good,

$$Y_t^n = \kappa K_t^n. \tag{1}$$

The coefficient κ is assumed to be the same in both industries. Consequently, the aggregate consumption good flow, $Y_t = Y_t^1 + Y_t^2$, is also proportional to the aggregate capital stock: $Y_t = \kappa K_t$. Consumption good flows are paid as dividends to shareholders. The dividend flow paid by n-firms is D_t^n. In aggregate, the dividend flow is equal to the consumption flow:

$$D_t^1 + D_t^2 = Y_t^1 + Y_t^2.$$

In the absence of capital production, capital and consumption flows naturally grow at the stochastic rate

$$\frac{dK_t^n}{K_t^n} = \frac{dY_t^n}{Y_t^n} = \mu_n dt + \sigma_n dz_{n,t}, \tag{2}$$

where z_1 and z_2 are Wiener processes with $cov(dz_{1,t}, dz_{2,t}) = \rho\, dt$, and $\rho \in (-1, 1)$, μ_n, and σ_n are constants, $n \in \{1, 2\}$. We impose the following parameter constraints.

Assumption 1 *(Non-degeneracy conditions) The firms satisfy* $\kappa > \delta$, $\mu_1 > \mu_2 - \sigma_2^2 + \rho\sigma_1\sigma_2$, $\mu_2 > \mu_1 - \sigma_1^2 + \rho\sigma_1\sigma_2$, $\sigma_1 - \rho\sigma_2 > 0$, *and* $\sigma_2 - \rho\sigma_1 > 0$.

These conditions will guarantee that, in equilibrium, both types of capital are in positive supply and it is optimal to invest in each sector.

We denote the total value of each firm by P_t^n, and define their price-to-capital and dividend-to-capital ratios as follows:

$$p_t^n = \frac{P_t^n}{K_t^n}, \qquad d_t^n = \frac{D_t^n}{K_t^n}, \tag{3}$$

where $n \in \{1,2\}$. The total value of the firms is $P_t = P_t^1 + P_t^2$. Let

$$s_t = \frac{K_t^1}{K_t} \tag{4}$$

be the ratio of sector 1 capital to the total amount of capital in the economy, which we denote the *capital share*. The aggregate price-to-capital ratio is

$$p_t = \frac{P_t}{K_t} = \frac{P_t^1 + P_t^2}{K_t^1 + K_t^2} = s_t p_t^1 + (1 - s_t) p_t^2. \tag{5}$$

In the absence of arbitrage, there exists a stochastic discount factor process (SDF), $\{M_t\}_{t\geq 0}$, such that the market value of a contingent claim with payoff X_T at time T is given by

$$V_t = E_t \left(\frac{M_T}{M_t} X_T \right) \tag{6}$$

at every $t \leq T$. Under complete markets, the firm makes decisions that maximize its market value, as defined by Equation (6).

1.2 Investment Technologies

A competitive venture capitalist (VC) firm, which we describe in the next section, can create new capital. The VC has access to a continuum of technologies, indexed by $w \in \mathbb{R}$, that require effort and capital as inputs, and creates new capital as output.

Definition 1 (w-technology) *For every $w \in \mathbb{R}$, the w-technology, $\mathcal{T}_w$, transforms one unit of effort, $s\, a\, \phi(w, s)$ units of 1-capital, and $(1 - s)\, a\, \phi(w, s)$ units 2-capital into w units of 1-capital output and $1 - w$ units of 2-capital output, where $a > 0$ and $\phi(w, s) = (w - s)^2$ for every $w \in \mathbb{R}$ and $s \in [0, 1]$.*

Under this definition, the capital input of a w-technology, $a\, \phi(w, s)$, is split in proportion to the capital share, so that $s\, a\, \phi(w, s)$ units of 1-capital and $(1 - s)\, a\, \phi(w, s)$ units of 2-capital are used as inputs. We interpret the total capital input in this investment technology, $a\, \phi(w, s)$, as a cost of capital transformation that is convex in the difference between the share of capital output, w, and the current capital share, s. The specific functional form $\phi(w, s) = a(w - s)^2$ allows for a tractable analysis with a representing the severity of the transformation friction and $(w - s)^2$ being the simplest convex function in the difference between current capital share and the capital share of output capital. We allow w to be outside the unit interval $[0, 1]$.[1]

The VC may only use a single technology at any given time $t \geq 0$, denoted by $w_{t},$. Let I_t represent the total investment level at t. The laws of motion for 1-capital and 2-capital are then given by:

$$\begin{aligned} dK_t^1 &= \mu_1 K_t^1 dt + \sigma_1 K_t^1 dz^1 + I_t \left[w_t - a s_t (w_t - s_t)^2 \right] dt, \\ dK_t^2 &= \mu_2 K_t^2 dt + \sigma_2 K_t^2 dz^2 + I_t \left[1 - w_t - a(1 - s_t)(w_t - s_t)^2 \right] dt. \end{aligned} \tag{7}$$

The more different the chosen investment technology is from the current capital share, the higher the capital transformation cost. This specification captures the fact that the resources needed for new investments are especially high when these investments differ significantly from the economy's current production type.

Consider, for example, an economy where K represents the total amount of physical capital used to grow crops, such as land and machinery. The shares s and $1-s$ could represent the fractions of these productive resources used to grow two types of crops. The shares could also represent the fractions of time the land and machines are used for alternating between two types of crops on the same land, rather than the fractions of land used for each crop. In these examples, it is natural that output will not be as high when the new technology is further away from the economy's existing mix. For example, if investments are used to increase the agricultural land area, machines that were used for existing land may need to be redeployed to cultivate the new land area. The more different the new technology is from the existing one in use, i.e., the larger is the difference between s and w, the higher is the cost in terms of forfeited growth of existing capital.

1.3 Venture Capitalist

The VC is a publicly traded firm in charge of capital transformation in the economy. In the context of the two-tree analogy, the VC firm is the gardener. It owns the patents to the linear short-term investment technologies $\mathcal{T}_w$, where $w \in \mathbb{R}$. The VC therefore owns all the economy's growth options.

At each point in time, the VC determines the policies I_t and w_t that maximize instantaneous

[1] When $w < 0$, we may interpret this as the capital input of 1-capital being $(-w)+as(w-s)^2$ and the capital output being zero. Similarly, when $w > 1$, we may interpret this as the capital input of 2-capital being $(w-1)+a(1-s)(w-s)^2$ and the capital output being zero.

value creation. It purchases $as_tI_t(w_t - s_t)^2$ units of 1-capital from 1-firms, and $a(1-s_t)I_t(w_t - s_t)^2$ units of 2-capital from 2-firms as production inputs, and hires households in the competitive labor market to exert the effort level I_t. The venture capitalist transforms the capital and creates a new 1-firm that holds $w_tI_t - as_tI_t(w_t - s_t)^2$ units of 1-capital, and a new 2-firm that holds $(1-w_t)I_t - a(1-s_t)I_t(w_t - s_t)^2$ units of 2-capital. It gives the fraction $1-\eta_t$ of the newly created firms to the household as compensation that matches the household's effort, and then sells the remaining η_t fraction of the firms in the market, generating a profit that is immediately paid out as a dividend to its owners.

The value of the VC firm at $t = 0$ is therefore equal to the future dividends it creates, $V_0^{VC} = V_0^1 + V_0^2$, where

$$V_0^1 = E_0\left(\int_0^\infty \frac{M_t}{M_0}\,\eta_t I_t\,\left\{p_t^1\left[w_t - a\,s_t\,(w_t - s_t)^2\right]\right\}dt\right), \tag{8}$$

$$V_0^2 = E_0\left(\int_0^\infty \frac{M_t}{M_0}\,\eta_t I_t\,\left\{p_t^2[1 - w_t - a\,(1-s_t)\,(w_t - s_t)^2]\right\}dt\right). \tag{9}$$

We will show that, in equilibrium, $\eta_t = 0$ at all points in time, leading to the market value of the VC firm $V_t^{VC} \equiv 0$.

1.4 Financial Markets

As previously mentioned, the market value of a representative publicly traded firm in each sector is P_t^n. The firm's market value represents the intrinsic value it creates via capital growth and dividends. There is also a short-term bond in zero net supply with price dynamics

$$dB_t = r_t^f B_t dt,$$

where r_t^f is the risk-free rate process.

1.5 Households

The household sector consists of a continuum of households of unit mass. Each household is infinitely-lived and has expected log utility:

$$U = E_0\left[\int_0^{+\infty} e^{-\delta t}\ln(\hat{C}_t - F_t)dt\right], \tag{10}$$

where $\hat{C}_t$ denotes consumption and F_t denotes the effort provided to the VC firm. We define *effective consumption* as consumption net of effort, $C_t = \hat{C}_t - F_t$.

Consider a household with wealth W_t that invests the amounts X_t^n, $n \in \{1, 2\}$, in 1-firms and 2-firms, is hired to exert effort $F_t = I_t$ for which it earns the compensation I_t in stock, and consumes at the rate $\hat{C}_t$. The household's instantaneous budget constraint is

$$\begin{aligned} dW_t &= X_t^1 \left(\frac{d_t^1}{p_t^1} dt + \frac{dP_t^1}{P_t^1} \right) + X_t^2 \left(\frac{d_t^2}{p_t^2} dt + \frac{dP_t^2}{P_t^2} \right) \\ &+ \left(W_t - X_t^1 - X_t^2 \right) r^f dt + I_t dt - \hat{C}_t dt. \end{aligned} \tag{11}$$

We note that the VC firm does not affect the budget constraint, since $V_t^{VC} \equiv 0$.

2 General Equilibrium

2.1 Definition

Aggregate effective consumption is given by $C_t = Y_t^1 + Y_t^2 - F_t$ or, equivalently,

$$C_t = Y_t - I_t = (\kappa - i_t) K_t, \tag{12}$$

where $i_t = I_t / K_t$ is the investment rate.

Under the investment process $\{I_t\}_t$ and the investment share process $\{w_t\}_t$, the dynamics of the aggregate capital are given by:

$$\begin{aligned} \frac{dK_t}{K_t} &= \left\{ s_t \mu_1 + (1 - s_t) \mu_2 + i_t \left[1 - a(w_t - s_t)^2 \right] \right\} dt + \sigma_1 s_t dz_t^1 + \sigma_2 (1 - s_t) dz_t^2, \\ &\overset{\text{def}}{=} \mu_K(s_t) dt + \sigma_K(s_t) dz_{K,t}, \end{aligned} \tag{13}$$

where

$$\sigma_K(s_t) = \left[\sigma_1^2 s_t^2 + \sigma_2^2 (1 - s_t)^2 + 2 s_t (1 - s_t) \rho \, \sigma_1 \, \sigma_2 \right]^{1/2}. \tag{14}$$

The capital share satisfies:

$$\begin{aligned} ds_t &= [i_t (w_t - s_t) - q_1(s_t)] dt + s_t (1 - s_t) \left(\sigma_1 dz_t^1 - \sigma_2 dz_t^2 \right), \\ &\overset{\text{def}}{=} \mu_s(s_t) dt + \sigma_s(s_t) dz_{s,t}, \end{aligned} \tag{15}$$

where

$$q_1(s) = s(1-s)\left[\mu_2 - \mu_1 + s\sigma_1^2 + (1-2s)\rho\sigma_1\sigma_2 - (1-s)\sigma_2^2\right]$$

for every $t \in [0, +\infty)$ and $s \in [0, 1]$. General equilibrium is defined as follows:

Definition 2 (General equilibrium) *The price of the 1-sector, $\{P_t^1\}_t$, the price of the 2-sector, $\{P_t^2\}_t$, the aggregate investment, $\{I_t\}_t$, and the investment share, $\{w_t\}_t$, define a general equilibrium if:*

- *the representative household maximizes the expected utility defined in Equation* (10)*, resulting in asset demands $X_t^1 \equiv P_t^1$, $X_t^2 \equiv P_t^2$, effort $F_t \equiv I_t$, and consumption $\hat{C}_t \equiv Y_t^1 + Y_t^2$, for all t;*
- *the VC firm chooses the investment level process $\{I_t\}_t$ and the investment share process $\{w_t\}_t$ that maximize shareholder value, V_0^{VC}.*

In the next sections, we derive general equilibrium in quasi-closed form.

2.2 Equilibrium Value Function, Pricing, and Investment

For notational convenience, we define the functions:

$$\begin{aligned}
A(t) &= \frac{1 - e^{-\delta t}}{\delta}, \\
q_0(s) &= \kappa + s\mu_1 + (1-s)\mu_2 - \frac{1}{2}\sigma_K^2(s), \\
q_2(s) &= \frac{1}{2}s^2(1-s)^2\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right),
\end{aligned}$$

for every $t \in [0, +\infty)$ and $s \in [0, 1]$. The following theorem characterizes the value function of the representative agent in general equilibrium.

Theorem 1 (Equilibrium value function) *The value function of the representative agent in equilibrium is separable in the aggregate capital stock and the capital share:*

$$U_t = \frac{\ln(K_t)}{\delta} + V(s_t).$$

The normalized utility, $V(s_t)$, satisfies

$$V(s) = \lim_{T \to \infty} W(T, s),$$

for every $s \in [0,1]$, where $W : \mathbb{R}_+ \times [0,1] \longrightarrow \mathbb{R}$ solves the PDE

$$\begin{aligned}\frac{\partial W}{\partial t} &= q_2(s)\,\frac{\partial^2 W}{\partial s^2} - q_1(s)\frac{\partial W}{\partial s} - \delta W \\ &\quad - \ln\left[A(t) + \frac{1}{4aA(t)}\left(\frac{\partial W}{\partial s}\right)^2\right] + \frac{\kappa}{4aA(t)}\left(\frac{\partial W}{\partial s}\right)^2 + A(t)q_0(s) - 1, \end{aligned} \tag{16}$$

with initial condition $W(0, s) = 0,\ \ s \in [0,1]$.

The value function of our infinite-horizon economy is obtained by taking the limit of the equilibrium value function of the economy with finite horizon T. From the value function, we obtain the stochastic discount factor, investment rate, and the aggregate price-to-capital ratio in general equilibrium.

Theorem 2 (Equilibrium pricing and investment) *In equilibrium, the stochastic discount factor satisfies*

$$M_t = \frac{e^{-\delta t}}{\delta K_t} Q(s_t), \tag{17}$$

where

$$Q(s) = 1 + \frac{\delta^2}{4a}[V'(s)]^2. \tag{18}$$

The investment rate is given by

$$i_t = \frac{4a(\kappa - \delta) + \delta^2 \kappa V'(s_t)^2}{4a + \delta^2 V'(s_t)^2}, \tag{19}$$

the investment technology by

$$w_t = s_t + \frac{\delta}{2a} V'(s_t), \tag{20}$$

and the aggregate price-to-capital ratio by

$$p_t = \frac{1}{1 + a(w_t - s_t)^2} = \frac{1}{Q_t}. \tag{21}$$

Moreover, the sectors' price-to-capital ratios satisfy the relationship:

$$p_t^1 - p_t^2 = 2a(w_t - s_t)p_t. \tag{22}$$

Each firm pays the dividend-to-capital ratio

$$d_t^n = \kappa + i_t(1 - a(w_t - s_t)^2)(p_t^n - p_t), \qquad n \in \{1, 2\}. \tag{23}$$

The VC firm breaks even at each point in time and has market value $V_t^{VC} \equiv 0$.

As shown in the proof, equilibrium is constructed by solving the social planner's problem that maximizes Equation (10) under the following constraints: $F_t = I_t$, $C_t = \kappa K_t - I_t$, and Equations (13) and (15). This determines the processes for the investment rate and share, $\{i_t\}_t$ and $\{w_t\}_t$, in turn defining the SDF, $\{M_t\}_t$, which is shown to be of the form given by Equation (17). We then verify that, under the SDF, the VC firm chooses the investment share and investment rate policies in accordance with the solution to the planner's problem. The VC firm ultimately breaks even so that $V_0^{VC} = 0$.

From Equation (5) and (22) it follows that $p_t^1 = [1 + 2a(w_t - s_t)(1 - s_t)]\, p_t$ and $p_t^2 = [1 - 2a(w_t - s_t)s_t]\, p_t$, which in turns implies that $s_t d_t^1 + (1 - s_t)d_t^2 = \kappa$. So, total dividends equal total consumption good flow $D_t^1 + D_t^2 = \kappa K_t = Y_t^1 + Y_t^2$.

We note that Theorem 2 implies that:

$$M_t = e^{-\delta t}\frac{1}{C_t} = \frac{e^{-\delta t}}{[\kappa - i(s_t)]\, K_t}, \tag{24}$$

and that

$$Q_t = \frac{\delta}{\kappa - i_t}, \qquad p_t = \frac{\kappa - i_t}{\delta}.$$

Limit Cases. Before we study the general case where $0 < a < \infty$, it is useful to understand how the equilibrium behaves in the limit cases where capital transformation is either costless ($a \to 0$)

or infinitely costly ($a \to \infty$).

The following proposition characterizes equilibrium when the reallocation of capital becomes costless.

Proposition 1 (Economy with costless reallocation of capital) *As $a \to 0$, equilibrium converges to what is effectively a one-sector economy. The capital share is constant,*

$$s_t \equiv s^0 = \frac{\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

The capital growth rate $\mu = s\mu_1 + (1-s)\mu_2$ and volatility, $\sigma = \sigma_K(s^0)$, are also constant and we obtain:

$$\frac{dK_t}{K_t} = \mu dt + \sigma dz_K.$$

The equilibrium investment rate is constant, $i_t = i^0 = \kappa - \delta$, and therefore the SDF is:

$$M_t^0 = \frac{e^{-\delta t}}{\delta K_t}. \tag{25}$$

The value function is also constant, $V(s) = [\mu + \kappa - \sigma^2/2 + \delta \ln(\delta) - \delta]/\delta^2$.

Because it is costless to transform capital, the VC firm instantaneously adjusts capital to maintain the same equilibrium that trades-off the riskiness and growth potential of each sector. The optimal capital share, s^0, lies on the mean-variance efficient frontier.

We next turn to the other limit case, where capital transformation is infinitely costly.

Proposition 2 (Economy without transformation of capital) *As $a \to \infty$, the economy converges to what is effectively a two-tree economy. The investment share is in the limit identical to the capital share at all times, $w_t = s_t$. The investment rate is constant, $i_t = i^\infty = \kappa - \delta$, and the SDF is given by Equation* (25).

Since the equilibrium investment rate is constant and the investment share is equal to the capital share, equilibrium consumption and price dynamics are qualitatively the same as in Cochrane et al. (2008). However, because the investment rate, $i^\infty = \kappa - \delta$, is strictly positive as per Assumption 1, the economy without capital transformation is not completely identical to the two-tree pure exchange economy described by Equation (2). In particular, the growth rate is higher than in the pure exchange economy:

$$\frac{dY_t^n}{Y_t^n} = (\mu_n + \kappa - \delta)dt + \sigma_n dz_{n,t}. \tag{26}$$

The rest of the economy then behaves like a two-tree exchange economy with the growth rate given by Equation (26).

The General Case. We now study the general interior case where capital transformation costs are positive and finite: $0 < a < \infty$. From Equations (17) and (25), the SDF can be written as

$$M_t = Q(s_t)M_t^0.$$

The function $Q(s_t)$ represents the adjustment to the SDF from the limit cases where $a = 0$ or $a = \infty$.[2] Because of $Q(s)$, the capital share s_t is a state variable that drives the SDF M_t and consequently the cross-section of asset prices.

In Figure 1, we study the equilibrium for different interior values of $a \in \{0.1, 1, 10\}$ and the limit cases $a = 0$ and $a = \infty$. The parameters are chosen so that 2-capital has lower growth and higher risk than 1-capital. Despite being an inferior investment opportunity, 2-capital is still demanded in positive amounts in equilibrium because of its diversification benefits (i.e., because Assumption 1 is satisfied).

The first panel of Figure 1 displays the value function as a function of the capital share s_t. The vertical line shows the value of the instantaneous mean-variance efficient capital share, s^0, corresponding to $a = 0$. Capital allocation is tilted toward 1-capital (i.e., the capital share s^0 is above 0.5) because of the sector's superior risk-return properties.

As the capital transformation friction a increases, the value function decreases and peaks further to the right, i.e. for higher values of the capital share s_t. We denote the optimal value of the capital share by s^*. The fact that s^* is greater than s^0 means the representative agent wishes to own *less* of 2-capital than in the frictionless case. The reason for this outcome is that, when a is high, it becomes costlier to adjust capital in a scenario where random events push s_t to be very low. The lack of capital redeployment is seen in the middle panel of Figure 1, which shows that for high values of a, the investment share w_t stays close to s_t. The low-s_t scenario is painful to the representative agent because at that point the economy is dominated by the low-growth and high-volatility 2-capital.

[2] Note that we are here using the SDF of the economy without transformation costs, using the capital in the a-economy. Of course, in general, the actual capital in the a-economy and the economy without frictions would differ.

Therefore, owning more of 1-capital becomes increasingly valuable.

When the capital share is at s^*, the economy is well-balanced. The invested capital is in the same proportion as the current capital share (i.e. $w_t = s_t$), so the VC firm does not reallocate real capital from one sector to another. The SDF is equal to M_t^0 and each sector has a price-to-capital ratio equal to one. However, when the capital share differs from s^*, the economy faces sector capital imbalance. Consider, for example, that sector 1 incurs a negative shock and its capital level falls below its optimal capital share. Its remaining capital stock becomes increasingly valued (i.e., $p_t^1 > p_t^2$). The VC is keen to supply more capital to that sector and bring it back to balance. This can be seen in Equation (20), which shows that $w_t - s_t = \delta V'(s_t)$ has the same sign as $V'(s_t)$.

The right panel of Figure 1 shows that the investment rate increases away from s^*. The high level of investment represents the increase in value of bringing the economy back toward the optimal share. The planner is thus willing to impose higher friction-costs when s_t is far away from s^*, both by increasing $w_t - s_t$, and by increasing the total level of investment I_t.

2.3 Capital Flows

We denote by $f_t = w_t - s_t$ the (signed) capital flow from sector 2 to sector 1 at time t. From Equation (20), the capital flow is given by:

$$f(s_t) = w_t - s_t = \frac{\delta}{2a} V'(s_t). \tag{27}$$

Figure 2 shows the capital flow as a function of the state, s_t, for different values of a. Several insights arise from the figure. First, there is a one-to-one mapping between the state variable s and the capital flow f. Capital flows are therefore informative about the level and degree of sector capital imbalance. For example, sector 1's capital flow is positive when its capital share is below its optimum value, s^*, zero when the capital share is at s^*, and negative when the capital share exceeds s^*.

Another insight from Equation (27) is that capital flows are persistent. Figure 2 shows that, the higher is the value of the friction a, the lower is the absolute value of the flow, $|f|$. It will typically take a while to return to the optimal capital share, s^*, especially when a is nonnegligible. We obtain the following empirical prediction.

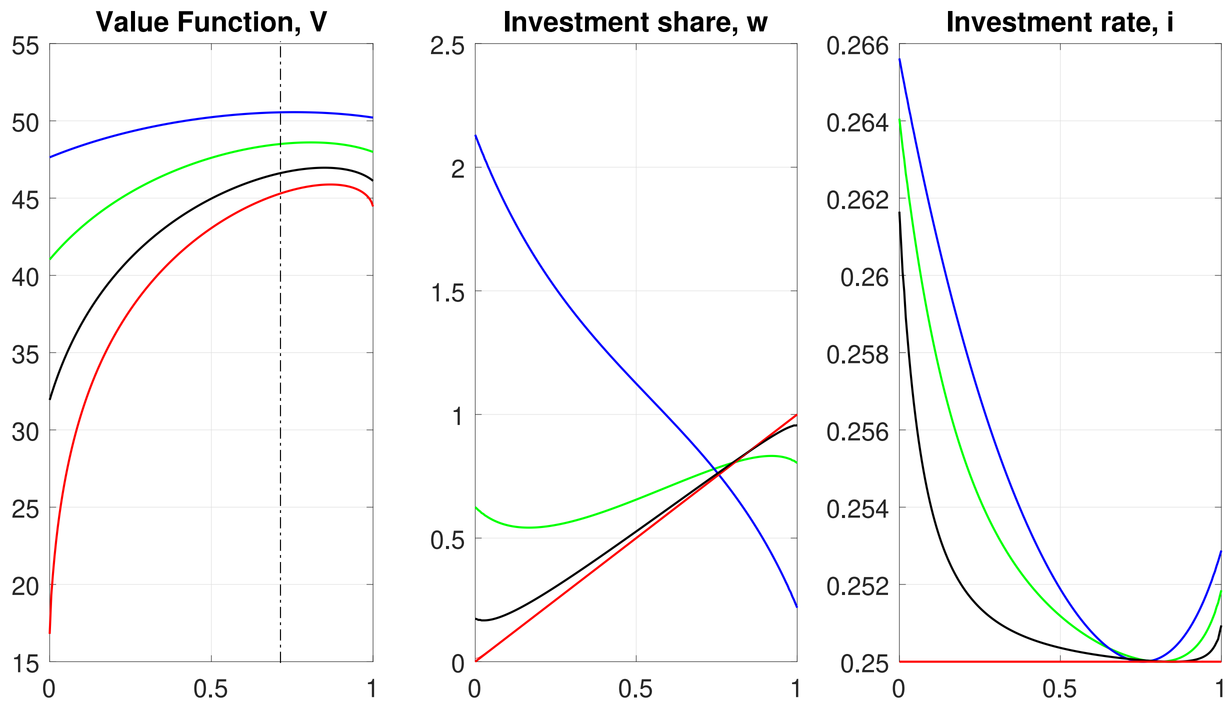

Figure 1: **Value function, investment share, and investment rate, for different values of a.** Parameters: $\delta = 0.05$, $\mu_1 = 0.1$, $\mu_2 = 0.09$, $\rho = 0$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $\kappa = 0.3$, $a \in \{0, 0.1, 1, 10, \infty\}$ (Bblue, green, black, and red). The investment rates in the limit cases are $i^0 = i^\infty = \kappa - \rho = 0.25$, and the Q functions are $Q \equiv 1$.

Prediction 1 *In the interior case where capital transformation costs are positive and finite:* $0 < a < \infty$*, the capital flow* f *is predictable and persistent.*

A third insight is that, because capital flows are informative about the sign and degree of the sector imbalance, they are also informative about asset prices, through the SDF. All equilibrium

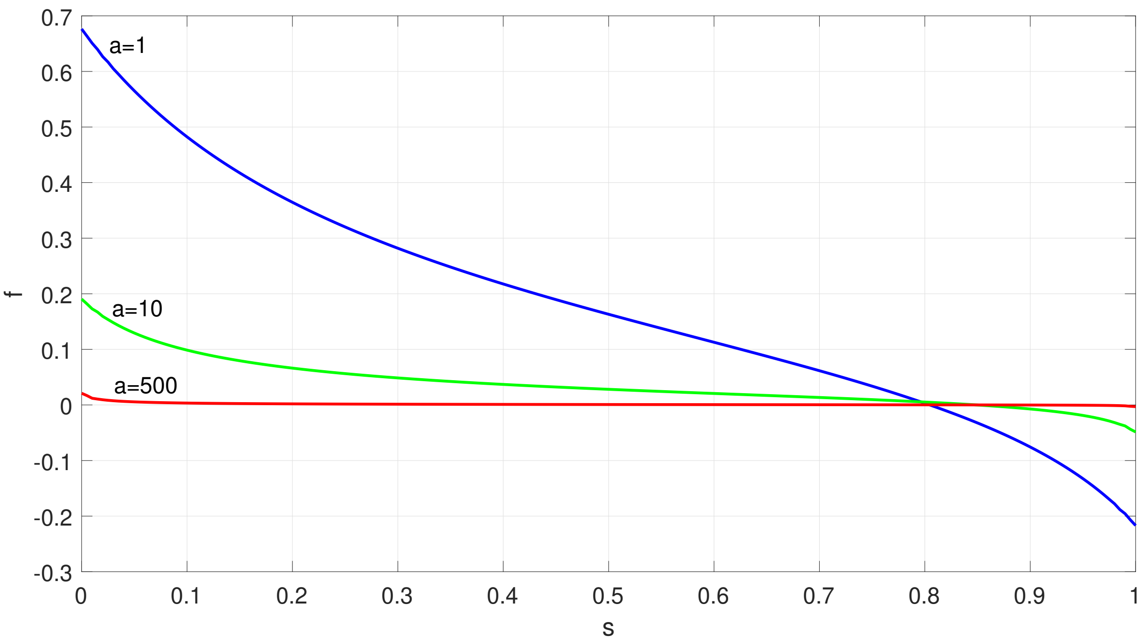

Figure 2: **Capital flow, f, for different values of the capital share, s.** Parameters: $\delta = 0.05$, $\mu_1 = 0.1$, $\mu_2 = 0.05$, $\rho = 0$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\kappa = 0.3$, $a \in \{1, 10, 500\}$.

quantities and prices can actually be re-expressed in terms of capital flows:

$$i_t = \kappa - \frac{\delta}{1 + af_t^2},$$

$$M_t = \frac{e^{-\delta t}}{\delta K_t}(1 + af_t^2),$$

$$Q(s_t) = 1 + af_t^2, \tag{28}$$

$$p(s_t) = \frac{1}{1 + af_t^2}, \tag{29}$$

$$p_t^1 = [1 + 2af_t(1 - s_t)]\, p_t = \frac{1 + 2af_t(1 - s_t)}{1 + af_t^2}, \tag{30}$$

$$p_t^2 = (1 - 2as_t f_t)p_t = \frac{1 - 2af_t s_t}{1 + af_t^2}, \tag{31}$$

$$d_t^1 = \kappa + 2ai_t f_t(1 - s_t)\frac{1 - af_t^2}{1 + af_t^2}, \tag{32}$$

$$d_t^2 = \kappa - 2ai_t f_t s_t \frac{1 - af_t^2}{1 + af_t^2}. \tag{33}$$

Equations (29), (30), and (31) imply a direct relation between the capital flow and the difference between the sectors' price-to-capital ratio.

$$\frac{p_t^1 - p_t^2}{p_t} = 2af_t. \tag{34}$$

This relation leads to the following prediction:

Prediction 2 *The sector with a high price-to-capital ratio has a positive capital flow, while the sector with a low price-to-capital ratio has a negative capital flow.*

An appealing property of equilibrium when the gardener is present ($0 < a < 0$) is that there exists a unique stationary share distribution. This is because the gardener transforms capital so that the share reverts back toward the interior when it gets close to the boundaries $s = 0$ and $s = 1$. The capital flows thus make the capital share mean-revert toward the interior, in contrast to the original two-trees economy, in which one tree always dominates in the long run. Our model therefore provides a microfoundation for the mean-reverting share processes that have been introduced elsewhere for multi-sector exchange economies, see, e.g., Santos and Veronesi (2006). We have:

Theorem 3 *In the interior case where costs are positive and finite:* $0 < a < \infty$*, there exists a unique stationary equilibrium capital share distribution.*

Consequently, all other important ratios, e.g., dividend shares and price-to-capital shares, also have unique stationary distributions.

2.4 Tobin's q

Consistent with q-theory under adjustment costs, our model links investment to a sector's price-to-book ratio, as the previous section explains. A distinct feature of our model is that the adjustment costs are associated with the transformation of capital between sectors, rather than with aggregate investments. Therefore, because new capital is created as a $(w, 1-w)$ mix of 1- and 2-capital, marginal q needs to be measured with respect to this new capital mix.

At the capital share s, the VC firm creates $w - as(w-s)^2$ units of 1-capital and $(1-w) - a(1-s)(w-s)^2$ units of 2-capital net, which amounts to $1 - a(w-s)^2$ units of new capital in total. One verifies that the price-to-capital ratio of the new capital generated by a unit of investment at the equilibrium w-mix is equal to one:

$$p^1[w - as(w-s)^2] + p^2[1 - w - a(1-s)(w-s)^2] = 1.$$

The value of one unit of w-capital mix is

$$2 - \frac{1}{1 + a(w-s)^2} > 1, \qquad s \neq s^*.$$

This is the marginal q before adjustment costs. Marginal q therefore increases when s is far from s^*, at which point the absolute flow $|f| = |w - s|$ is high.

Average q before adjustment costs is the q of the s-mix rather than the w-mix. It can be written as

$$\frac{1}{1 + a(w-s)^2} < 1, \qquad s \neq s^*.$$

In contrast to marginal q, average q *decreases* when the capital share is far away from s^*.

The wedge between marginal q and average q in our model is generated by capital transformation costs rather than by convex costs of aggregate investments. In our model, the adjustment costs are zero if and only if the capital mix is the same as the current capital share, $w = s$, in which case marginal q and average q are both equal to one. The wedge is thus not driven by the *size* of aggregate investments, but rather by the magnitude of the *flow* of capital transformation.

3 Asset Pricing Implications

We investigate the model's main asset pricing implications.

3.1 Sources of Risk and Return

Given Equation (17), the dynamics of the stochastic discount factor are given by:

$$\begin{aligned}
\frac{dM_t}{M_t} &= -\delta dt - \frac{p'}{p}\left(ds - \sigma_{sK}dt\right) - \frac{dK}{K} - \sigma_K^2 dt \left(\frac{1}{2}\frac{p''}{p} - \left(\frac{p'}{p}\right)^2\right)\sigma_s^2 dt \\
&= -\underbrace{\left(\delta + \frac{p'}{p}(\mu_s - \sigma_{sK}) + \mu_K - \sigma_K^2 + \left(\frac{1}{2}\frac{p''}{p} - \left(\frac{p'}{p}\right)^2\right)\sigma_s^2\right)}_{r_f} dt - \sigma_K dz_K - \frac{p'}{p}\sigma_s dz_s \\
&= -r_f dt - \sigma_K dz_K - \underbrace{\frac{p'}{p}}_{\lambda^s}\sigma_s dz_s \\
&= -r_f dt - \sigma_K dz_K - \lambda^s \sigma_s dz_s, \qquad (35)
\end{aligned}$$

where an expression for $\sigma_{sK}dt = \langle ds, \frac{dK}{K}\rangle$ is provided in the Appendix.

The first two terms on the right-hand-side of Equation (35) are familiar from standard consumption-based asset pricing models. The coefficient σ_K is the market price of a unit of consumption risk, keeping the capital share constant. The coefficient takes this form because $\hat{C}_t = \kappa K_t$ and the household's coefficient of relative risk aversion is equal to one.

The third term on the right-hand-side, $\lambda^s \sigma_s dz_s$, represents a second pricing factor regarding risk associated with sector capital imbalance. When there is a "capital share shock" so that the economy becomes more imbalanced, more effort is exerted to move back toward a balanced outcome, which in turn increases marginal utility. The coefficient λ^s is the risk premium associated with one unit of this risk, keeping aggregate capital constant.

The dynamics of the stochastic discount factor in Equation (35) imply a two-factor pricing model. If we consider an asset with instantaneous returns

$$\frac{dV}{V} = \mu dt + \varphi_1 dz_K + \varphi_2 dz_s + \varphi_3 d\hat{z}, \qquad (36)$$

where $\hat{z}$ is uncorrelated with z_K and z_s, then the following conditional two-factor expected return relation holds:

$$\mu = r^f + \varphi_1 \sigma_K \left(1 + \lambda^s \frac{\sigma_{sK}}{\sigma_K^2}\right) + \varphi_2 \sigma_s \left(\frac{\lambda^s + \sigma_{sK}}{\sigma_s^2}\right). \qquad (37)$$

The first factor provides compensation for exposure to consumption risk, while the second factor provides compensation for exposure to sector capital imbalance risk.

3.2 Capital Flows and the Cross-Section of Stock Returns

We next investigate the relation between capital flows and the cross-section of stock returns. Consider a portfolio that is long one dollar of 1-firms and short one dollar of 2-firms, i.e., the portfolio $\mathbf{w} = (X^1, X^2) = (1, -1)$. By definition, the expected payoff of this long-short portfolio is

$$r^e dt \quad = \quad E\left[\left(\frac{d^1}{p^1}dt + \frac{dP^1}{P^1}\right) - \left(\frac{d^2}{p^2}dt + \frac{dP^2}{P^2}\right)\right]. \tag{38}$$

The following proposition characterizes the expected payoff of $\mathbf{w}$.[3]

Proposition 3 *The expected payoff of the long-short portfolio* $\mathbf{w}$*,* r^e*, is*

$$r^e dt \quad = \quad \hat{\lambda}^s \left(\sigma_{sK} + \frac{p_s}{p}\sigma_s^2\right) dt, \qquad \textit{where} \tag{39}$$

$$\hat{\lambda}^s \quad = \quad \frac{d}{ds}\left[\ln\left(\frac{s p^1(s)}{(1-s)p^2(s)}\right)\right], \tag{40}$$

and has the following properties:

1. $r^e < 0$ *for small* s*, with limit value* $-(\sigma_2^2 - \rho\sigma_1\sigma_2)$ *when* $s \to 0$*.*

2. $r^e > 0$ *for large* s*, with limit value* $\sigma_1^2 - \rho\sigma_1\sigma_2$ *when* $s \to 1$*.*

3. r^e *is increasing in* s *outside some interval around* s^**, i.e., when* $s \notin [\underline{s}, \bar{s}]$*, where* $0 < \underline{s} < s^* < \bar{s} < 1$*.*

The proposition shows that, when the economy is sufficiently out of balance (i.e. outside some interval around s^*), there is a monotonically increasing relation between the average return of the long-short portfolio and the capital share s. This has clear implications for the cross-section of stock returns. When sector 1 is below its optimal capital share, the long-short portfolio has a negative average payoff, meaning that 1-firms have *lower* expected returns than 2-firms. Conversely, when sector 1 is above its optimal capital share, the long-short portfolio has a positive expected payoff, meaning that 1-firms have *higher* expected returns than 2-firms.

[3] Since this is a long-short portfolio that has a zero cost, we refer to the expected payoff instead of the expected return.

Why do 1-firms generate a lower average return than 2-firms when sector 1 is below its optimal capital share? Two effects are at play. First, 1-firms represent a small proportion of the aggregate economy and therefore are less systematic than 2-firms. Second, investing in 1-firms is valuable because they provide a hedge against the sector capital imbalance risk factor. More specifically, as we explained in Section 2.2, if sector 1 continues to incur negative shocks, the economy will become even more out of balance and the price-to-capital ratio of 1-firms will go up further, while the price-to-capital ratio of 2-firms will go down. By investing in 1-firms, investors are therefore hedging against this high-marginal-utility state.

By the same logic, when sector 1 is above its optimal capital share, the risk profile of 1- and 2-firms flips. The 2-firms now have *higher* expected returns than 1-firms because they are less systematic and also provide a hedge against the sector capital imbalance risk factor.

Altogether, these results allow us to characterize the properties of a long-short portfolio that is conditioned on the sign of the capital flow, $\mathbf{w}^f = \text{sign}(f_t)(-1, 1)$. Unlike the portfolio $\mathbf{w}$ that is always long 1-firms and short 2-firms, the portfolio $\mathbf{w}^f$ is long the capital-outflow sector and short the capital-inflow sector. As a consequence of Proposition 3 and the fact that capital flows are high when the economy is far from s^*, the long-short portfolio $\mathbf{w}^f$ has a high expected payoff when unsigned capital flows are high. We obtain the following prediction.

Prediction 3 *The expected payoff of the long-short portfolio* $\mathbf{w}^f$*, i.e., the portfolio that holds a long position in the sector with negative capital flow and a short position in the sector with positive capital flow, is high when unsigned capital flows,* $|f_t|$*, are large.*

3.3 Aggregate Equity Pricing and Risk

We obtain a pricing relation with respect to the aggregate (equity) portfolio, which holds the fraction $x_t = \frac{p_t^1 K_t^1}{p_t K_t} = \frac{P_t^1}{P_t}$ in 1-firms and the fraction $1 - x_t = \frac{p_t^2 K_t^2}{p_t K_t} = \frac{P_t^2}{P_t}$ in 2-firms at date t. In other words, the aggregate portfolio is $m_t = (x_t, 1 - x_t)$. We refer to x_t as the *financial share.*

From Equations (28) and (29), it follows that the stochastic discount factor can be re-written as $M_t = \frac{e^{-\delta t}}{\delta P_t}$. The corresponding SDF dynamics are:

$$\frac{dM_t}{M_t} \quad = \quad -\delta dt + \left(\frac{dP_t}{P_t}\right)^2 - \frac{dP_t}{P_t}. \tag{41}$$

Because the return on the aggregate portfolio, $\frac{dP_t}{P_t}$, perfectly captures the stochastic component of the SDF, a conditional CAPM holds with respect to the aggregate portfolio.

Specifically, consider an asset with return dynamics given by Equation (36). The expected return on this asset can be written as:

$$\mu dt = r_f dt + \left(\frac{dV}{V}\right)\left(\frac{dP}{P}\right) = r_f dt + \sigma_{V,P} dt. \tag{42}$$

Because the instantaneous total return on the aggregate portfolio is equal to

$$\frac{dV_m}{V_m} = \frac{dP}{P} + \frac{\kappa - i}{p} dt = \mu_m dt + \sigma_m dz_m,$$

where $\sigma_m^2 = (\lambda^s)^2 p^2 \sigma_s^2 + \sigma_K^2 - 2\sigma_{s,K}\lambda^s p$, we obtain from Equation (42) that the aggregate risk premium is equal to σ_m^2,

$$\mu_m = r_f + \sigma_m^2,$$

which is consistent with equilibrium under log-utility investor preferences. In addition, we obtain the conditional CAPM pricing relation for an arbitrary asset with total value process V:

$$\mu = r_f + \beta(\mu_m - r_f). \tag{43}$$

where $\beta = \sigma_{V,P}/\sigma_m^2$. This result is again consistent with the fact that log-utility investors hold instantaneously mean-variance efficient portfolios.

The instantaneous variance of the aggregate portfolio is

$$\begin{aligned} \sigma_m^2 dt &= \left\langle \frac{dP}{P}, \frac{dP}{P} \right\rangle \\ &= \left(\frac{p'}{p}\right)^2 \sigma_S^2 dt + 2\frac{p'}{p}\sigma_{SK} dt + \sigma_K^2 dt. \end{aligned}$$

The first term is nonnegative and minimized at s^*, whereas the middle term is negative between $\hat{s}$ and s^*, and positive outside of that interval. Both terms tend to zero as s approaches the boundaries ($s = 0$ and $s = 1$). By contrast, the third term is strictly convex in s with its minimum at $\hat{s}$. As a consequence, we obtain the following prediction.

Prediction 4 *Aggregate volatility, σ_m, is minimized between $\hat{s}$ and s^*, and is high when unsigned capital flows, $|f_t|$, are large.*

Intuitively, when the capital share is far from s^*, the economy is out of balance and less diversified. The aggregate volatility therefore increases when capital flows are large.

3.4 IPOs and Public Equity Pricing

We next obtain a pricing relation with respect to the publicly traded portfolio, i.e., the market portfolio. As is well known from Roll (1977), when the market portfolio differs from the aggregate portfolio, an asset can have non-zero CAPM alpha with respect to the market portfolio of publicly listed firms, even though the conditional CAPM holds with respect to the aggregate portfolio.

We consider a setting where newly created firms are held privately for a period before they enter the market via initial public offerings (IPOs). Assuming that the n-capital owned by publicly traded n-firms is $\hat{K}_t^n$, $n = 1, 2$, respectively, the *publicly traded capital share* is

$$\hat{s}_t = \frac{\hat{K}_t^1}{\hat{K}_t^1 + \hat{K}_t^2},$$

and the *publicly traded financial share* is

$$\hat{x}_t = \frac{p_t^1 \hat{K}_t^1}{p_t^1 \hat{K}_t^1 + p_t^2 \hat{K}_t^2}.$$

The publicly traded portfolio is $\hat{m}_t = (\hat{x}_t, 1 - \hat{x}_t)$ with market value $\hat{P}_t$. It follows that $\hat{x}_t > x_t$ if and only if $\hat{s}_t > s_t$.

To obtain a pricing relation with respect to the publicly traded portfolio $\hat{m}$, we proceed as follows. The total return on the n-firm is

$$\frac{dV^n}{V^n} = \frac{d(p^n K^n)}{p^n K^n} + \frac{d^n}{p^n} dt = \mu_n^P dt + \sigma_n^P dz_n^P. \tag{44}$$

The instantaneous returns of the aggregate portfolio and the publicly traded portfolio can therefore be written as

$$\frac{dV_{m,t}}{V_{m,t}} = x \frac{dV_t^1}{V_t^1} + (1 - x) \frac{dV_t^2}{V_t^2} = \mu_m dt + \sigma_m dz_m. \tag{45}$$

$$\frac{dV_{\hat{m},t}}{V_{\hat{m},t}} = \hat{x} \frac{dV_t^1}{V_t^1} + (1 - \hat{x}) \frac{dV_t^2}{V_t^2} = \mu_{\hat{m}} dt + \sigma_{\hat{m}} dz_{\hat{m}}. \tag{46}$$

Together, Equations (43) and (45) imply that the expected return of each firm is equal to:

$$\begin{aligned}
\mu_1^P &= r_f + x(\sigma_1^P)^2 + (1-x)\sigma_{1,2}^P, \\
\mu_2^P &= r_f + (1-x)(\sigma_2^P)^2 + x\sigma_{1,2}^P,
\end{aligned}$$

where $\sigma_{1,2}^P dt = \langle \frac{dV^1}{V^1}, \frac{dV^2}{V^2} \rangle$ is the instantaneous covariance between returns on 1-firms and 2-firms. The expected return of each firm is a function of (i) its share in the market portfolio, (ii) its variance, and (iii) its covariance with the other firm.

When the market return is (incorrectly) used to price assets using the CAPM model given by Equation (43), the CAPM alpha of the n-firm, α_n, arises from the difference between the true expected returns and incorrect required rates

$$\alpha_n = (\mu_n - r_f) - \beta_{n,\hat{m}} (\mu_{\hat{m}} - r_f)$$

where

$$\begin{aligned}
\beta_{1,\hat{m}} =& \frac{1}{\sigma_{\hat{m}}^2} \left[\hat{x} \left(\sigma_1^P\right)^2 + (1-\hat{x})\, \sigma_{1,2}^P \right], \\
\beta_{2,\hat{m}} =& \frac{1}{\sigma_{\hat{m}}^2} \left[\hat{x}\sigma_{1,2}^P + (1-\hat{x}) \left(\sigma_2^P\right)^2 \right], \\
\sigma_{\hat{m}}^2 =& \hat{x}^2 \left(\sigma_1^P\right)^2 + 2\hat{x}(1-\hat{x})\, \sigma_{1,2}^P + (1-\hat{x})^2 \left(\sigma_2^P\right)^2, \\
\mu_{\hat{m}} - r_f =& \hat{x} \left(x \left(\sigma_1^P\right)^2 + (1-x)\, \sigma_{1,2}^P \right), \\
&+ (1-\hat{x}) \left(x\sigma_{1,2}^P + (1-x) \left(\sigma_2^P\right)^2 \right).
\end{aligned}$$

Expanding these terms, we obtain a simple formula for the CAPM alpha for each firm.

Proposition 4 *When the publicly traded portfolio $\hat{m} = (\hat{x}, 1-\hat{x})$ is used to price assets rather than the aggregate portfolio $m = (x, 1-x)$, the observed alphas of 1- and 2-firms are*

$$\begin{aligned}
\alpha_1 = \mu_1^P - \hat{\mu}_1^P &= \frac{(\sigma_1^P \sigma_2^P)^2 - \sigma_{12}^2}{\sigma_{\hat{m}}^2} (x-\hat{x})(1-\hat{x}), \\
\alpha_2 = \mu_2^P - \hat{\mu}_2^P &= \frac{(\sigma_1^P \sigma_2^P)^2 - \sigma_{12}^2}{\sigma_{\hat{m}}^2} (\hat{x}-x)\hat{x}.
\end{aligned}$$

As a consequence of this Proposition, the sector that is over-represented in the publicly traded portfolio has a negative CAPM alpha, whereas the sector that is under-represented in the publicly traded portfolio has a positive alpha.

A well-known stylized fact in the IPO literature is that firms with high valuation ratios are pushed more aggressively toward IPOs than firms with low valuation ratios (Helwege and Liang, 2004; Lerner, 1994; Lowry, 2003; Pagano et al., 1998)). In our setting, p^1 and p^2 represent the valuation ratios of firms in the two sectors. From Equations (30) and (31), it follows that 1-firms have high valuation ratios when the capital share is low ($s_t < s^*$) and, conversely, 2-firms have high valuation ratios when the capital share is high ($s_t > s^*$).

For simplicity, we assume that high-valuation firms are introduced to the market immediately, whereas low valuation firms are kept private indefinitely. It then follows that $\hat{x} > x$ when $s < s^*$, and $\hat{x} < x$ when $s > s^*$. In other words, the publicly traded portfolio overweights 1-firms when the capital share is low and overweights 2-firms when the capital share is high. Altogether, these points lead to the following prediction:

Prediction 5 *Firms in sectors with capital inflows have negative CAPM alphas whereas firms in sectors with capital outflows have positive CAPM alphas.*

In addition to this relation between capital flows and CAPM alphas, we obtain a two-factor pricing model that includes both the market factor and the sector capital imbalance factor. From Equations (41), (45), and (46), we rewrite the stochastic discount factor as

$$\begin{aligned} \frac{dM_t}{M_t} &= = -r_f dt - \sigma_m dz_m, && (47) \\ &= -r_f dt - \sigma_{\hat{m}} dz_{\hat{m}} - (x - \hat{x}) \underbrace{\left(\frac{d}{ds} \left[\ln \left(\frac{s p^1(s)}{(1-s) p^2(s)} \right) \right] \right)}_{\hat{\lambda}^s} \sigma_s dz_s. && (48) \end{aligned}$$

The second term on the right-hand-side of Equation (48) is a publicly observed market factor.

The third term, $(x - \hat{x})\hat{\lambda}^s \sigma_s dz_s$, corresponds to an additional risk premium that arises when the model is based on the publicly traded portfolio rather than the aggregate portfolio. This term depends on the coefficient $\hat{\lambda}^s$ because the aggregate portfolio can be viewed as the combination of the publicly traded portfolio and the long-short portfolio introduced in Section 3.2. When the publicly traded portfolio is equal to the aggregate portfolio, $\hat{x} = x$, this risk premium is zero.

4 Empirical Findings

We now turn to the data and empirically test the predictions of the model.

4.1 Data

We use data on daily stock market returns on listed US firms from CRSP, which is available at the Wharton WRDS library. The sample covers the period from January 1976 to December 2023. We focus on the returns of ordinary common stock shares and exclude firms in utility and financial sectors, as in Fama and French (1993) and many others.[4] Daily data on asset pricing factor returns are obtained from Ken French's data library and Hou-Xue-Zhang's q-factors data library.[5]

Firm-level accounting data come from Compustat, which is also available at the Wharton WRDS library. We collect the data at the quarterly frequency but construct variables at the annual end-of-year frequency to avoid seasonality issues. For stock variables such as a firm's book value, we use the last available observation for each calendar year. For flow variables such as income expense, we add up the variable's values from the four most recent quarters to obtain the total flow over the past calendar year. Because firms sometimes change their fiscal years, we require that, in any given year, the firm has been using the same fiscal year end for at least 5 consecutive quarters. This removes duplicate observations that result from changes in the fiscal year end.

For every firm and every year, we merge the CRSP daily stock return data in that year with the Compustat accounting data corresponding to the end of the previous year. For example, the book value of firm n on June 1st, 2021 corresponds to the firm's book value reported in Compustat at the end of 2020.

Figure 3 shows the number of firms in our sample between 1976 and 2023. The sample includes at least 2,000 stocks every year. The decline over the past two decades corresponds to the delisting trend that is well-known in the literature (Dodge et al., 2017).

[4]Ordinary common shares are identified as shrcd 10 or 11, and we use the link flags LC, LU, or LS to link CRSP with Compustat. Shares are aggregated at the firm-date level. We attribute to each firm (i) the total market equity across all ordinary share classes and (ii) the return of the largest share class by market capitalization.

[5]These data libraries can be accessed online at `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html` and `https://global-q.org`.

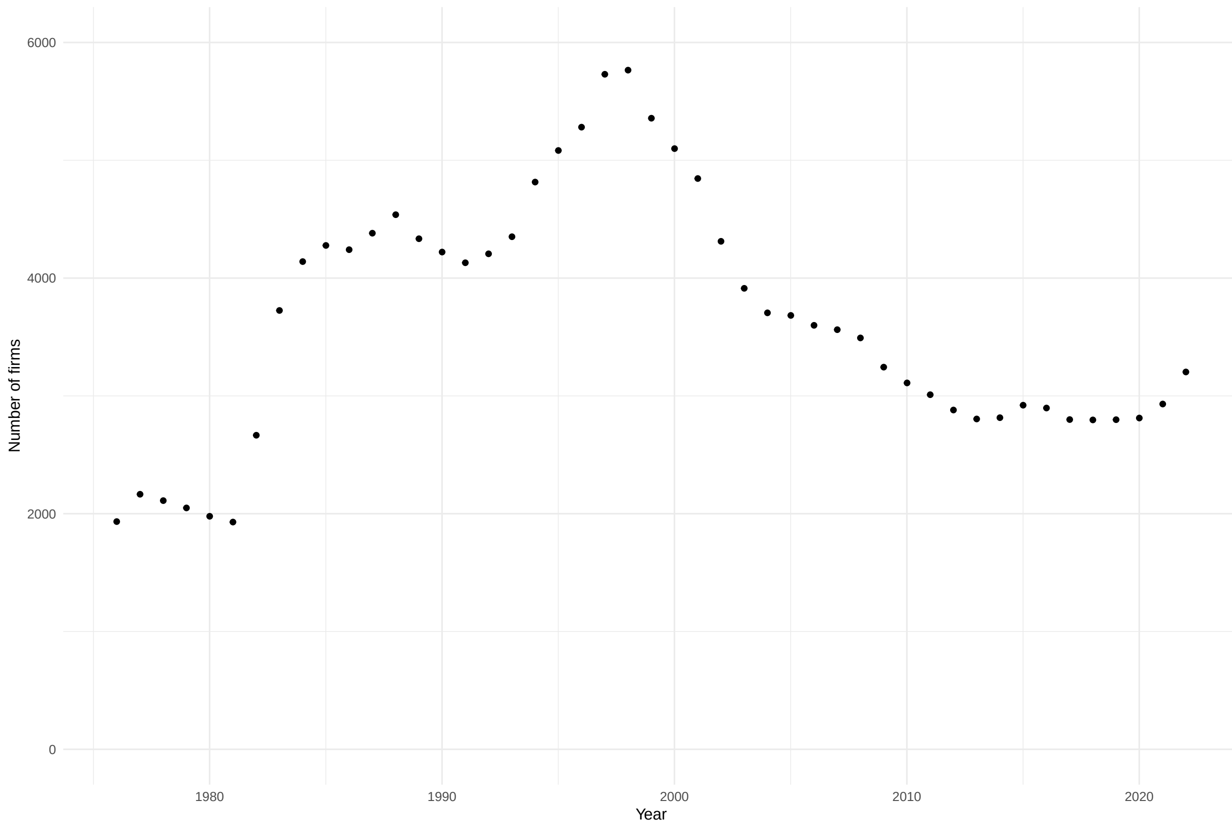

Figure 3: **Number of US listed stocks, 1976-2023.** The data is obtained from a sample of US firms in the combined CRSP-Compustat database, excluding firms in utility and financial sectors.

4.2 Real Capital Flows

We present a simple methodology to measure real capital flows from the data. For every firm n, we denote by $BE_{n,t}$ and by $ME_{n,t}$ the book and market values of the firm's equity at time t.[6] We also denote by $BA_{n,t}$ and by $MA_{n,t}$ the book and market values of the firm's net operating assets, which include all assets except for cash holdings and other short-term financial assets.[7] We focus on a firm's net operating assets as our measure of real capital. This measure is most appropriate to calculate real capital flows because an increase in net operating assets over time corresponds to an increase in real assets and not to a build up of cash reserves.

[6] The book value is calculated as follows: (i) Shareholder equity (seqq) less preferred equity (pstkrq or pstkq), (ii) Common equity (ceqq), or (iii) Asset less liability less preferred equity, adjusted by deferred tax credit (txditcq) if applicable.

[7] The book value of net operating assets is calculated as the difference between total assets (atq) and cash holdings and short-term financial assets (cheq). We refer to $BA_{n,t}$ as real capital. The market value of real capital, $MA_{n,t}$, is calculated as the sum of the market value of equity, $ME_{n,t}$, and the difference between a firm's book value of net operating assets, $BA_{n,t}$ and its book value net of equity, $BE_{n,t}$.

Aggregate Capital Flows For every firm n, we denote by $BA^p_{n,t}$ the one-year lagged book value of the firm's net operating assets and refer to it as *previous-capital*.[8] We categorize a positive difference between current capital and previous-capital as *in-capital*:

$$BA^i_{n,t} = \left[BA_{n,t} - BA^p_{n,t}\right]_+ ,$$

and a negative difference as *out-capital*:

$$BA^o_{n,t} = \left|\left[BA_{n,t} - BA^p_{n,t}\right]_-\right| .$$

The firm's capital at time t can thus be broken down as previous-capital plus in-capital minus out-capital:

$$BA_{n,t} = BA^p_{n,t} + BA^i_{n,t} - BA^o_{n,t}. \tag{49}$$

We denote by $BA_t = \sum_n BA_{n,t}$ the total capital in our sample of of publicly traded firms at time t. Similarly, we denote by $BA^j_t = \sum_n BA^j_{n,t}$, $j \in \{p, i, o\}$, the total values of previous-, in-, and out-capital. The terms BA^i_t and BA^o_t represent the total values of real capital flows at time t.

Our methodology for calculating real capital flows has several desirable properties. It calculates capital flows from firm-level information and reduces dimensionality by summarizing capital flows in and out of multiple sectors into (i) one portfolio regrouping all capital inflows, and (ii) one portfolio regrouping all capital outflows. Both portfolios are well-behaved and, together with the previous-capital portfolio, add up to the total capital in our sample of publicly traded firms. Additionally, the method is data-driven and agnostic on whether capital flows take place within or across standard industry classifications. The method therefore allows us to bypass such standard industry classifications which may not fully capture how the U.S. economy has transformed over the past 50 years (see Bergsman et al. (1975), Fort and Klimek (2016), and Keil (2017)).

Market Value of Assets For each firm n, we take the book value of each type of capital (in-, out-, previous-) and scale it up by the firm's market-to-book ratio. We obtain the market value

[8] The lagged value corresponds to the firm's book value of real capital from 4 fiscal quarters ago

$MA^j_{n,t}$:

$$MA^j_{n,t} = BA^j_{n,t} \frac{MA_{n,t}}{BA_{n,t}}, \qquad j \in \{p, i, o\}. \tag{50}$$

The market value of the in-, out-, and previous-capital portfolios is $MA^j_t = \sum_n MA^j_{n,t}$. Similarly, the market value of all firms in our sample is $MA_t = \sum_n MA_{n,t}$.

Market Value of Equity To obtain the market value of a firm's equity invested in each type of capital, we take the market value of each type of capital, $MA^j_{n,t}$, and multiply it by the firm's equity-to-asset ratio:

$$ME^j_{n,t} = MA^j_{n,t} \frac{ME_{n,t}}{MA_{n,t}}, \qquad j \in \{p, i, o\}. \tag{51}$$

We then sum up these market values across all firms to obtain the market value of equity invested in each portfolio: $ME^j_t = \sum_n ME^j_{n,t}$. Similarly, the market value of all firms in our sample is $ME_t = \sum_n ME_{n,t}$. This group of firms weighted by their equity market capitalization proxies for the publicly traded portfolio defined in Section 3. We hereafter refer to it as the market portfolio.

Equity Returns The daily rate of equity return of the market portfolio is the weighted average of the daily rate of equity return of each firm inside the market portfolio:

$$r_{m,t} = \frac{1}{ME_{t-1}} \sum_n \left[ME_{n,t-1} \cdot r_{n,t} \right]. \tag{52}$$

Similarly, the rate of equity return of the previous-, in-, and out-capital portfolios is defined as:

$$r_{j,t} = \frac{1}{ME^j_{t-1}} \sum_n \left[ME^j_{n,t-1} \cdot r_{n,t} \right], \qquad j \in \{p, i, o\}. \tag{53}$$

Combining Equations (52) and (53), we obtain that the rate of equity return of the market portfolio is equal to the weighted average of the rate of return on the previous-, in-, and out-capital portfolios:

$$
\begin{aligned}
r_{m,t} =& \frac{1}{ME_{t-1}} \sum_n \left[\left(\frac{BA^p_{n,t-1}}{BA_{t-1}} + \frac{BA^i_{n,t-1}}{BA_{t-1}} - \frac{BA^o_{n,t-1}}{BA_{t-1}} \right) ME_{n,t-1} \cdot r_{n,t} \right] && (54) \\
=& \frac{ME^p_{t-1}}{ME_{t-1}} \cdot r_{p,t} + \frac{ME^i_{t-1}}{ME_{t-1}} \cdot r_{i,t} - \frac{ME^o_{t-1}}{ME_{t-1}} \cdot r_{o,t}. && (55)
\end{aligned}
$$

4.3 Empirical Results

We present several results about real capital flows that are consistent with the model's key predictions.

Predictability of real capital flows. The model predicts that it is optimal to transform the capital over several periods in order to minimize the transformation costs (Prediction 1). As a result, capital flows should be predictable and persistent.

Each quarter, we study whether a firm that previously was in the in-capital portfolio stays in the in-capital portfolio or switches to the out-capital portfolio. Conversely, we study whether a firm that was previously in the out-capital portfolio stays in the out-capital portfolio or switches to the in-capital portfolio. Figure 4 plots the sample probabilities one, two, three, and more quarters ahead ("lead"). Because some firms exit the sample over time, the probabilities we estimate are conditional on staying in the sample.

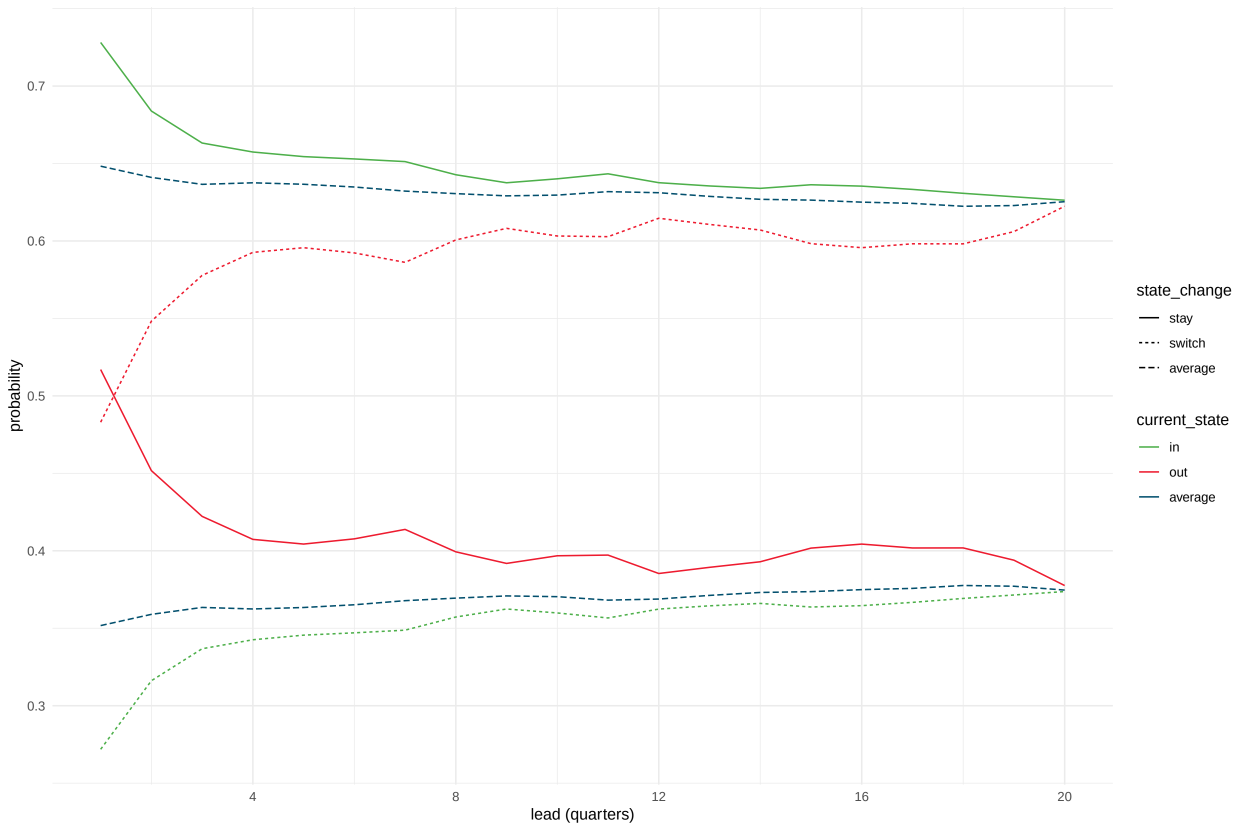

Figure 4: **Predictability of capital flows.** This Figure displays Markov probabilities of firms joining the in- and out-capital portfolios. Green lines depict the probabilities for a firm in the in-capital portfolio to stay in the in-capital portfolio (solid) or switch to the out-capital portfolio (dotted). Red lines depict the probabilities for a firm in the out-capital portfolio to stay in the out-capital portfolio (solid) or switch to the in-capital portfolio (dotted). Blue lines depict the unconditional average probabilities.

The green lines depict the probabilities for a firm in the in-capital portfolio in quarter t to either stay in the in-capital portfolio (solid) or switch to the out-capital portfolio (dotted) in the quarters that follow. By construction, the probabilities of the green lines add up to one for every lead. Similarly, the red lines depict the probabilities for a firm in the out-capital portfolio in quarter t to either stay in this portfolio (solid) or switch to the in-capital portfolio (dotted). The blue lines depict the sample average probabilities of a firm being in either in the in- or out-capital portfolio. A firm has a 62.5% probability of being in the in-capital portfolio at time t and a 37.5% probability of being in the out-capital portfolio.

After one year, a firm in the in-capital portfolio has a significantly higher probability of staying in the in-capital portfolio (75%) than switching over to the out-capital portfolio (25%). Similarly, a firm in the out-capital portfolio has a significantly higher probability of staying in the out-capital portfolio (55%) than switching to the in-capital portfolio (45%). Over time, however, the probabilities of remaining in the same portfolio drop and converge to the unconditional probabilities.

These findings altogether show evidence that capital flows are persistent, in line with the model's predictions.

Price-to-Capital Ratios The model predicts that capital flows towards where it is most valued (Prediction 2). Therefore, the sector with positive capital flows should have a higher price-to-capital ratio p_t than the sector with negative capital flows.

We compare each year the price-to-capital ratios of the previous-, in- and out-capital portfolios. Figure 5 plots the ratio MA_t^j/BA_t^j for each type of portfolio $j \in \{p, i, o\}$. In almost every year, the market-to-book ratio of the in-capital portfolio (green) is significantly higher than that of previous-capital (orange), which is itself higher than that of the out-capital portfolio (red). Capital therefore flows toward high market-to-book ratio firms and away from low market-to-book ratio firms.

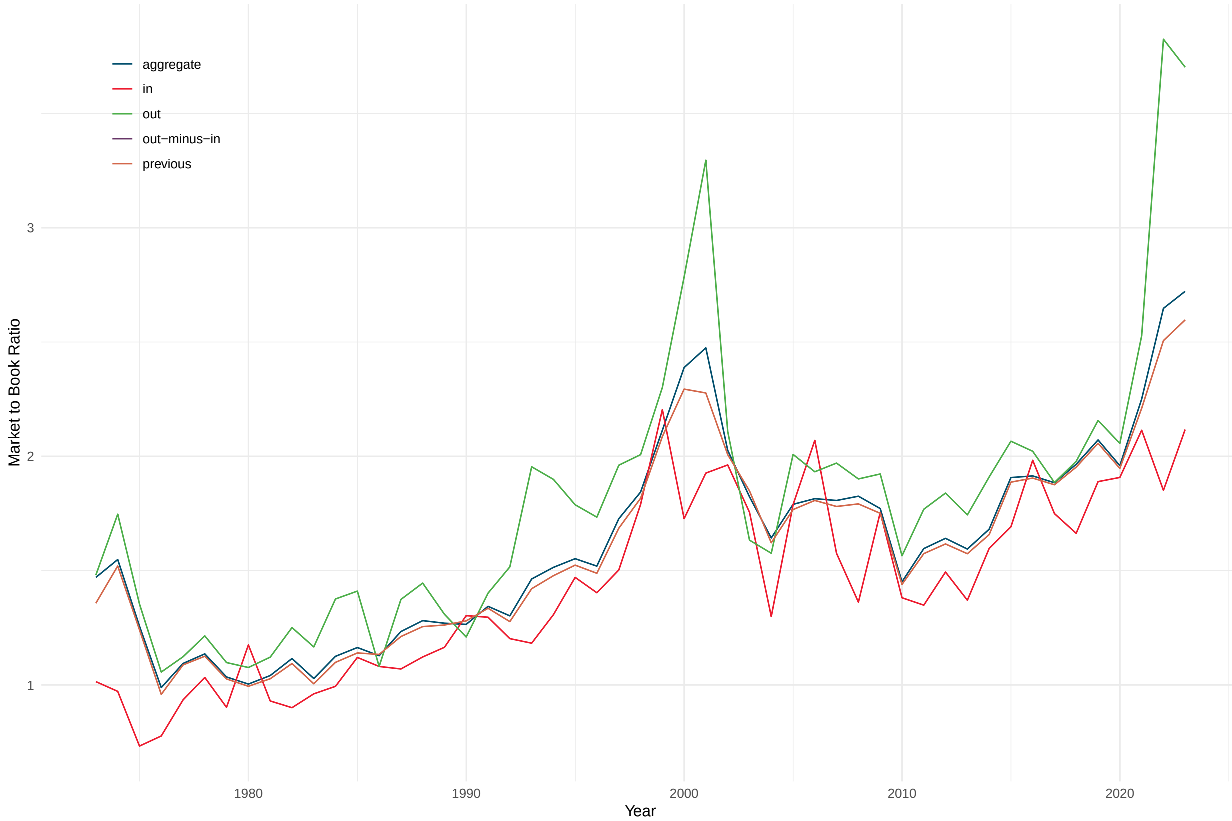

Figure 5: **Market-to-book ratios of previous-, in-, and out-capital.** This Figure displays the market-to-book ratio of the aggregate portfolio and the previous-, in-, and out-capital portfolios every year between 1976 and 2023.

The blue and orange lines depicts the price-to-capital ratios of the aggregate-capital and previous-capital portfolios. Not surprisingly, the market portfolio has a slightly higher price-to-capital ratio than the previous-capital portfolio because it is long the in-capital portfolio, which has a high price-to-capital ratio, and short the out-capital portfolio, which has a low price-to-capital ratio.

These facts are altogether consistent with the prediction from the model that capital flows away from where it is valued the least and towards where it is valued the most.

Capital Flows and Stock Returns The model predicts that a portfolio that is long the out-capital portfolio and short the in-capital portfolio has a high expected return (Prediction 3).

In Figure 6, we plot the cumulative performance of the in-capital, out-capital, previous-capital, and market portfolios. The out-capital portfolio (red) significantly and consistently outperforms the previous-capital portfolio (orange), which itself outperforms the in-capital portfolio (green). A dollar invested at the end 1976 in the out-capital portfolio (red) generates $825.56 at the end of 2023. By contrast, the same dollar invested in the previous-capital and in-capital portfolios only generates $215.75 and $59.38 at the end of 2023 respectively.

A portfolio constructed by buying one dollar of the out-capital portfolio, financed by selling one dollar of the in-capital portfolio can be viewed as a proxy for the long-short portfolio introduced in the previous section, constructed so that the position is long in the sector with high expected returns and short in the sector with low expected returns. The portfolio has an annualized yield of 7.15%, which is significantly higher than zero (t=3.54). The portfolio has a slightly negative market beta of -0.1, so a market model would predict the yield to be negative.

Figure 6: **Cumulative performance of previous-, in-, and out-capital portfolios.** This Figure displays the dollar value of a dollar invested in 1976 in each of the previous-, in-, and out-capital portfolios every year from 1976 to 2023.

Altogether, capital flows negatively predict future returns. This fact is consistent with the prediction from the model that sectors with capital outflows generate higher average returns than sectors with capital inflows.

CAPM Alpha In the model, a two-factor pricing model arises where the first factor is captured by the return of the publicly-traded market portfolio and the second factor reflects exposure to sector capital imbalance risk. This second factor can be directly recovered by constructing a portfolio that is long capital outflows and short capital inflows. Because this second factor has a risk premium not captured by the publicly traded portfolio, it should generate a positive CAPM alpha (Prediction 5).

In Table 1 we run CAPM time-series regressions of daily returns of the different portfolios against the daily returns of the market factor that we obtain from Ken French's data library. Column 1 shows the alpha of the market portfolio constructed from our dataset on the market factor. Not surprisingly, the alpha is close to zero and the R-square coefficient is close to 100%. The market portfolio of US firms in our sample is therefore representative of the market portfolio commonly used in asset pricing tests.

Columns 2 and 3 report the alpha of the out-capital and in-capital portfolios. Consistent with the model's prediction, the out-capital portfolio has positive alpha (t-stat of 2.9) and the in-capital portfolio portfolio has negative alpha (t-stat of 3.2). Moreover, the alpha of the long-short out-minus-in capital portfolio (Column 4) is large and amounts to 0.079 per year. Its t-value of 3.9 exceeds the threshold of 3 recommended by Harvey et al. (2015). This result is consistent with the prediction from the model that a portfolio constructed from real capital flows can be used as a pricing factor.

In Table 2, we regress the return of each portfolio on the market, value (HML), size (SMB), profitability (RMW), and investment (CMA) factors introduced in Fama and French (2015). The alpha of the out-minus-in factor loses statistical significance at the 5% level and has near unit (0.88) exposure to the investment factor. This finding suggests a strong linkage between the out-minus-in factor and the widely used investment factor, which is consistent with the prediction from the model that a long-short portfolio constructed from real capital flows can be used as a pricing factor in conjunction with the market factor.

5 Conclusion

Corporate investments transform capital and changes the economy's capital mix. We develop a parsimonious general-equilibrium model that builds on Cochrane et al. (2008)'s two-tree framework

Table 1: **CAPM Regressions** This table reports time-series regressions of daily returns of multiple stock portfolios on the daily returns of the market (MKT) factor. Stock portfolios include the aggregate-capital portfolio, the out-capital portfolio, the in-capital portfolio, the out-minus-in long-short portfolio, and the previous-capital portfolio. For each regression, we report the value of the intercept (alpha), its t-value, and the adjusted R^2 coefficient.

	Dependent variable:				
	aggregate-capital	out-capital	in-capital	out-minus-in	previous-capital
	(1)	(2)	(3)	(4)	(5)
alpha	0.0005	0.054***	−0.025***	0.079***	0.007**
	0.161	2.939	−3.198	3.917	2.247
mkt_excess	1.010***	0.991***	1.093***	−0.102***	0.991***
	949.303	146.394	386.759	−13.827	865.290
Observations	12,712	12,712	12,712	12,712	12,712
R^2	0.986	0.628	0.922	0.016	0.983
Adjusted R^2	0.986	0.628	0.922	0.016	0.983
Note:					*p<0.1; **p<0.05; ***p<0.01

and introduces a venture capital firm — the gardener — that creates and transforms capital. Our model leads to a rich set of equilibrium properties, linking capital flows to asset prices, and providing a micro-foundation for the empirically successful investment factor.

Our framework is consistent with both q-theory models of the firm and consumption-based models of investors. Consistent with q-theory, corporate investments are positively related to a firm's price-to-capital ratio. Consistent with consumption-based asset pricing theory, a long-short factor constructed from capital flows proxies for risks associated with sector capital imbalance and prices the cross-section of equity returns, in addition to the standard consumption risk factor, in a two-factor representation. Because the risk premium of this factor is not fully captured by the publicly traded market portfolio, the capital flow factor generates CAPM alpha, suggesting an explanation for the empirical success of the investment factor.

Table 2: **FF5 Regressions** This table reports time-series regressions of daily returns of multiple stock portfolios on the daily returns of the market (MKT), value (HML), size (SMB), profitability (RMW), and investment (CMA) factors obtained from Ken French's data library. Stock portfolios include the market portfolio constructed from our data, the out-capital portfolio, the in-capital portfolio, the out-minus-in long-short portfolio, and the previous-capital portfolio. For each regression, we report the value of the intercept (alpha), its t-value, and the adjusted R^2 coefficient.

	Dependent variable:				
	aggregate-capital	out-capital	in-capital	out-minus-in	previous-capital
	(1)	(2)	(3)	(4)	(5)
alpha	0.004*	0.045**	0.012**	0.034*	0.003
	1.746	2.576	2.463	1.833	1.046
mkt_excess	1.006***	1.027***	1.016***	0.011	1.005***
	1,137.951	144.992	537.672	1.430	987.595
hml	−0.153***	−0.170***	−0.262***	0.092***	−0.132***
	−88.236	−12.260	−70.765	6.331	−66.450
smb	−0.013***	0.220***	0.089***	0.131***	−0.025***
	−7.953	16.772	25.433	9.541	−13.385
rmw	0.027***	−0.295***	−0.195***	−0.100***	0.080***
	11.742	−16.267	−40.360	−5.251	30.541
cma	0.064***	0.536***	−0.344***	0.880***	0.160***
	21.975	23.115	−55.593	36.265	48.171
Observations	12,712	12,712	12,712	12,712	12,712
R^2	0.992	0.661	0.971	0.175	0.989
Adjusted R^2	0.992	0.661	0.971	0.174	0.989
Note:					*p<0.1; **p<0.05; ***p<0.01

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A Proofs

Proof of Theorems 1 and 2 :

We solve the social planner's problem, max U_t, where U_t is the representative agent's expected utility at t, to characterize the efficient allocation of investments and transformation. We then verify that the solution defines equilibrium, which is effectively an application of the second welfare theorem in our environment.

It is convenient study the finite horizon economy, $T < \infty$, and then take the limit as $T \to \infty$, since existence and uniqueness is straightforward for the finite horizon economy. By proceeding this way, we avoid potential issues with bubble solutions and transversality conditions. The planner's problem, taking into account that $F_t = I_t$ for all t, is

$$\hat{V}(t, K, s) = \max_{\{w_t\}_t, \{i_t\}_t} E\left[\int_0^T e^{-\delta t}\Big(\ln(K_t) + \ln(\kappa - i_t)\Big)dt\right], \tag{56}$$

where capital and the capital shares evolves according to (13, 15), and sector-capital according to (7). Note that the consumption flow $\ln(K_t(\kappa - i_t))$ is taken net of effort, corresponding to the $i_t K_t$ term, which represents the agent's disutility of effort under the constraint $F_t = I_t$. As is well-known, because of the separability of the logarithmic utility function, the solution is on the form

$$\hat{V}(t, K, s) = A(T - t)\ln(K) + V(t, s),$$

where

$$A(z) \stackrel{\text{def}}{=} \frac{1 - e^{-\delta z}}{\delta}.$$

Itô calculus gives us

$$ds^2 = s^2(1-s)^2(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho)dt \stackrel{\text{def}}{=} \sigma_s^2 dt,$$

$$\left(\frac{dK}{K}\right)^2 = (\sigma_1^2 s^2 + \sigma_2^2(1-s)^2 + 2\sigma_1\sigma_2\rho s(1-s))dt \stackrel{\text{def}}{=} \sigma_K^2 dt,$$

and

$$\left\langle ds\ , \frac{dK}{K}\right\rangle = s(1-s)(s\sigma_1^2 - (1-s)\sigma_2^2 + (1-2s)\sigma_1\sigma_2\rho)dt \stackrel{\text{def}}{=} \sigma_{sK} dt.$$

Defining μ_k and μ_s as the drift terms of $\frac{dK}{K}$ and ds, we can write

$$\begin{aligned} \frac{dK}{K} &= \mu_k dt + \sigma_k dz_K, \\ ds &= \mu_s dt + \sigma_s dz_s. \end{aligned}$$

The Bellman equation associated with V is then:

$$\begin{aligned} 0 = \max_{w,i} V_t, dt + \ln(\kappa - i)dt - \delta V dt + V' E[ds] \\ + \frac{1}{2} V''(ds)^2 + A(T-t) E\left[\frac{dK}{K}\right] - \frac{A(T-t)}{2}\left(\frac{dK}{K}\right)^2, \end{aligned}$$

with the first-order conditions (using the notation $A = A(T-t)$):

$$\begin{aligned} w(t,s) &= s + \frac{1}{2aA} V_s(t,s), \\ i(t,s) &= \frac{4a\kappa A^2 - 4aA + \kappa V_s(t,s)^2}{4aA^2 + V_s(t,s)^2}, \end{aligned}$$

which, when plugged into the PDE, leads to

$$\begin{aligned} 0 &= V_t + q_2(s) V_{ss} - q_1(s) V_s - \delta V \\ &\quad - \ln\left(A + \frac{1}{4aA}\hat{V}_s^2\right) + \frac{\kappa}{4aA} V_s^2 + A q_0(t,s) - 1, \qquad (57) \\ q_0(s) &= \kappa + s\mu_1 + (1-s)\mu_2 - \frac{1}{2}\left(s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)\rho\sigma_1\sigma_2\right) \\ &= \kappa + s\mu_1 + (1-s)\mu_2 - \frac{1}{2}\sigma_K^2 \\ q_1(s) &= s(1-s)\left(\mu_2 - \mu_1 + s\sigma_1^2 + (1-2s)\rho\sigma_1\sigma_2 - (1-s)\sigma_2^2\right) \\ &= s(1-s)(\mu_2 - \mu_1) + \sigma_{sK} \\ q_2(s) &= \frac{1}{2}\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right) s^2 (1-s)^2 \\ &= \frac{\sigma_s^2}{2}. \qquad (58) \end{aligned}$$

The terminal condition for V are

$$V(s,T) \equiv 0. \qquad (59)$$

One also verifies that the Hessian of the optimization over w and i is negative definite when the first order conditions are satisfied, with eigenvalues $\lambda_1 = -\frac{1}{(\kappa-i)^2}$, $\lambda_2 = -\frac{2a}{\delta}$. The solution of the social planner's problem is thus on the form stated in the theorem (16). The SDF defined by the representative agent's marginal utility is therefore

$$M_t = e^{-\delta t} u'(C_t) = e^{-\delta t} \frac{1}{(\kappa - i_t)K_t} = e^{-\delta t} \frac{1}{(\kappa - i_t)K_t} = M_t = \frac{e^{-\delta t}}{\delta K_t} Q(s), \qquad (60)$$

where $Q_t = \frac{\delta}{\kappa - i_t}$. This is the candidate SDF for equilibrium.

We let $T \to \infty$, to get $A \equiv \frac{1}{\delta}$, $V(s) = \lim_{t\to\infty} V(t,s)$, and

$$\hat{V}(t,K,s) \equiv \frac{\ln(K)}{\delta} + V(s).$$

This in turn leads to the time-independent optimal policies

$$w_t = w(s_t) = s_t + \frac{\delta}{2a} V'(s_t), \tag{61}$$

$$i_t = i(s_t) = \frac{4a(\kappa - \delta) + \delta^2 \kappa V'(s_t)^2}{4a + \delta^2 V'(s_t)^2}. \tag{62}$$

It follows that

$$V'(s_t) = \frac{2a}{\delta}(w_t - s_t),$$

and also that

$$Q_t = \frac{\delta}{\kappa - i_t} = 1 + a(w_t - s_t)^2.$$

We use (60) to define the following candidate sector price functions

$$P_t^n = E_t\left[\int_t^\infty \frac{M_q}{M_t}(\kappa - i_q) K_q^n dq\right],$$

and the corresponding capital market price function $P_t = P_t^1 + P_t^2$. These prices thus represent the values of the dividends net investments, paid out by the two types of capital, under the candidate SDF and the optimal investment rate and share policies. It follows that

$$P_t = E_t\left[\int_t^\infty \frac{M_q}{M_t}(\kappa - i_q) K_q dq\right] = \frac{\kappa - i_t}{\delta} K_t = \frac{1}{1 + a(w_t - s_t)^2} K_t. \tag{63}$$

So, the candidate market price-to-capital ratio is

$$p_t = \frac{1}{1 + a(w_t - s_t)^2}.$$

We next focus on the candidate sector price functions, defined net of dilution. Recall that $\hat{V}(K,s)$ solves the social planner's problem, and $K = K_1 + K_1$. We have

$$\begin{aligned}
\frac{d\hat{V}}{dK_1} &= \frac{\partial \hat{V}}{\partial K} + V' \frac{ds}{dK_1} \\
&= \frac{1}{\delta K} + V' \frac{1-s}{K}, \\
\frac{d\hat{V}}{dK_2} &= \frac{\partial \hat{V}}{\partial K} + V' \frac{ds}{dK_2} \\
&= \frac{1}{\delta K} - V' \frac{s}{K}.
\end{aligned}$$

The candidate SDF determines the price of the nth sector as the contribution to marginal utility of the representative investor from a marginal unit of n-capital, i.e.,

$$
\begin{aligned}
M_0 p^1 &= M_0 \frac{1}{M_0} E_0 \left[\int_0^\infty M_t \frac{dC_t}{dK^1} dt \right] \\
&= \frac{d\hat{V}}{dK_1} = \frac{1}{\delta K_0} + V' \frac{1 - s_0}{K_0}, \\
M_0 p^2 &= M_0 \frac{1}{M_0} E_0 \left[\int_0^\infty M_t \frac{dC_t}{dK^2} dt \right] \\
&= \frac{1}{\delta K_0} - V' \frac{s_0}{K_0}.
\end{aligned}
$$

Noting that $M_0 = \frac{1}{(\kappa - i_0)K_0} = \frac{1}{\delta K_0 p}$, and that the formula holds for arbitrary t because of time invariance, this leads to:

$$
\begin{aligned}
p_t^1 &= p_t(1 + (1 - s_t)\delta V'(s_t)), \\
p_t^2 &= p_t(1 - s_t \delta V'(s_t)).
\end{aligned}
$$

It then follows that

$$
\frac{p_t^1 - p_t^2}{p_t} = \delta V'(s_t).
$$

Note also that since

$$
\begin{aligned}
p_0^1 K_0^1 &= \frac{1}{M_0} E_0 \left[\int_0^\infty M_t (\kappa - i_t) s_t K_t dt \right] \\
&= \delta p_0 K_0 E_0 \left[\int_0^\infty e^{-\delta t} s_t dt \right],
\end{aligned}
$$

and

$$
p_0^1 K_0^1 = p_0 s_0 K_0 (1 + (1 - s_0)\delta V'(s_0)),
$$

and $\frac{s_0}{\delta} = \int_0^\infty e^{-\delta t} s_0 dt$, it follows that

$$
E_0 \left[\int_0^\infty e^{-\delta t}(s_t - s_0) dt \right] = s_0(1 - s_0)V'(s_0), \tag{64}
$$

which is a result we will use. Equivalently, from integration by parts,

$$
E_0 \left[\int_0^\infty e^{-\delta t} ds_t \right] = \delta s_0 (1 - s_0) V'(s_0). \tag{65}
$$

To derive the expressions for the dividends, d^n, we define the dividends net capital flows, $Z^n = d^n - \kappa$.

The SDF formula for the price-to-capital ratio, p^1 at time 0, is then such that

$$p^1 = \kappa dt + Z^1 dt + E\left[\frac{M_{dt}}{M_0}(p^1 + dp^1)\left(1 + \frac{d\hat{K}^1}{\hat{K}^1}\right)\right]. \tag{66}$$

Here, $\hat{K}$ follows the dynamics

$$\frac{d\hat{K}^1}{\hat{K}^1} = \mu_1 dt + \sigma_1 dz^1,$$

i.e., it is the organic growth of the firms, not taking into account the growth generated by the VC firm's capital transformation. Using Ito's lemma, this leads to the following expression for Z^1:

$$\begin{aligned}
0 &= -\left(\delta + (1-s)(\mu_2 - \mu_1) + i(1 - af^2) + \frac{\sigma_{sK}}{s} + \frac{p_s}{p}\left(\mu_s + \frac{\sigma_s^2}{s}\right)\right)p^1 \\
&+ \frac{1}{2}\left(p^1_{ss} - \frac{p^1}{p}p_{ss}\right)\sigma_s^2 \\
&+ \left(\mu_s + \frac{\sigma_s^2}{s}\right)p^1_s \\
&- \frac{p_s}{p}\left(p^1_s - \frac{p_s}{p}p^1\right)\sigma_s^2 \\
&+ \kappa + Z^1
\end{aligned}$$

We note that

$$p^1_s - \frac{p_s}{p}p^1 = 2a((1-s)f' - f)p,$$

allowing us to write

$$\begin{aligned}
0 &= -\left(\delta + (1-s)(\mu_2 - \mu_1) + i(1 - af^2) + \frac{\sigma_{sK}}{s}\right)p^1 \\
&+ \frac{1}{2}\left(p^1_{ss} - \frac{p^1}{p}p_{ss}\right)\sigma_s^2 \\
&+ 2a((1-s)f' - f)p\left(\mu_s + \frac{\sigma_s^2}{s}\right) \\
&- 2a((1-s)f' - f)p_s\sigma_s^2 \\
&+ \kappa + Z^1
\end{aligned}$$

We also have

$$p^1_{ss} - \frac{p^1}{p}p_{ss} = 4a((1-s)f' - f)p_s - 4af'p + 2a(1-s)f''p,$$

implying

$$
\begin{aligned}
0 &= -\left(\delta + (1-s)(\mu_2 - \mu_1) + i(1-af^2) + \frac{\sigma_{sK}}{s}\right)(1 + 2af(1-s))p \\
&+ a\left((1-s)f'' - 2f'\right)p\sigma_s^2 \\
&+ 2a((1-s)f' - f)p\left(\mu_s + \frac{\sigma_s^2}{s}\right) \\
&+ \kappa + Z^1
\end{aligned}
$$

We arrive at

$$
\begin{aligned}
\frac{\kappa + Z^1}{p} &= \left(\delta + (1-s)(\mu_2 - \mu_1) + i(1-af^2) + \frac{\sigma_{sK}}{s}\right)(1 + 2af(1-s)) \\
&- a\left((1-s)f'' - 2f'\right)\sigma_s^2 \\
&- 2a((1-s)f' - f)\left(\mu_s + \frac{\sigma_s^2}{s}\right).
\end{aligned}
$$

A similar argument for p^2 gives us

$$
\begin{aligned}
0 &= -\left(\delta - s(\mu_2 - \mu_1) + i(1-af^2) - \frac{\sigma_{sK}}{1-s} + \frac{p_s}{p}\left(\mu_s - \frac{\sigma_s^2}{1-s}\right)\right)p^2 \\
&+ \frac{1}{2}\left(p_{ss}^2 - \frac{p^2}{p}p_{ss}\right)\sigma_s^2 \\
&+ \left(\mu_s - \frac{\sigma_s^2}{1-s}\right)p_s^2 \\
&- \frac{p_s}{p}\left(p_s^2 - \frac{p_s}{p}p^2\right)\sigma_s^2 \\
&+ \kappa + Z^2
\end{aligned}
$$

which together with

$$
\begin{aligned}
p_s^2 - \frac{p_s}{p}p^2 &= -2a(sf' + f)p, \\
p_s^2 s - \frac{p^2}{p}p_{ss} &= -4a(sf' + f)p - 4af'p - 2asf'',
\end{aligned}
$$

leads to

$$
\begin{aligned}
\frac{\kappa + Z^2}{p} &= \left(\delta - s(\mu_2 - \mu_1) + i(1-af^2) - \frac{\sigma_{sK}}{1-s}\right)(1 - 2afs) \\
&- a\left(sf'' + 2f'\right)\sigma_s^2 \\
&- 2a(sf' + f)\left(\mu_s - \frac{\sigma_s^2}{1-s}\right).
\end{aligned}
$$

Note that

$$sZ^1 + (1-s)Z^2 \equiv 0, \tag{67}$$

so aggregate dividends net consumption flow is at all point in time equal to zero. The sector-level deviation of dividends from the sector-level comsumption flow accoutns for the externality associated with having unblananced sectors. The too small sector (compared with the balanced outcome, s^*) is subsidized by the too large sector, so that the present value of discounted n-dividends is equal to the marginal value of n-capital, p^n.

It follows that

$$\begin{aligned} Z^1 &= (1-s)(Z^1 - Z^2), \\ Z^2 &= -s(Z^1 - Z^2), \end{aligned}$$

so Z^1 and Z^2 immediately follow from the difference $Z^1 - Z^2$, which can be written as

$$\begin{aligned} -\frac{Z^1 - Z^2}{p} &= (\mu_1 - \mu_2)(1 + 2a(1-2s)f) - \frac{\sigma_{sK}}{s(1-s)}(1 + 2a(1-2s)f) \\ &- 2af(\delta + i(1 - af^2)) + af''\sigma_s^2 + 2a\left(f'\mu_s - f\frac{\sigma_s^2}{s(1-s)} + (1-2s)f'\frac{\sigma_s^2}{s(1-s)}\right). \end{aligned} \tag{68}$$

We can now derive the following result

Proposition A.1

$$Z^1 - Z^2 = 2aif(1 - af^2)p \tag{69}$$

Proof: The result follows from the following lemma:

Lemma 1 *The following condition holds:*

$$\sigma_s^2 f'' + 2(q_2' - q_1)f' - 2q_1' f - 2\delta f - \frac{2\delta}{1 + af^2} f' f + 2\kappa f' f + \frac{q_0'}{a} = 0. \tag{70}$$

Proof: Differentiate the ODE for $V(s) = W(\infty, s)$, defined by (16), and use the fact that $V' = \frac{2a}{\delta} f$. ∎

By differentiation, one now derives the following formulas:

$$\begin{aligned} q_0'(s) &= \mu_1 - \mu_2 - \frac{\sigma_{sK}}{s(1-s)} \\ q_1'(s) &= (1-2s)\left(\mu_2 - \mu_1 + \sigma_{sK} s(1-s)\right) + \frac{\sigma_s^2}{s(1-s)} \\ q_2'(s) &= \frac{1-2s}{s(1-s)}\sigma_s^2, \end{aligned}$$

which together with $\mu_s = i(s)f(s) - q_1(s)$ can be plugged into (70) to get

$$\begin{aligned}
-af''\sigma_s^2 &= +2a\left(\frac{1-2s}{s(1-s)}\sigma_s^2 + \mu_s - if\right) f' \\
&- 2af\left((1-2s)\left(\mu_2 - \mu_1 + \sigma_{sK} s(1-s)\right) + \frac{\sigma_s^2}{s(1-s)} + \delta\right) \\
&- \frac{2\delta}{1+af^2} f' f + 2\kappa f' f + \mu_1 - \mu_2 - \frac{\sigma_{sK}}{s(1-s)}.
\end{aligned}$$

Replacing $af''\sigma_s^2$ in (70) with (minus) the RHS above yields

$$-\frac{Z^1 - Z^2}{p} = 2af\left(-i(1-af^2) + \left(i - \kappa + \frac{\delta}{1+af^2}\right) ff'\right).$$

Finally, using that $p = \frac{1}{1+af^2}$, and $\kappa - i = \delta p$, we arrive at

$$\begin{aligned}
Z^1 - Z^2 &= 2aif\left(1 - af^2\right) p, \\
Z^1 &= 2aif(1-s)\left(1 - af^2\right) p, \\
Z^2 &= -2aifs\left(1 - af^2\right) p.
\end{aligned}$$

Thus, Proposition A.1 holds. Note that we can write

$$\begin{aligned}
Z^1 &= i(1-af^2)(p^1 - p), \\
Z^2 &= i(1-af^2)(p^2 - p).
\end{aligned}$$

We next turn to the behavior of the VC firm under the candidate SDF and pricing functions. From (8,9), it follows that the VC firm's objective at time t is to maximize $v_t^{VC} dt$, where

$$\begin{aligned}
v_t^{VC} &= \left(p_t^1\left[w_t - as_t(w_t - s_t)^2\right] + p_t^2\left[1 - w_t - a(1-s_t)(w_t - s_t)^2\right] - 1\right) I_t, \\
&= \left(p_t^1 w_t + (1-w_t)p_t^2 - ap_t(w_t - s_t)^2 - 1\right) I_t \\
&= \left((p_t^1 - p_t^2)(w_t - s_t) + p_t(1 - a(w_t - s_t)^2) - 1\right) I_t.
\end{aligned}$$

Here, we have used that $p_t = s_t p_t^1 + (1-s_t)p_t^2$. Given $I_t > 0$, the VC-firm optimal choice of w_t satisfies

$$(w_t - s_t) = \frac{p_t^2 - p_t^1}{2ap_t} = \frac{\delta}{2a} V'(s_t),$$

consistent with (61). The instantaneous contribution to value is then

$$v_t^{VC} = (p_t(1 + a(w_t - s_t)^2) - 1) I_t = 0,$$

regardless of $I_t \geq 0$, so the VC-firm is willing to choose $I_t = i_t K_t$ at all points in time, and since it never generates any surplus, it must choose $\zeta_t \equiv 0$, always using 100% of the newly created firms to compensate the household for its effort. So the VC firms value is zero

$$V_0^{VC} = E_0 \left[\int_0^\infty \frac{M_t}{M_0} v_t^{VC} dt \right] = 0.$$

One still need to show that the representative agent's asset demand and consumption are consistent with the candidate equilibrium SDF. First, because of the log-utility specification, the household with time-t wealth $W_t = P_t = p_t K_t = \frac{\kappa - i_t}{\delta} K_t$ will choose net consumption $C_t = \delta W_t = (\kappa - i_t) K_t = Y_t^1 + Y_t^2 - I_t$, so $\hat{C}_t = Y_t^1 + Y_t^2$, i.e., the consumption market clears under the candidate SDF.

Finally, the fact that the representative agent's asset demand leads to market clearance in the capital market follows from standard arguments (see, e.g., Lucas (1978)), since the candidate SDF defined by (60) is the representative agent's marginal utility of aggregate consumption at all points in time in all states.

∎

Proof of Propositions 1 and 2:

Proposition 1: When $a \to 0$, the social planner's problem converges to one in a friction-free economy, when capital can be costlessly transformed, so that the planner can effectively choose s_t at any time t, see (7). This, in turn reduces the planner's problem to a portfolio choice problem equivalent to that in Merton (1969), with portfolio weights $\frac{K^1}{K^1+K^2}$ and $\frac{K^2}{K^1+K^2}$, respectively, with the solution given in the proposition.

Proposition 2: When $a \to \infty$, it becomes prohibitively expensive for the planner to choose any other policy that $w_t = s_t$ for all t, so the only policy variable available for the planner is the choice of $\{I_t\}_t$, leading to capital dynamics

$$\begin{aligned} dK_t^1 &= \mu_1 K_t^1 dt + \sigma_1 K_t^1 dz^1 + I_t s_t dt, \\ dK_t^2 &= \mu_2 K_t^2 dt + \sigma_2 K_t^2 dz^2 + I_t (1 - s_t) dt, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \frac{dK_t^1}{K_t^1} &= (\mu_1 + i_t) dt + \sigma_1 dz^1, \\ \frac{dK_t^2}{K_t^2} &= (\mu_2 + i_t) dt + \sigma_2 dz^2. \end{aligned}$$

and instantaneous net dividends $\ln((\kappa - i_t) K_t) dt$.

With $w_t = s_t$, investments no longer play a role in moving the capital share. Indeed, one verifies the following dynamics:

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(s_t \mu_1 + (1 - s_t) \mu_2 + i_t\right) dt + \sigma_1 s_t dz^1 + \sigma_2 (1 - s_t) dz^2, \\ ds_t &= -q_1(s_t) dt + s_t (1 - s_t) \left(\sigma_1 dz^1 - \sigma_2 dz^2\right). \end{aligned}$$

Because of the utility specification, the Bellman equation is again on the form

$$\hat{V}(t, K, s) = \frac{\ln(K)}{\delta} + V(s),$$

where V satisfies

$$\begin{aligned} 0 = &\max_i \ln(\kappa - i)dt - \delta V dt + V' E[ds] \\ &+ \frac{1}{2}V''(ds)^2 + \frac{1}{\delta}E\left[\frac{dK}{K}\right] - \frac{1}{2\delta}\left(\frac{dK}{K}\right)^2, \end{aligned}$$

and since only $\ln(\kappa - i)$ and $E\left[\frac{dK}{K}\right]$ depend on i, one derives the first-order condition

$$i_t \equiv \kappa - \delta.$$

It follows immediately that this leads to a two-trees economy, with the growth rate implied by the chosen constant investment share, i.

∎

Proof of Theorem 3: Define the processes $R_t = \frac{K_{1t}}{K_{2t}} = \frac{s_t}{1-s_t} \in \mathbb{R}_{++}$ and $r_t = \log(R_t) \in \mathbb{R}$. It follows that $s_t = \frac{1}{1+\frac{1}{R_t}} = \frac{1}{1+e^{-r_t}}$.

Ito's lemma gives us

$$dr_t = \left(\mu_* + f\left(s_t\right) e^{r_t}\left(1 + e^{-r_t}\right)^2\right) dt + \sigma_* dz_*,$$

where $\mu_* = \mu_1 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2 + \frac{\sigma_*^2}{2}$ and $\sigma_*^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. Define the scale function

$$v(r) = e^{\xi(r)}, \qquad \xi(r) \stackrel{\text{def}}{=} -2\int_0^r \frac{\mu_* + f\left(\frac{1}{1+e^{-q}}\right) e^q \left(1 + e^{-q}\right)^2}{\sigma_*^2} dq. \tag{71}$$

The flow function, f, is decreasing, with $f(s^*) = 0$ for $0 < s^* < 1$, and $f(0_+) > 0$, $f(1_-) < 0$, from which it follows that $\xi(r) \sim e^{|r|}$ for large $|r|$. From the argument in Karlin and Taylor (1981), p. 221, it follows that r_t has a unique stationary distribution, as does consequently also the capital share, s_t.

∎

Proof of Proposition 3: The SDF pricing formula for the two sectors can be written as

$$E\left[\frac{dP^n}{P^n}\right] + \frac{\kappa + Z^n}{p^n}dt \quad = \quad r_f dt - E\left[\frac{dM}{M}\left(\frac{p_s^n}{p^n}ds + \frac{d\hat{K}^n}{\hat{K}^n}\right)\right]. \tag{72}$$

We have

$$
\begin{aligned}
E\left[\frac{dM}{M}\left(\frac{p_s^1}{p^1}ds+\frac{d\hat{K}^1}{\hat{K}^1}\right)\right] &= -\left(\frac{p_s^1}{p^1}\sigma_{sK}+\frac{p_s}{p}\frac{p_s^1}{p^1}\sigma_s^2\right)dt \\
&- \left(\sigma_1^2 s+\rho\sigma_1\sigma_2(1-s)+\frac{p_s}{p}s(1-s)(\sigma_1^2-\rho\sigma_1\sigma_2)\right)dt, \\
E\left[\frac{dM}{M}\left(\frac{p_s^2}{p^2}ds+\frac{d\hat{K}^2}{\hat{K}^2}\right)\right] &= -\left(\frac{p_s^2}{p^2}\sigma_{sK}+\frac{p_s}{p}\frac{p_s^2}{p^2}\sigma_s^2\right)dt \\
&- \left(\sigma_2^2(1-s)+\rho\sigma_1\sigma_2 s+\frac{p_s}{p}s(1-s)(\rho\sigma_1\sigma_2-\sigma_2^2)\right)dt.
\end{aligned}
$$

So,

$$
\begin{aligned}
E\left[\frac{dP^1}{P^1}\right]-E\left[\frac{dP^2}{P^2}\right]+\left(\frac{\kappa+Z^1}{p^1}-\frac{\kappa+Z^2}{p^2}\right)dt &= \left(\frac{p_s^1}{p^1}\sigma_{sK}+\frac{p_s}{p}\frac{p_s^1}{p^1}\sigma_s^2\right)dt \\
&+ \left(\sigma_1^2 s+\rho\sigma_1\sigma_2(1-s)+\frac{p_s}{p}s(1-s)(\sigma_1^2-\rho\sigma_1\sigma_2)\right)dt, \\
&- \left(\frac{p_s^2}{p^2}\sigma_{sK}+\frac{p_s}{p}\frac{p_s^2}{p^2}\sigma_s^2\right)dt \\
&- \left(\sigma_2^2(1-s)+\rho\sigma_1\sigma_2 s+\frac{p_s}{p}s(1-s)(\rho\sigma_1\sigma_2-\sigma_2^2)\right)dt \\
&= \frac{\sigma_{sK}}{s(1-s)}dt+\frac{p_s}{p}\sigma_s^2\frac{1}{s(1-s)}dt+\left(\frac{p_s^1}{p^1}-\frac{p_s^2}{p^2}\right)\sigma_{sK}dt \\
&+ \frac{p_s}{p}\sigma_s^2\left(\frac{p_s^1}{p^1}-\frac{p_s^2}{p^2}\right)dt,
\end{aligned}
$$

and the total expected payoff generated of buying a dollar of 1-capital and funding it with shortselling a dollar of 2-capital (the portfolio $(X^1,X^2)=(1,-1)$) is then:

$$
\begin{aligned}
r^e dt &= \left(E\left[\frac{dP^1}{P^1}\right]+\frac{\kappa+Z^1}{p^1}dt\right)-\left(E\left[\frac{dP^2}{P^2}\right]+\frac{\kappa+Z^2}{p^2}dt\right) \\
&= \frac{\sigma_{sK}}{s(1-s)}dt+\frac{p_s}{p}\frac{\sigma_s^2}{s(1-s)}dt+\left(\frac{p_s^1}{p^1}-\frac{p_s^2}{p^2}\right)\sigma_{sK}dt+\frac{p_s}{p}\left(\frac{p_s^1}{p^1}-\frac{p_s^2}{p^2}\right)\sigma_s^2 dt, \\
&= -\hat{\lambda}^s\left(\sigma_{sK}+\frac{p_s}{p}\sigma_s^2\right)dt, \qquad (73)
\end{aligned}
$$

where

$$\begin{aligned}\hat{\lambda}^s &= -\left(\frac{p_s^1}{p^1} - \frac{p_s^2}{p^2} + \frac{1}{s(1-s)}\right)\\ &= -\frac{d}{ds}\left[\ln\left(\frac{sp^1(s)}{(1-s)p^2(s)}\right)\right]\\ &= -\left(\frac{f' + 2af^2}{1+2af(1-2s)-4a^2f^2s(1-s)} + \frac{1}{s(1-s)}\right),\end{aligned}$$

The behavior of $\hat{\lambda}^s$ can be inferred from the following lemma:

Lemma 2

- *The function* $s\frac{p^1(s)}{p(s)}$ *is increasing in* s*, zero at* $s=0$*, and one at* $s=1$*.*
- *The function* $(1-s)\frac{p^2(s)}{p(s)}$ *is decreasing in* s*, one at* $s=0$*, and zero at* $s=0$*.*

The lemma immediately implies that $\hat{\lambda}^s(s) > 0$ when $0 < s < 1$.

Proof of lemma: Note first that $s\frac{p^1(s)}{p(s)} + (1-s)\frac{p^2(s)}{p(s)} \equiv 1$, so the first part of the lemma immediately implies the second. We have

$$s\frac{p^1(s)}{p(s)} = s(1+2af(1-s)),$$

which immediately implies the function's behavior at $s=0$ and $s=1$. To show that the function is increasing, we note that $f(s) = \frac{\delta}{2a}V'(s)$, see (20), so we have

$$\begin{aligned}s\frac{p^1(s)}{p(s)} &= s + \delta s(1-s)V'(s)\\ &= s + \delta E\left[\int e^{-\delta t}(s_t - s)dt\middle| s_0 = s\right]\\ &= s + \delta E\left[\int e^{-\delta t}s_t dt\middle| s_0 = s\right] - s\delta E\left[\int e^{-\delta t}dt\right]\\ &= \delta E\left[\int e^{-\delta t}s_t dt\middle| s_0 = s\right]. \end{aligned}\tag{74}$$

Here, we used the expectation result (64).

Now, since the diffusion process for s follows a time-invariant Markov structure, stochastic monotonicity implies that $E[s_t|s_0]$ is increasing in s_0. It follows that $s\frac{p^1(s)}{p(s)}$ is increasing in s, and thus the lemma holds. ∎

The functional form of $\hat{\lambda}^s$ also immediately implies that the function behaves like $\frac{1}{s(1-s)}$ close to boundaries. We study the remaining term, $\sigma_{sK} + \frac{p_s}{p}\sigma_s^2$. First, $\frac{p_s}{p} = -\frac{2aff'}{1+af^2}$, which is a decreasing function with unique root at s^*. Also,

$$\sigma_{sK} = s(1-s)\underbrace{\left(\sigma_1^2 s - \sigma_2^2(1-s) + (1-2s)\rho\sigma_1\sigma_2\right)}_{\ell(s)},$$

where it is easy to verify that $\ell(s)$ is linearly increasing in s, and zero at $\hat{s} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$.

We can write:

$$\sigma_{sK} + \frac{p_s}{p}\sigma_s^2 = s(1-s)\left(s\sigma_*^2 - \sigma_2^2 + \sigma_1\sigma_2\rho + s(1-s)\frac{p_s}{p}\sigma_*^2\right) \stackrel{\text{def}}{=} s(1-s)G(s),$$

where $\sigma_*^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 > 0$. It then follows immediately that $G(0) < 0$, $G'(0) > 0$, $G(1) > 0$, $G'(1) < 0$. Altogether, r^e is therefore negative and decreasing close to the $s = 0$ boundary, and positive and increasing close to the $s = 1$ boundary, with limit values $r^e(0+) = -(\sigma_2^2 - \sigma_1\sigma_2\rho)$, and $r^e(1-) = \sigma_1^2 - \rho\sigma_1\sigma_2$. The properties of r^e as stated in the proposition are therefore satisfied.

∎

Proof of Proposition 4:

Substituting, the expression for the firm-a alpha simplifies to

$$\alpha_1 \cdot \sigma_{\hat{m}}^2 = \left[x \cdot \left(\sigma_1^P\right)^2 + (1-x) \cdot \sigma_{1,2}^P\right] \cdot \sigma_{\hat{m}}^2 \tag{75}$$

$$- \left[\hat{x} \cdot \left(\sigma_1^P\right)^2 + (1-\hat{x}) \cdot \sigma_{1,2}^P\right] \cdot (\mu_{\hat{m}} - r_f) \tag{76}$$

$$= \left[(1-\hat{x})\, x - \hat{x}\,(1-x)\right](1-\hat{x})\left(\sigma_1^P\right)^2\left(\sigma_2^P\right)^2 \tag{77}$$

$$\left[2\hat{x}\,(1-x) - (\hat{x}\,(1-x) + (1-\hat{x})\,x)\right](1-\hat{x})\left(\sigma_{1,2}^P\right)^2 \tag{78}$$

$$= \left[x - \hat{x}\right](1-\hat{x})\left(\left(\sigma_1^P\right)^2\left(\sigma_2^P\right)^2 - \left(\sigma_{1,2}^P\right)^2\right) \tag{79}$$

$$\alpha_1 = (x - \hat{x})(1-\hat{x})\,\frac{\left(\sigma_1^P\right)^2\left(\sigma_2^P\right)^2 - \left(\sigma_{1,2}^P\right)^2}{\sigma_{\hat{m}}^2}. \tag{80}$$

Since the market portfolio must have zero alpha, $\alpha_{\hat{m}} = 0 = \hat{x}\alpha_1 + (1-\hat{x})\,\alpha_2$, we get

$$\alpha_2 = (\hat{x} - x)\,\hat{x}\,\frac{\left(\sigma_1^P\right)^2\left(\sigma_2^P\right)^2 - \left(\sigma_{1,2}^P\right)^2}{\sigma_{\hat{m}}^2}$$

∎