

Informational Efficiency and Asset Prices in Large Markets*

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Abstract

We study informational efficiency in a general equilibrium economy with asymmetric information and data generated by investors trading many derivative securities. Investors have logarithmic preferences and are informed, uninformed, or biased; biased investors hold heterogeneous noisy beliefs. Because data contain endogenously overlapping information, equilibria can be fully or partially revealing, and more data need not yield more knowledge. Consistent with evidence, derivatives and interest rates reveal valuable macroeconomic information. In partially revealing equilibria, informational efficiency is U-shaped in output volatility. At low volatility, information comes primarily from securities paying when output is near its expected level, and at high volatility it comes from those paying in extreme output states. With sufficiently many assets, trading data reveal not only private information but also the entire cross-section of noisy beliefs. By contrast, markets with CARA investors remain partially revealing: information is compressed into two statistics, regardless of the dataset's dimension.

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1. Introduction

In the age of big data, vast datasets offer new opportunities to uncover information about macroeconomic and financial fundamentals. Yet the abundance of data does not necessarily translate into greater knowledge because observations often contain overlapping information. Such overlap arises throughout economics and finance, where decisions, contracts, and market signals reflect common fundamentals and noise. Financial data are special in this regard because information and noise enter prices endogenously through investors' simultaneous trading across a large number of securities, causing common macroeconomic information and the effects of investor preferences and beliefs to appear in multiple market signals. Understanding the informational efficiency of financial markets therefore requires quantifying the distinct information contained in large, entangled sets of trading data.

Markets for state-contingent claims, including non-redundant derivatives and prediction-market contracts, provide a tractable and economically relevant laboratory for studying informational efficiency in the presence of endogenous overlap across data sources. A large menu of options generates a rich cross-section of signals, but options with similar strikes, maturities, or underlying assets may reveal closely related information. How these opposing effects determine the total informativeness of such markets remains relatively unexplored.

In this paper, we show theoretically that the amount of distinct information in market data is determined endogenously in equilibrium by macroeconomic conditions, investor preferences, beliefs, behavioral biases, and the number of traded securities. Depending on these features, trading data may fully or partially reveal investors' private information.

Using derivatives markets as an application, we show that trading data from these markets reveal useful information about fundamentals, consistent with the evidence (e.g., [Easley, O'Hara, and Srinivas, 1998](#); [Pan and Poteshman, 2006](#); [Chakravarty, Gulen, and Mayhew, 2004](#); [Kacperczyk and Pagnotta, 2019](#)), but both the amount of information and the contracts that contain the most useful information vary sharply with macroeconomic conditions. The measure of informational efficiency has a U shape in output volatility. When output volatility is low, information comes primarily from securities that pay when output is close to its expected level, whereas when volatility is high, it comes primarily from securities that pay in extreme output events. Informational efficiency also varies non-monotonically with belief dispersion, investor overconfidence, and the number of noise sources. We further show that interest rates contain information about future output (e.g., [Harvey, 1988](#); [Ang, Piazzesi, and Wei, 2006](#)), but vary non-monotonically with output volatility and change sharply near the volatility thresholds separating fully and partially revealing equilibria.

We adopt logarithmic preferences for investors, departing from the constant absolute risk aversion (CARA) preferences used in workhorse models of asymmetric information.

This departure is essential for addressing the evidence described above because, as discussed below, the CARA benchmark often implies that derivatives reveal no information beyond that contained in the prices of the underlying assets, despite the intuition that trades in derivatives reflect investors’ expectations. The resulting model provides a new, parsimonious, and tractable setting with closed-form solutions for analyzing asymmetric information with endogenously overlapping market signals. We achieve tractability by exploiting the limit with many data sources and developing new analytical tools based on operator theory.

The paper also establishes a close connection between the literatures on asymmetric information and heterogeneous beliefs. The latter literature typically assumes that investor beliefs are common knowledge. We show that, when the number of traded assets in the economy is sufficiently large, trading data not only reveal the private information of informed investors but also permit identification of the entire cross-section of exogenous noisy beliefs held by unsophisticated investors, thereby endogenizing the common-knowledge assumption. However, when full revelation fails, noisy beliefs confound private signals in the data.

Our analysis is set in a general equilibrium economy with two dates and multiple states of the world. A representative firm produces exogenous output at the current and final dates, and the probabilities of states depend on a random productivity shock. Informed investors know this shock, while uninformed investors try to infer it from market data. Noise in market signals arises endogenously from multiple groups of biased investors who hold uninformative beliefs about the shock, which are i.i.d. across these groups. The investors trade all possible state-contingent Arrow–Debreu securities and maximize expected logarithmic utility across the two dates. Their portfolio weights are proportional to their subjective state probabilities. Consequently, information in trading data is summarized by sufficient statistics given by weighted averages of informed and biased investors’ probabilities for each state.

When productivity shocks are binary, the squared signal-to-noise ratio of the most informative combination of sufficient statistics endogenously emerges as a measure of informational efficiency. Interestingly, this measure is closely related to the effective sample size of signals, a concept widely used in statistics and machine learning to adjust the number of observations for common information (e.g., [Berger, Bayarri, and Pericchi, 2014](#); [Murphy, 2023](#)). This connection provides intuition on how informational overlap reduces market efficiency.

We first consider an economy where the number of traded securities is finite and greatly exceeds the number of biased investor types, and the state probabilities belong to the same exponential family, which includes many distributions used in the asymmetric information literature. In this setting, trading generates enough signals to identify both the true shock and the entire cross-section of biased beliefs. This result also holds under constant relative risk aversion (CRRA) preferences.

The latter result stands in stark contrast to models with CARA investors (e.g., [Bren-](#)

nan and Cao, 1996; Vanden, 2008), which imply that derivatives reveal no new information beyond what is contained in the prices of the underlying assets. Moreover, when our model yields a fully revealing equilibrium, an otherwise identical model with CARA investors produces a partially revealing equilibrium in which all market information is compressed into two scalar statistics, irrespective of a dataset’s dimension. This contrast suggests that CARA-based models may underestimate the information in markets where investors may have CRRA preferences, as commonly assumed in macroeconomics and macro-finance.

Next, we consider a large population economy in which the number of biased investor types is large relative to the number of traded assets. When the number of assets is finite, the productivity shock and biased beliefs can no longer be separately identified from market data. A large population of investors with idiosyncratic noisy beliefs captures the diversity of opinions in financial markets (Atmaz and Basak, 2018; Giglio, Maggiori, Stroebel, and Utkus, 2021). Following the evidence, we also assume that informed investors are scarce (Tetlock, 2005). We model this scarcity by assuming that their mass is inversely proportional to the square root of the number of biased investor types and, for tractability, study the limit as this number becomes large. Although idiosyncratic noise then diversifies away, the informed signal weakens at the same rate, so the equilibrium, in general, remains partially revealing. The resulting economy is tractable and admits closed-form solutions.

In the limit of a large market with many traded securities, and hence many information sources, the equilibrium may be either fully or partially revealing depending on the extent of information overlap. Partial revelation occurs when biased investors are overconfident about their beliefs, in the sense that their perceived output volatility is lower than the true volatility, and when the true output volatility is lower than the breadth of biased investors’ trades. This breadth combines the breadth of an individual investor’s trading with average disagreement across investors.

Intuitively, when the breadth of noise trading is small, biased investors trade a narrow range of securities. Consequently, noise cannot conceal informed trading, especially in securities that correspond to tail events and are held in small quantities by biased investors, giving rise to a fully revealing equilibrium. Thus, the model predicts that lower dispersion of biased beliefs increases informational efficiency. Surprisingly, very high dispersion of beliefs also improves informational efficiency. This is because the noise from biased investors spreads so thinly across the state space that the informed signal becomes easier to detect. Therefore, the dependence of informational efficiency on belief dispersion is non-monotone.

The equilibrium is also fully revealing when the true output volatility is smaller than that of the biased investors (i.e., biased investors are not overconfident). In this case, informed trading is localized in securities that pay in more likely states around expected output and stands out against the background noise. Therefore, lower overconfidence improves market

efficiency. Yet lower overconfidence also broadens the dispersion of biased trades, increasing noise. Consequently, overconfidence has ambiguous effects on informational efficiency.

We also derive interest rates and show that they predict next-period output. After observing trading data generated by a bad productivity shock, uninformed investors assign a higher probability to low future output. This raises the price of the risk-free bond and lowers the interest rate. Finally, we quantify the informational rent of informed investors as the fraction of initial wealth that an otherwise identical uninformed investor would give up to become informed. This rent vanishes under full revelation and reaches about 3% in our quantitative illustration when the equilibrium is partially revealing.

Related Literature. This paper is related to a large literature on noisy rational expectations equilibrium (REE) models pioneered by [Grossman and Stiglitz \(1980\)](#), [Hellwig \(1980\)](#), [Admati \(1985\)](#) and others. These studies introduced models with CARA preferences and normally distributed asset payoffs, which remain prevalent in the literature on asymmetric information due to their tractability (e.g., [Vives, 2008](#); [Dávila and Parlato, 2021](#); [Glebkina, Gondhi, and Kuong, 2021](#); [Kurlat and Veldkamp, 2015](#)). [Banerjee \(2011\)](#) studies a CARA-normal model that nests heterogeneous beliefs and rational expectations. [Breugem, Buss, and Peress \(2020\)](#) study a CARA-normal model with endogenous interest rates.

The work closest to ours is [Malamud \(2015\)](#), who studies a noisy REE with Arrow–Debreu securities in a continuous-space, complete-market economy with exogenous noise and interest rates, and shows that equilibria with non-CARA preferences are fully revealing. Our paper differs both in market structure and in economic implications. Specifically, we study an economy with endogenous noise and interest rates, allow for finitely many states, and show how the analysis can be extended to incomplete markets. More importantly, learning from a continuum of trading data is neither necessary nor sufficient for full revelation in our model. Moreover, the economic mechanisms leading to fully revealing equilibria in our model differ from the separation of smooth information from rough noise in [Malamud \(2015\)](#). Consequently, the economic channels in the two papers are complementary.

The paper contributes to the limited literature on REE models with asymmetric information and non-CARA preferences, as well as to the literature on the informational role of interest rates. [Albagli, Hellwig, and Tsyvinski \(2024\)](#) study an REE model with risk-neutral investors and position limits. [Peress \(2004\)](#) analyzes wealth effects in a noisy REE model with non-CARA preferences, using small-risk approximations. [Avdis and Glebkina \(2023\)](#) examine wealth distribution in a model with heterogeneously informed investors and non-CARA preferences. [Azarmsa \(2026\)](#) studies wealth inequality in a complete-market economy with logarithmic, heterogeneously informed investors and solves an example with two states of the world and three investor types. [Sackin \(2026\)](#) studies a production economy with behavioral logarithmic investors in which interest rates serve as noisy signals about produc-

tivity, with noise introduced through a financial multiplier. [Allen \(1981\)](#) finds conditions under which fully revealing equilibria are generic, but does not classify specific economies as fully or partially revealing, and does not quantify informational efficiency in the latter case.

The paper is also related to the literature that relaxes the assumption of normally distributed asset payoffs while maintaining CARA preferences for investors. [Breon-Drish \(2015\)](#) studies an economy with CARA investors and asset returns drawn from distributions belonging to the exponential family. [Chabakauri, Yuan, and Zachariadis \(2022\)](#) characterize derivative redundancy in a CARA–multinomial economy with exogenous noise; our model instead endogenizes noise through unobserved heterogeneous beliefs and shows that trading data from sufficiently many derivatives identify both the private shock and the entire cross-section of those beliefs. [Pálvölgyi and Venter \(2015\)](#) prove equilibrium uniqueness in multi-asset settings without normality.

[Han \(2018\)](#) and [Detemple, Rindisbacher, and Robertson \(2020, 2022\)](#) study dynamic models in which asset payoffs are not normally distributed. [Brennan and Cao \(1996\)](#), [Vanden \(2008\)](#), [Smith \(2019\)](#), and [Huang, Yueshen, and Zhang \(2024\)](#), among others, find that derivatives disclose no information beyond that contained in the prices of underlying assets. In contrast, our investors are logarithmic and derivatives contribute to market efficiency.

The paper is also related to the literature studying the effects of heterogeneous investor beliefs on asset prices (e.g., [Atmaz and Basak, 2018](#); [Basak, 2005](#); [Brunnermeier, Simsek, and Xiong, 2014](#); [Chabakauri, 2015](#); [Chabakauri and Han, 2020](#); [Martin and Papadimitriou, 2022](#)), as well as to work on recovering the wealth-weighted distribution of beliefs from time-series variation in asset prices (e.g., [Hugonnier and Nik Nejad, 2025](#)). Our contribution to this literature is twofold. First, we relax the standard assumptions that beliefs are common knowledge and that investors “agree to disagree.” Second, we derive conditions under which the unweighted beliefs can be recovered from the cross-section of trading data.

The paper also contributes to the emerging literature on the role of big data in economics and finance (e.g., [Baley and Veldkamp, 2025](#); [Begenau, Farboodi, and Veldkamp, 2018](#); [Farboodi and Veldkamp, 2020](#)) by exploring the implications of big data for informational efficiency. Our findings introduce a novel insight: the amount of information revealed through big data is influenced by the extent of information overlap across data sources.

2. Model

2.1. Securities Markets and Information Structure

We consider an economy with two dates, $t = 0$ and $t = 1$, and M states $\omega_1, \dots, \omega_M$ at the terminal date, where $M \geq 2$. The economy is populated by competitive informed and un-

informed investors, labeled I and U , respectively, and N different groups of unsophisticated biased investors who act as noise traders. The masses of informed and uninformed investors in the economy are μ_I and μ_U , respectively, and the mass of each group of biased investors is μ_B/N . The total mass of all investors is normalized to one, $\mu_I + \mu_U + \mu_B = 1$.

There is one representative firm in the economy, interpreted as a [Lucas \(1978\)](#) tree, that produces D_0 units of consumption good at date $t = 0$ and $D_m > 0$ units in state ω_m at the final date, per capita. The investors can trade M Arrow–Debreu securities with payoffs given by indicator functions $1_{\{\omega=\omega_m\}}$, equal to one in state ω_m and zero otherwise. The state- m security is in net supply of D_m units (per capita) for all states ω_m . As a result, the financial market is complete. In practice, these securities can be synthesized as portfolios of non-redundant stock options that cannot be replicated by trading only bonds and stocks, e.g., due to jumps in stock prices or an inability to trade continuously. Conversely, derivatives and claims on the Lucas tree can be assembled as portfolios of Arrow–Debreu securities.

The probabilities of states $\pi_{I,m}(\varepsilon) > 0$ depend on a macroeconomic *productivity shock* ε drawn by Nature before date-0 from a distribution with the probability density function (PDF) $\varphi_\varepsilon(x)$. This parameter may represent a realization of an economic shock that affects the probabilities of states so that when ε is higher, larger output is more likely. In some applications below, ε determines the expected output growth. The informed investors know shock ε , whereas uninformed investors only know the prior PDF $\varphi_\varepsilon(x)$ over shock ε . The uninformed investors update their prior PDF after observing publicly available information, as discussed below, and then trade based on their posterior probabilities $\pi_{U,m}$.

Biased investors are unsophisticated and misinformed, and may have behavioral biases such as overconfidence. They hold exogenous heterogeneous beliefs about the probabilities of states $\pi_{B,m}(\nu_n)$, where belief parameters ν_n are i.i.d. random variables across groups $n = 1, \dots, N$ of these investors, and are drawn from a continuous distribution with PDF $\varphi_\nu(x)$, which is common knowledge. The belief parameters are independent of shock ε . The probability functions $\pi_{B,1}(z), \dots, \pi_{B,M}(z)$ are linearly independent and strictly positive. The description of biased investors is consistent with the evidence in [Giglio, Maggiori, Stroebl, and Utkus \(2021\)](#) that retail investor beliefs are characterized by heterogeneous and persistent individual fixed effects, and that they disagree strongly about expected cash flows.

The beliefs of biased investors are not directly observable and therefore may introduce noise into asset prices, as in the related literature with CARA investors (e.g., [Dávila and Parlato, 2021](#); [Detemple, Rindisbacher, and Robertson, 2020](#)). Biased investors do not seek additional information and hold dogmatic beliefs, as in the literature on heterogeneous beliefs ([Atmaz and Basak, 2018](#); [Basak, 2005](#); [Brunnermeier, Simsek, and Xiong, 2014](#); [Chabakauri, 2015](#); [Chabakauri and Han, 2020](#); [Martin and Papadimitriou, 2022](#)). Our model, however, departs from this literature by relaxing the usual assumptions that investors know each other's

beliefs and mutually agree to disagree. In particular, uninformed investors do not observe the beliefs of others and recognize that informed investors possess superior information.

Standard models with CARA investors often incorporate exogenous normally distributed price-inelastic noise traders. However, the latter approach is not feasible in realistic economies where investors have infinite disutility of zero or negative consumption, as in our model. This is because such investors would not be able to absorb exogenous demand exceeding total asset supply. Another challenge with exogenous noise in multi-asset markets is that there are many degrees of freedom for determining correlations of noise across assets. This abundance of choices presents a difficulty in selecting an economically plausible structure for noise. Introducing optimizing biased investors circumvents these problems by generating price-elastic noisy demands and imposing economic discipline on their cross-sectional structure.

2.2. Investors' Optimization and Definition of Equilibrium

All investors start with an equal allocation of each Arrow–Debreu security and output D_0 at $t = 0$, and hence, have the same initial wealth $W_0 > 0$. They allocate this wealth to date-0 consumption C_0 and a portfolio of Arrow–Debreu securities, which determines their final-date consumption $C_1(\omega)$. By $\theta_{i,m}$ we denote the fraction of wealth W_0 that type- i investor allocates to security m , and by p_m the date-0 price of state- m Arrow–Debreu security.

The investors have logarithmic preferences, which are widely used in the literature to obtain closed-form characterizations of equilibrium (e.g., [Basak and Cuoco, 1998](#); [Kyle and Xiong, 2001](#); [Martin and Papadimitriou, 2022](#)). These preferences deliver a parsimonious setting with empirically plausible informational roles for derivatives and interest rates. Each investor of type i solves the following optimization problem:

$$\max_{C_{i,0}, C_{i,1}(\omega), \theta_{i,1}, \dots, \theta_{i,M}} \ln(C_{i,0}) + \beta \mathbb{E}^i[\ln(C_{i,1}(\omega))], \quad (1)$$

where $\beta > 0$ and $\mathbb{E}^i[\cdot]$ is the expectation under the beliefs of type- i investor, subject to the self-financing budget constraints

$$C_{i,0} = (1 - \theta_{i,1} - \dots - \theta_{i,M})W_0, \quad (2)$$

$$C_{i,1}(\omega) = \frac{\theta_{i,1}W_0}{p_1}1_{\{\omega=\omega_1\}} + \dots + \frac{\theta_{i,M}W_0}{p_M}1_{\{\omega=\omega_M\}}, \quad (3)$$

where $1_{\{\omega=\omega_m\}}$ is the payoff of state- m security, and $\theta_{i,m}W_0/p_m$ is the number of state- m securities in the investor's portfolio, $i = I, U$ for informed and uninformed investors, respectively, and $i = 1, \dots, N$ for biased investors.

The posterior probabilities of uninformed investors conditional on their *information set* $\mathcal{F}_U$ are given by $\pi_{U,m} = \mathbb{E}[\pi_{I,m}(\varepsilon)|\mathcal{F}_U]$. This information set consists of asset prices and the

combined demands of informed and biased investors, as in the related literature (Banerjee and Green, 2015; Detemple, Rindisbacher, and Robertson, 2020; Kreps, 1977; Malamud, 2015). These combined demands provide a tractable proxy for trading information beyond prices, such as limit-order-book information, open interest, trading volumes, and order imbalances. Call auctions used by stock exchanges to determine closing prices provide a concrete example of a price-determination mechanism in which investors can condition their orders on publicly disclosed aggregate order imbalances. In such auctions, orders designated for execution at the close are pooled and executed at a common price. Before the auction clears, exchanges, such as the NYSE and Nasdaq, disseminate preliminary aggregate order imbalances, facilitating the provision of offsetting liquidity.¹ Investors supplying such liquidity can condition their orders on the disclosed imbalances. In our model, the latter investors correspond to uninformed investors, while the pre-clearance imbalance in security m corresponds to the combined demands of informed and biased investors net of their pre-trade holdings $(\mu_I + \mu_B)D_m$, thereby allowing conditioning of trades on those combined demands.

Because information is incorporated into prices through trades, trading data may contain incremental information in addition to prices, consistent with the evidence (e.g., Beber, Brandt, and Kavajecz, 2011; Blume, Easley, and O'Hara, 1994; Chabakauri, Fos, and Jiang, 2026). Such incremental information arises when prices are non-invertible functions of the underlying information. Consequently, restricting the information set $\mathcal{F}_U$ to prices alone can be overly restrictive. We discuss some special cases with invertible prices in Section 3 below.

Definition of Equilibrium. *A competitive noisy general rational expectations equilibrium is a vector of risky asset prices $P(\varepsilon, \nu)$, posterior probabilities $\pi_{U,m} = \mathbb{E}[\pi_{I,m}(\varepsilon)|\mathcal{F}_U]$ of the uninformed investors, and optimal consumptions $C_{0,i}^*$ and $C_{1,i}^*(\omega)$ and portfolios θ_i^* that solve optimization problem (1) subject to self-financing budget constraints (2) and (3), taking asset prices as given, and satisfy the following market clearing conditions for all $m = 1, \dots, M$:*

$$\mu_I C_{0,I}^* + \mu_U C_{0,U}^* + \frac{\mu_B}{N} (C_{0,1}^* + C_{0,2}^* + \dots + C_{0,N}^*) = D_0, \quad (4)$$

$$\mu_I C_{1,I}^*(\omega_m) + \mu_U C_{1,U}^*(\omega_m) + \frac{\mu_B}{N} (C_{1,1}^*(\omega_m) + C_{1,2}^*(\omega_m) + \dots + C_{1,N}^*(\omega_m)) = D_m, \quad (5)$$

$$\mu_I \frac{\theta_{I,m}^* W_0}{p_m} + \mu_U \frac{\theta_{U,m}^* W_0}{p_m} + \frac{\mu_B}{N} \left(\frac{\theta_{1,m}^* W_0}{p_m} + \dots + \frac{\theta_{N,m}^* W_0}{p_m} \right) = D_m. \quad (6)$$

¹See New York Stock Exchange, “NYSE Opening and Closing Auctions” (2024), and Nasdaq, “The Nasdaq Opening and Closing Crosses: Frequently Asked Questions” (2025).

3. Characterization of equilibrium

We start with a general characterization of investors' optimal consumptions, portfolios, and asset prices. We observe that, given prices and investors' beliefs, the financial market is effectively complete. By completeness and absence of arbitrage, the prices of Arrow–Debreu securities are given by $p_m = \pi_m^{\text{RN}}/(1 + r_f)$, where r_f is the risk-free interest rate and π_m^{RN} are the risk-neutral probabilities. The investors' first-order conditions (FOCs) then equate their marginal utilities and state price densities $p_m/\pi_{i,m}$, where $\pi_{i,m}$ are investor i 's state probabilities. We solve for the optimal consumptions from the FOCs, and then find optimal portfolios θ_i^* from the budget constraint (3). Proposition 1 summarizes the results.

Proposition 1 [Optimal consumptions and portfolios]. *Optimal date-0 consumptions of all investors are given by $C_{0,i}^* = W_0/(1+\beta)$. Optimal date-1 consumptions of the informed, uninformed, and type- n biased investors in state ω_m are given by*

$$C_{1,I}^*(\omega_m) = \ell W_0 \frac{\pi_{I,m}(\varepsilon)}{p_m}, \quad C_{1,U}^*(\omega_m) = \ell W_0 \frac{\pi_{U,m}}{p_m}, \quad C_{1,n}^*(\omega_m) = \ell W_0 \frac{\pi_{B,m}(\nu_n)}{p_m}, \quad (7)$$

respectively, and their optimal fractions of wealth invested in asset m are given by

$$\theta_{I,m}^* = \ell \pi_{I,m}(\varepsilon), \quad \theta_{U,m}^* = \ell \pi_{U,m}, \quad \theta_{n,m}^* = \ell \pi_{B,m}(\nu_n), \quad (8)$$

where $\ell = \beta/(1 + \beta)$, and $n = 1, \dots, N$ are indices for the biased-investor types.

Proposition 1 demonstrates that portfolios (8) are proportional to the investors' subjective probabilities, formalizing the standard textbook insight (e.g., Hull, 2022) that investors' derivatives trading strategies reflect their beliefs about the likelihoods of future states of the world. Conveniently, the portfolio weights are independent of prices. This independence generates sufficient statistics, employed by uninformed investors to update their beliefs, that are separable in shock ε and noisy beliefs ν , as elaborated below. Such separability is key to the tractability of our model. Portfolios (8) also imply that the asset demands of investors I and B , given by $\theta_{I,m}^* W_0/p_m$ and $\theta_{n,m}^* W_0/p_m$, are downward sloping functions of prices p_m .

Next, we derive the *sufficient statistics* that summarize the information of the uninformed investors. Portfolios (8) imply that the combined asset demand of informed and biased investors for security m is given by $\ell W_0 (\mu_I \pi_{I,m}(\varepsilon) + \mu_B (\pi_{B,m}(\nu_1) + \dots + \pi_{B,m}(\nu_N))/N)/p_m$. The latter demands and prices p_m reveal the following sufficient statistics:

$$s_m = \mu_I \pi_{I,m}(\varepsilon) + \mu_B \frac{\pi_{B,m}(\nu_1) + \dots + \pi_{B,m}(\nu_N)}{N}. \quad (9)$$

We note that these statistics sum up to $\mu_I + \mu_B$ since probabilities sum to one, and hence, statistic s_M is redundant. We denote the vector of the remaining statistics by $s = (s_1, \dots, s_{M-1})^\top$.

The sufficient statistics (9) represent weighted average beliefs of informed and biased investors, and are separable in the shock ε and noise. These statistics constitute a large, high-dimensional, and entangled dataset of trading signals, which uninformed investors use to update their beliefs about ε according to $\pi_{U,m} = \mathbb{E}[\pi_{I,m}(\varepsilon)|s]$. Because statistics are correlated and contain overlapping information, the total amount of information they reveal is unclear. The extent of the correlation, and hence, the amount of information revealed, depends on economic parameters that determine the probabilities in (9). Filtering information about ε is further complicated by the fact that the joint distribution of the noise components of the sufficient statistics in (9) is not available in closed form.

Next, we characterize the equilibrium. By substituting the optimal consumptions and portfolios from Proposition 1 into the market-clearing conditions, we obtain asset prices and the risk-free interest rate in terms of the posterior probabilities $\mathbb{E}[\pi_{I,m}(\varepsilon)|s]$. Proposition 2 summarizes these results and provides the foundation for the analytical characterization of equilibrium in the sections that follow.

Proposition 2 [Equilibrium]. *The prices of Arrow–Debreu securities and the risk-free rate are given by*

$$p_m = \frac{\beta \bar{\pi}_m(s)}{D_m/D_0}, \quad (10)$$

$$r_f = \frac{1/\beta}{\sum_{m=1}^M \bar{\pi}_m(s)(D_0/D_m)} - 1, \quad (11)$$

where $\bar{\pi}_m(s)$ is the average probability across investors as a function of statistics s , given by

$$\bar{\pi}_m(s) = s_m + \mu_U \mathbb{E}[\pi_{I,m}(\varepsilon)|s]. \quad (12)$$

Equations (10) and (12) show that prices are, in general, highly nonlinear functions of the sufficient statistics s through posterior beliefs $\mathbb{E}[\pi_{I,m}(\varepsilon)|s]$, which are not available in closed form. Therefore, price invertibility with respect to the statistics cannot be guaranteed, except in two special cases. The first is the fully revealing equilibrium discussed in Remark 3 in Section 4. The second is the case in which uninformed investors have zero mass, $\mu_U = 0$. In that case, prices in (10) are proportional to the sufficient statistics and directly reveal them. This second case is economically useful, because it can be interpreted as a situation in which the uninformed agents are econometricians of negligible total mass studying the market.

We note that non-invertibility of prices implies non-existence of a price-only REE.² Restoring existence requires including both prices and asset demands net of uninformed

²Indeed, market clearing implies $s_m = (D_m/D_0)p_m/\beta - \mu_U \mathbb{E}[\pi_{I,m}(\varepsilon)|P]$. In a price-only REE, the right-hand side is a function of prices, and hence, prices must reveal s almost surely. A non-invertible price map therefore cannot constitute such an equilibrium.

demands into the information set $\mathcal{F}_U$ of uninformed investors, as in the related literature (Banerjee and Green, 2015; Detemple, Rindisbacher, and Robertson, 2020; Kreps, 1977; Malamud, 2015), because such demands contain information which is not in the prices. The economic rationale for conditioning trades on these demands is discussed in Section 2.2.

Remark 1 [Extension to incomplete markets]. The complete-market assumption can be relaxed without altering the key structure of the model. The Online Appendix, Section I.1, extends the sufficient statistics in (9) and the results of Proposition 2 to an incomplete-market economy in which investors can freely trade Arrow–Debreu securities for states $1 \leq m \leq M$, but can only trade securities for states $M < m \leq \widehat{M}$ jointly as a bundled index portfolio. Such bundles may arise as insurance contracts against default and economic shocks, as a consequence of discrete option strike prices, or due to limits on contractibility or verification that preclude trading fine state-contingent claims. The sufficient statistics in this extension retain the same functional form as in (9), and hence, the informational efficiency can be analyzed using the same tools as in the complete-market economy.

Remark 2 [Data dimension M as a parameter]. In our model, the number of traded securities M equals the number of states of the economy. There are two justifications for considering M as a parameter. First, in the incomplete-market economy discussed above, the number of freely traded securities M is smaller than the number of states, and hence, may change as a result of financial market development or the emergence of new risk-sharing and prediction markets. Second, even in complete-market economies, technological innovation can generate new investment opportunities and risks, thereby expanding both the state space and the menu of traded securities. More broadly, forces such as climate change, increasing frequency of natural disasters, and geopolitical uncertainty may further expand the state space. These new states can be spanned through the issuance of derivatives or through the creation of betting markets.

4. Fully revealing markets and limits of CARA models

In this section, we outline conditions for a fully revealing equilibrium in a sufficiently large market where the number of securities greatly exceeds the number of biased investor types. By contrast, in an otherwise similar economy with CARA preferences, all market information is compressed into two scalar statistics, independently of the data dimension. This contrast justifies our focus on logarithmic preferences, which are more prevalent in macroeconomic research, whereas CARA preferences dominate the asymmetric information literature.

4.1. Conditions for fully revealing equilibria

An equilibrium is considered *fully revealing* if for any two vectors of shocks and beliefs $(\varepsilon, \nu_1, \dots, \nu_N)$ and $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ that give rise to identical sufficient statistics (9), it follows that $\varepsilon = \tilde{\varepsilon}$. Consequently, the shock can be uniquely identified from the data.

We now provide an intuitive explanation for the emergence of fully revealing equilibria. Suppose there exist two vectors the of shock and belief realizations, $(\varepsilon, \nu_1, \dots, \nu_N)$ and $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$, that result in identical sufficient statistics (9). Consequently, they satisfy the following system of $M - 1$ equations with $2N + 2$ unknown variables:

$$\mu_I \pi_{I,m}(\varepsilon) + \mu_B \frac{\pi_{B,m}(\nu_1) + \dots + \pi_{B,m}(\nu_N)}{N} = \mu_I \pi_{I,m}(\tilde{\varepsilon}) + \mu_B \frac{\pi_{B,m}(\tilde{\nu}_1) + \dots + \pi_{B,m}(\tilde{\nu}_N)}{N}, \quad (13)$$

$m = 1, \dots, M - 1$. This system is overdetermined when the number of equations is large and exceeds the number of unknown variables, and hence, generically unsolvable unless $(\varepsilon, \nu_1, \dots, \nu_N) = (\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ (up to a permutation of beliefs ν_i and $\tilde{\nu}_i$ in Equation (13)).

Proposition 3 below shows that equilibrium is fully revealing in economies where probability distributions belong to the exponential family, which includes normal, log-normal, gamma, and many other distributions. We focus on this family for three reasons. First, canonical models of asymmetric information with CARA preferences almost exclusively rely on exponential-family distributions (see the discussion in Breon-Drish, 2015). Second, the contrast between the implications of our model and a CARA-based model in Section 4.2 is sharpest for this class of distributions. Finally, exponential-family distributions arise endogenously in settings where investors update state probabilities after receiving signals about economic fundamentals (e.g., Chabakauri, Yuan, and Zachariadis, 2022).

Proposition 3 [Fully revealing equilibrium]. *Suppose the informed and biased investors have probabilities from the same exponential family, given by*

$$\pi_{I,m}(z) = \pi_{B,m}(z) = \exp\{a(x_m) + zx_m - c(z)\}, \quad (14)$$

albeit with different parameters z across investors ($z = \varepsilon$ for the informed and $z = \nu_n$ for the n -th group of biased investors), where $a(x_m)$ and $c(z)$ are functions of the state variable x_m and parameter z , respectively, the values $x_1, \dots, x_M$ are distinct, and $\mu_I > 0$ and $\mu_B > 0$. If $2N + 1 < M$ and $\mu_I \neq \mu_B n/N$ for every integer $n = 1, \dots, N$, then the equilibrium is fully revealing. Moreover, any two vectors that generate identical sufficient statistics have the same shock, and their biased-belief vectors are permutations of one another.

First, we discuss probabilities (14). In the equation for probabilities, x_m stands for distinct outcomes of a random variable such that $x(\omega_m) = x_m$. This variable may, for example, represent log-output growth, $x(\omega) = \ln(D(\omega)/D_0)$. A positive or negative shock z

then captures whether the states with high output become more or less likely, respectively. The scaling factor $c(z)$ ensures that probabilities sum up to one.

The key implication of Proposition 3 is a new condition for the identification of beliefs. In particular, under the stated dimensionality conditions, not only the shock ε but also the entire cross-section of biased investors' beliefs ν_n can be recovered from market data, up to a permutation. This result provides a microfoundation for the common assumption in the literature on heterogeneous beliefs that investors know each other's beliefs.

Next, we explain condition $\mu_I \neq \mu_B n/N$ in Proposition 3. If $\mu_I = \mu_B n/N$, then the equilibrium is not fully revealing when $\nu_1 = \dots = \nu_n$. In this case, ε and ν_n enter s_m symmetrically and therefore cannot be separately identified. For $n \geq 2$, this case, however, is non-generic because the event $\nu_1 = \nu_2 = \dots = \nu_n$ has zero probability.

Remark 3 [Extension to CRRA preferences and invertible prices]. Proposition I.4 in the Online Appendix extends Proposition 3 to economies with CRRA preferences. The conditions for a fully revealing equilibrium are essentially the same as in Proposition 3. Consequently, full revelation is generic. Proposition I.4 also shows that for economies with CRRA preferences asset prices are almost surely invertible functions of $\{\varepsilon, \nu_1, \dots, \nu_N\}$, and hence, the information contained in the combined demands of informed and biased investors is redundant. For logarithmic preferences the invertibility can be established analogously.

4.2. Comparison with CARA models

We now juxtapose our fully revealing market with logarithmic investors against markets with CARA investors and exponential-family probabilities. In contrast to Proposition 3, equilibria under CARA preferences are always partially revealing, and all information can be compressed into two scalar statistics, independently of the number of traded securities.

The economy considered here follows standard models with CARA preferences, extended to allow consumption at date 0 in addition to the final date. This extension is introduced for realism and better comparison with the previous section. Other features of the economy, such as (effectively) complete markets and heterogeneous noisy beliefs, have been used in the related literature (e.g., Brennan and Cao (1996), Chabakauri, Yuan, and Zachariadis (2022), Dávila and Parlato (2021), Detemple, Rindisbacher, and Robertson (2020)). The exponential-family probabilities $\pi_{I,m}(\varepsilon) = \exp\{a(x_m) + \varepsilon x_m - c(\varepsilon)\}$ are also standard and arise endogenously when investors receive signal ε about the state variables x_m (see, e.g., Breon-Drish, 2015; Chabakauri, Yuan, and Zachariadis, 2022).

We study an economy with the same investor groups, probability structures, and output as described in Section 2.1. However, the investors now have CARA utility with absolute

risk aversion parameter γ and solve the following intertemporal optimization:

$$\max_{C_{0,i}, C_{1,i}(\omega)} -e^{-\gamma C_{0,i}} + \beta \mathbb{E}^i \left[-e^{-\gamma C_{1,i}(\omega)} \right], \quad (15)$$

subject to self-financing budget constraints (2) and (3). Proposition A.1 in the Appendix finds optimal final-date consumptions and the optimal quantities $n_{i,m}^*$ of securities. As in Section 3, substituting consumptions (A.7) in the Appendix into date-1 consumption clearing (5), using date-0 consumption clearing (4), and simplifying, we find sufficient statistics:

$$s_m = \mu_I \ln(\pi_{I,m}(\varepsilon)) + \mu_B \frac{\ln(\pi_{B,m}(\nu_1)) + \cdots + \ln(\pi_{B,m}(\nu_N))}{N}. \quad (16)$$

Substituting exponential-family probabilities (14) into Equation (16), we find statistics:

$$s_m = x_m \left(\mu_I \varepsilon + \mu_B \frac{\nu_1 + \cdots + \nu_N}{N} \right) - \left(\mu_I c(\varepsilon) + \mu_B \frac{c(\nu_1) + \cdots + c(\nu_N)}{N} \right) + (\mu_I + \mu_B) a(x_m). \quad (17)$$

We observe that all statistics s_m linearly depend on the same two functions of shock ε and noises ν , and hence, the information in these statistics can be compressed to two statistics

$$\tilde{s}_1 = \mu_I \varepsilon + \mu_B \frac{\nu_1 + \cdots + \nu_N}{N}, \quad (18)$$

$$\tilde{s}_2 = \mu_I c(\varepsilon) + \mu_B \frac{c(\nu_1) + \cdots + c(\nu_N)}{N}. \quad (19)$$

As established formally in Proposition 4, the shock ε is almost surely not identified from the sufficient statistics (18) and (19), and hence, the equilibrium is not fully revealing. A striking property of the latter statistics is that the equilibrium is always partially revealing irrespective of the number of traded securities M . This result is in sharp contrast with fully revealing equilibria in Proposition 3. We also note that the nonlinear statistic $\tilde{s}_2$ in (19), to the best of our knowledge, is new to the literature and emerges due to date-0 consumption clearing, which standard models with CARA preferences usually do not have.

We note that the degeneracy of sufficient statistics (16) is due to the special property that both CARA utility and distributions (14) have exponential structures. Consequently, the uncertainty factor $x_m z - c(z)$ in the expression for probabilities (14), where $z = \varepsilon$ for the informed and $z = \nu_n$ for biased investors, and an investor's wealth are substitutes in the expected utility function. That is, the optimal portfolios and consumptions are the same as in an economy where investors receive wealth endowments $-(x_m z - c(z) + c(0))/\gamma$ at the final date but have shock $z = 0$ in probabilities (14). This substitutability and market completeness allow investors to construct portfolios that offset the uncertainty factor, similar to Chabakauri, Yuan, and Zachariadis (2022). The investors' holdings are then linear functions of z and the scaling parameter $c(z)$, giving rise to sufficient statistics (18)–(19).

We summarize the results on equilibrium with CARA investors in Proposition 4 below.

Proposition 4 [Equilibrium with CARA investors]. *Suppose all investors have CARA preferences, the probabilities of informed and biased investors belong to the exponential family (14), the values $x_1, \dots, x_M$ are distinct, $M, N \geq 2$, $\mu_I, \mu_B > 0$, and the distributions of ε and ν have PDFs with interval supports. Then, all information in the sufficient statistics (16) is summarized by the two scalar statistics (18)–(19), independently of the number of traded securities M , and the equilibrium is partially revealing.*

Next, we argue that CARA preferences and exponential-family probabilities are fine-tuned to deliver a partially revealing equilibrium irrespective of the dimensionality of data. Consider, for example, more general probabilities, given by $\pi_{I,m}(z) = \pi_{B,m}(z) = \exp\{a(x_m) + P_k(zx_m) - c(z)\}$, where $P_k(x)$ is a polynomial of degree k with non-zero coefficients, and $M \geq k + 1$. Substituting these probabilities into Equation (16) yields $k + 1$ sufficient statistics, one of which is given by (19) and the others are given by $\tilde{s}_l = \mu_I \varepsilon^l + \mu_B (\nu_1^l + \dots + \nu_N^l)/N$, where $l = 1, \dots, k$. As a result, deviating from either CARA preferences or exponential-family probabilities (14) can increase the volume of the revealed information.

We also observe that linearizing the probabilities (14) around $z = 0$ as $\pi_{I,m}(z) = \pi_{B,m}(z) \approx \pi_{B,m}(0)[1 + (x_m - c'(0))z]$ and substituting them into the sufficient statistics of log-utility investors (9) yields degenerate and non-revealing statistics $s_m = \pi_{B,m}(0)[\mu_I + \mu_B + (x_m - c'(0))\tilde{s}_1]$ that depend only on $\tilde{s}_1$. This analysis pins down the source of full revelation in Proposition 3 to the nonlinearity of the investors' trading strategies (8).

5. Revealing and non-revealing markets with many biased investors and big data

In this section, we investigate economies populated by a substantial number of biased investors. We derive a closed-form characterization of the financial market's informational efficiency as N approaches infinity. The limit in which the number of traded securities M also approaches infinity is particularly tractable and is called the *big data economy*. Our findings suggest that in the big data economy the market can be either fully or partially revealing depending on subtle economic conditions on model parameters.

5.1. Constructing non-revealing equilibria

A notable implication of the model is that the amount of information revealed can be non-monotone with the number of biased investor types. Holding aggregate investor masses fixed, Proposition I.5 in the Online Appendix formally shows that, for every finite $N \geq M - 1$, the equilibrium is partially revealing, although the equilibrium characterization is not available in closed form due to the complexity of statistics (9). Full revelation, however, can arise at

both extremes. When N is small relative to M , Proposition 3 yields full revelation under its conditions. In the large- N limit, the noise generated by biased investors in the statistics (9) diversifies away by the law of large numbers because beliefs ν_n are i.i.d., and hence, full revelation returns. Thus, increasing the number of biased investor types can move the market from full to partial revelation and then to full revelation again.

The large- N economy above captures the diversity of opinions in financial markets (Atmaz and Basak, 2018; Giglio, Maggiori, Stroebel, and Utkus, 2021). It does not, however, capture the scarcity of informed investors. Accurate forecasting is rare even among experts (Tetlock, 2005), and hence, the mass of informed investors is likely to be small. Full revelation may therefore fail even when diversification reduces the noise. We model this case by setting $\mu_I = \hat{\mu}_I/\sqrt{N}$, which leads to a tractable Gaussian-signal economy in the large- N limit.

Under this specification, we replace the sufficient statistics s_m with the affine informationally equivalent transformations $\hat{s}_m = \sqrt{N}(s_m/\mu_B - \mathbb{E}[\pi_{B,m}(\nu)])$, where $m = 1, \dots, M-1$, and $\mathbb{E}[\pi_{B,m}(\nu)]$ are expectations with respect to ν . After simplifying, we find that

$$\hat{s}_m = \frac{\hat{\mu}_I}{\mu_B} \pi_{I,m}(\varepsilon) + \frac{(\pi_{B,m}(\nu_1) - \mathbb{E}[\pi_{B,m}(\nu)]) + \dots + (\pi_{B,m}(\nu_N) - \mathbb{E}[\pi_{B,m}(\nu)])}{\sqrt{N}}. \quad (20)$$

Next, we apply the central limit theorem to (20). We observe that probability vectors $(\pi_{B,1}(\nu_n), \dots, \pi_{B,M-1}(\nu_n))^\top$ are i.i.d. across biased investors, due to independence of belief parameters ν_n . Consequently, the second terms of statistics (20) have a joint normal distribution in the limit $N \rightarrow \infty$. To further simplify the exposition and derive closed-form solutions, we consider a parsimonious setup in which shock ε can take only two values, normalized to $\varepsilon_L = 0$ and $\varepsilon_H = 1$. These values might correspond to situations in which mean output growth is zero ($\mathbb{E}[\ln(D_1/D_0)] = 0$) or positive ($\mathbb{E}[\ln(D_1/D_0)] > 0$), respectively. The corresponding prior probabilities of ε_L and ε_H are given by $\varphi_L > 0$ and $\varphi_H > 0$, respectively, and $\varphi_L + \varphi_H = 1$. In the case of this binary shock, the probabilities can be exactly written as $\pi_{I,m}(\varepsilon) = \pi_{I,m}(\varepsilon_L) + (\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L))\varepsilon$. Consequently, after subtracting the free term $\eta\pi_{I,m}(\varepsilon_L)$ from the statistics (20), we rewrite them as follows:

$$\hat{s}_m = f_m \varepsilon + \xi_m, \quad \xi \sim \mathcal{N}(0, \Sigma_\xi), \quad (21)$$

where $f_m = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L))$, $m = 1, \dots, M-1$, and not all f_m are simultaneously zero; $\eta = \hat{\mu}_I/\mu_B$ is the mass parameter of informed investors relative to biased investors, and $\Sigma_\xi \in \mathbb{R}^{(M-1) \times (M-1)}$ is a symmetric cross-sectional covariance matrix with elements equal to covariances $\text{cov}(\pi_{B,m}(\nu), \pi_{B,k}(\nu))$ with respect to beliefs ν . As with standard asymptotic arguments, the large- N limit is interpreted as an approximation to economies with finite but large N (e.g., $N = 1,000,000$, $\mu_I = 0.1\%$, and $\mu_B = 50\%$ imply $\hat{\mu}_I = 1$ and $\eta = \hat{\mu}_I/\mu_B = 2$).

Equations (20) and (21) highlight the economic mechanism under which biased beliefs give rise to noise in financial markets. The equilibrium with statistics (21) is partially

revealing for $\eta \in (0, +\infty)$. The intuition is that when N is large, the noise introduced by biased investors is reduced through diversification. However, because the mass of informed investors is small, discerning information about ε remains challenging. The model also nests fully revealing and fully non-revealing equilibria when $\eta = +\infty$ and $\eta = 0$, respectively. The former equilibrium emerges when μ_I does not approach zero as N grows, whereas the latter arises when $\mu_I = o(1/\sqrt{N})$, implying that μ_I converges to zero too fast.

We call the economy in which uninformed investors make inferences from statistics (21) a *Gaussian-signal economy*. Our subsequent focus is on the total amount of information revealed in this economy. Section I.4 in the Online Appendix studies conditions for the convergence of asset prices to those in the Gaussian-signal economy.

We now derive the *measure of informational efficiency*. Let $\hat{\pi}_L(\hat{s}) = \text{Prob}(\varepsilon = \varepsilon_L | \hat{s})$ be the posterior probability that uninformed investors assign to shock ε_L . Given that the signals depend on random beliefs of biased investors, this probability is inherently random. Consider the expected posterior probability conditioned on the true shock being ε_L , denoted by $\mathbb{E}[\hat{\pi}_L(\hat{s}) | \varepsilon = \varepsilon_L]$. Intuitively, the latter expected probability being close to one indicates that the uninformed investors, on average, correctly assign high probability $\hat{\pi}_L(\hat{s})$ to shock $\varepsilon = \varepsilon_L$. Consequently, we will say that the *equilibrium is close to fully revealing* when $\mathbb{E}[\hat{\pi}_L(\hat{s}) | \varepsilon = \varepsilon_L] \geq 1 - q$ and $\text{var}[\hat{\pi}_L(\hat{s}) | \varepsilon = \varepsilon_L] \leq q$ for a sufficiently small q . We note that the latter condition on variance, added for completeness, is redundant. Because $\hat{\pi}_L(\hat{s}) \in [0, 1]$, the expected value of $\hat{\pi}_L(\hat{s})$ close to one implies that its variance is close to zero.³ Hence, the expected posterior probability measures *proximity of equilibrium to fully revealing*.

Lemma 1 below derives the proximity measure $\mathbb{E}[\hat{\pi}_L(\hat{s}) | \varepsilon = \varepsilon_L]$ analytically, first for general exogenous signals about shock ε and then specifically for the endogenous sufficient statistics (21). The same lemma also defines a convenient measure of informational efficiency.

Lemma 1 [Informational efficiency].

1) For the binary shock $\varepsilon \in \{0, 1\}$, consider exogenous signals $\hat{s}_m = f_m \varepsilon + \xi_m$, where $m = 1, \dots, M - 1$, and not all f_m are zero. The noise vector $\xi = (\xi_1, \dots, \xi_{M-1})^\top$ is independent of ε and normally distributed with mean zero and covariance matrix $\Sigma_\xi \succ 0$. Among all linear signal combinations whose coefficient on ε is normalized to one, let $(w^*)^\top \hat{s}$ be the combination with the smallest noise variance. Thus, w^* solves

$$w^* = \underset{w^\top \tilde{f}=1}{\text{argmin}} w^\top \Sigma_\xi w, \quad (22)$$

where $\tilde{f} \in \mathbb{R}^{M-1}$ is a vector with $M - 1$ elements f_m . Then, $w^* = \Sigma_\xi^{-1} \tilde{f} / (\tilde{f}^\top \Sigma_\xi^{-1} \tilde{f})$, and the combination $(w^*)^\top \hat{s}$ contains all the information in $\hat{s}$ about ε . The noise precision of this

³Formally, if $\mathbb{E}[X] \geq 1 - q$ and $X \in [0, 1]$, then $\text{var}(X) \leq \mathbb{E}[X](1 - \mathbb{E}[X]) \leq 1 - \mathbb{E}[X] \leq q$.

combination is given by $J_M^* = 1/(w^*)^\top \Sigma_\varepsilon w^*$. Moreover, the proximity measure is given by

$$\mathbb{E}[\widehat{\pi}_L(\widehat{s})|\varepsilon = \varepsilon_L] = g(\sqrt{J_M^*}), \quad (23)$$

where $\widehat{\pi}_L(\widehat{s}) = \text{Prob}(\varepsilon = \varepsilon_L|\widehat{s})$, the function $g(y)$ is increasing, satisfies $g(0) = \varphi_L > 0$, $g(y) \rightarrow 1$ as $y \rightarrow \infty$, and is given by Equation (A.20) in the Appendix.

2) For statistics (21), $J_M^* = f^\top \mathcal{K}^{-1} f$, where $f \in \mathbb{R}^M$ has M elements $f_m = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L))$ and $\mathcal{K} \in \mathbb{R}^{M \times M}$ has elements $\mathbb{E}[\pi_{B,m}(\nu)\pi_{B,k}(\nu)]$, for $m, k = 1, \dots, M$. The expectation is taken with respect to the distribution of beliefs ν .

Part 1 of Lemma 1 introduces the *measure of informational efficiency* J_M^* and gives it a direct statistical interpretation for a general Gaussian signal vector of which sufficient statistics (21) are a special case. The lemma shows that J_M^* is the noise precision, or equivalently the squared signal-to-noise ratio, of the most informative linear combination of the $M - 1$ signals with the loading on ε normalized to one. This combination is sufficient for ε in the sense that observing it yields the same posterior as observing the full signal vector. Because $g(y)$ in (23) is increasing and converges to one, a larger J_M^* implies greater informational efficiency, and the equilibrium approaches full revelation of shock ε as $J_M^* \rightarrow \infty$.

We next show that the measure of informational efficiency J_M^* , which arises endogenously in our model, turns out to be closely related to the *effective sample size*, a concept widely used in statistics and machine learning to adjust the raw number of signals for the information they have in common (e.g., Berger, Bayarri, and Pericchi, 2014; Murphy, 2023). This connection is best illustrated using exogenous signals of the form $\widehat{s}_m = \varepsilon + \tilde{\xi}_m$, where the noise vector is jointly Gaussian, and every noise has variance σ^2 . This structure is convenient because every signal contains the same amount of information about ε . One signal has noise precision $1/\sigma^2$, whereas the full signal vector has noise precision J_M^* . We therefore define $\text{ESS}_M = \sigma^2 J_M^*$ as the equivalent number of independent signals, each with noise variance σ^2 , whose combined precision equals J_M^* . Lemma A.3 in the Appendix derives several examples, summarized in Table 1, showing that ESS_M behaves intuitively as the effective number of observations.

In particular, Table 1 shows that $\text{ESS}_M = M - 1$ when the noises are independent, $\text{ESS}_M = M - 1 - k$ when k signals can be exactly replicated using the remaining non-redundant $M - 1 - k$ signals with i.i.d. noises, and $\text{ESS}_M = (M - 1)/(1 + (M - 2)\rho)$ when all noises have the same pairwise correlation $\rho \in [0, 1]$. In the latter case, positive correlation reduces the effective number of observations below their raw number, and the effective sample size approaches one as $\rho \uparrow 1$. For every fixed $\rho > 0$, $\text{ESS}_M \rightarrow 1/\rho$ as $M \rightarrow \infty$. Table 1 also shows that averaging two signals with perfectly negatively correlated noises eliminates the noise. The signal vector is then fully revealing and $\text{ESS}_M = \infty$, meaning that the latter pair contains as much information as infinitely many independent signals. If the noises are independent but have different variances σ_m^2 , we use the most precise signal as

Case	Signal and noise structure	Effective sample size
1.	i.i.d. noises, each with variance σ^2	$M - 1$
2.	$M - 1 - k$ signals with i.i.d. noises of variance σ^2 ; k signals exactly replicable from them	$M - 1 - k$
3.	Common variance σ^2 and pairwise correlation $\rho \geq 0$	$\frac{M - 1}{1 + (M - 2)\rho}$
4.	Two noises with variance σ^2 and correlation -1	$+\infty$
5.	Independent noises with variances σ_m^2	$\sum_{m=1}^{M-1} \frac{\sigma_{\min}^2}{\sigma_m^2} \in [1, M - 1]$

Table 1

Effective sample size under alternative noise structures

This Table reports the effective sample size for $M - 1$ signals $\hat{s}_m = \varepsilon + \tilde{\xi}_m$ under alternative noise structures. In Cases 1–4, ESS is measured in units of one signal with noise variance σ^2 . In Case 5, it is measured in units of the most precise signal, where $\sigma_{\min}^2 = \min_m \sigma_m^2$.

the benchmark. Letting $\sigma_{\min}^2 = \min_m \sigma_m^2$, we define $\text{ESS}_M = \sigma_{\min}^2 J_M^* = \sum_{m=1}^{M-1} \sigma_{\min}^2 / \sigma_m^2$, which satisfies $1 \leq \text{ESS}_M \leq M - 1$. More generally, ESS requires a benchmark signal that defines the unit of comparison. With a unit-variance signal as the benchmark, $\text{ESS}_M = J_M^*$, so J_M^* is the equivalent number of independent unit-variance signals whose combined information matches that contained in the signal vector.

Part 2 of Lemma 1 derives the measure of informational efficiency J_M^* for the equilibrium with sufficient statistics (21). Finding this measure, given by $J_M^* = f^\top \mathcal{K}^{-1} f$, reduces to solving a system of linear equations, given by $\mathcal{K}v = f$, where $\mathcal{K}$ is a symmetric, positive-definite, and invertible matrix (by Lemma A.2 in the Appendix). Hence, the informational efficiency can be expressed as $J_M^* = f^\top v$. Finding solution v in practice, however, is challenging because matrix $\mathcal{K}$ is near-singular. Such near-singularity arises because, when M is large, tail events ω_m have small probabilities, and hence, matrix $\mathcal{K}$ has many rows and columns with near-zero elements, resulting in a near-singularity.

We tackle the challenge of solving the system $\mathcal{K}v = f$ by employing Tikhonov regularization, widely used in machine learning and empirical asset pricing. In particular, we look for $v(\tau_M)$ that solves the following problem:

$$v(\tau_M) = \underset{v}{\operatorname{argmin}} \|\mathcal{K}v - f\|_2^2 + \tau_M v^\top \mathcal{K}v, \quad (24)$$

where $\tau_M > 0$ is a small parameter. The first-order condition for (24) is given by

$$\mathcal{K}v(\tau_M) + \tau_M v(\tau_M) = f. \quad (25)$$

The system of equations (25) is significantly easier to solve than the original system with $\tau_M = 0$. We then introduce the *regularized measure of informational efficiency* $J_M(\tau_M) = f^\top v(\tau_M)$, such that $J_M(0) = J_M^*$. Lemma 2 below establishes important properties of $J_M(\tau_M)$.

Lemma 2 [Lower bound on informational efficiency]. *The regularized informational efficiency $J_M(\tau_M)$ is a continuous and decreasing function of τ_M for all $\tau_M \geq 0$. Therefore, $J_M(\tau_M)$ is a lower bound on the informational efficiency J_M^* :*

$$J_M(\tau_M) \leq J_M^*. \quad (26)$$

Lemma 2 provides an easily computable lower bound $J_M(\tau_M)$ for the measure of informational efficiency. If this bound is large for some $\tau_M > 0$, then the equilibrium is close to fully revealing by Lemma 1. As the regularization parameter τ_M shrinks to zero, the regularized measure $J_M(\tau_M)$ converges to the efficiency measure J_M^* .

5.2. Big data limit

In this subsection, we characterize the measure of informational efficiency in an economy with a large number of states/securities M . This economy captures some aspects of a big data economy such as the availability of many endogenously entangled signals (sufficient statistics). We therefore use this setting to study the tradeoff between a large number of information sources and endogenous overlap across them. For a particular specification of state probabilities, we obtain a closed-form expression for its continuous-state limit $J_\infty(\tau)$. We then take the limit $\tau \rightarrow 0$ and derive conditions for fully and partially revealing equilibria.

We define the probabilities of states as discretizations of continuous-space PDFs $\pi_I(x; \varepsilon)$ and $\pi_B(x; \nu)$, where $x \in \mathbb{R}$. The state probabilities of investors are then given by

$$\pi_{I,m}(\varepsilon) = \frac{\pi_I(x_m; \varepsilon) \Delta x}{\sum_{k=1}^M \pi_I(x_k; \varepsilon) \Delta x}, \quad \pi_{B,m}(\nu) = \frac{\pi_B(x_m; \nu) \Delta x}{\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x}, \quad (27)$$

where $x_m = -A_M + m\Delta x$ and $\Delta x = 2A_M/M$ is the *granularity of the state-space*. We assume that $A_M \rightarrow \infty$ and $\Delta x \rightarrow 0$ as $M \rightarrow \infty$, so that the probabilities (27) converge to distributions $\pi_I(x; \varepsilon)$ and $\pi_B(x; \nu)$, respectively. As the dimension M increases, investors gain access to more sources of information. However, adjacent statistics (9) become increasingly similar, thereby revealing similar information. Therefore, the net impact on informational efficiency is unclear without additional analysis.

The regularized system (25) in the discrete-space economy with $\tau_M = \tau\Delta x$, where $\tau > 0$ is fixed, has a continuous-state counterpart (see Lemma A.4 in the Appendix) given by the following *Fredholm integral equation of the second kind*:

$$\int_{-\infty}^{\infty} K(x, y) v(y; \tau) dy + \tau v(x; \tau) = f(x), \quad (28)$$

where $f(x) = \eta(\pi_I(x; \varepsilon_H) - \pi_I(x; \varepsilon_L))$ and $K(x, y) = \mathbb{E}[\pi_B(x; \nu)\pi_B(y; \nu)]$ is the expectation with respect to belief parameter ν . Our methodological contribution is to show how Equation (28) arises in the limit and how it can be solved in closed form. By contrast, its matrix analogue (25) is more difficult to solve even numerically, as discussed in Section 5.1.

We solve Equation (28) analytically for normal distributions given by

$$\pi_I(x; \varepsilon) = \frac{\exp(-0.5(x - \mu\varepsilon)^2/\sigma_I^2)}{\sqrt{2\pi\sigma_I^2}}, \quad \pi_B(x; \nu) = \frac{\exp(-0.5(x - \nu)^2/\sigma_B^2)}{\sqrt{2\pi\sigma_B^2}}, \quad (29)$$

where $\varepsilon \in \{\varepsilon_L = 0, \varepsilon_H = 1\}$, state variable x is the log-output growth $x = \ln(D_1/D_0)$, $\mu > 0$, $\sigma_B > 0$, $\sigma_I > 0$, and beliefs ν represent expected log-output growth. The shocks ε_L and ε_H have unconditional probabilities $\varphi_L > 0$ and $\varphi_H > 0$, respectively. Beliefs are also normally distributed in the cross-section of biased investors, $\nu \sim \mathcal{N}(0, \sigma_\nu^2)$, as in [Atmaz and Basak \(2018\)](#). The proposition below presents the kernel $K(x, y)$ and the solution of Equation (28).

Proposition 5 [Fredholm integral equation and informational efficiency]. *Suppose the state probabilities from the viewpoints of informed and biased investors are given by Equations (29), respectively, and beliefs ν are normally distributed in the cross-section of biased investors, $\nu \sim \mathcal{N}(0, \sigma_\nu^2)$. Then, the following statements hold.*

1) *The Fredholm integral equation (28) has a Gaussian kernel given by*

$$K(x, y) = \frac{\hat{\sigma}}{2\pi\sigma_B^2\sigma_\nu} \exp\left(-\frac{\hat{\sigma}^2}{2\sigma_B^4}(x - y)^2 - \frac{\hat{\sigma}^2}{2\sigma_B^2\sigma_\nu^2}(x^2 + y^2)\right), \quad (30)$$

where $\hat{\sigma} = (2/\sigma_B^2 + 1/\sigma_\nu^2)^{-1/2}$.

2) *The integral operator in (28) has the following set of eigenfunctions and eigenvalues:*

$$\psi_n(x) = \sqrt{a} \exp(-0.5a^2x^2) \frac{H_n(ax)}{\sqrt{2^n n! \sqrt{\pi}}}, \quad (31)$$

$$\lambda_n = \frac{1}{\sqrt{\pi\sigma_B^2\sigma_\nu} + \hat{\sigma}} \frac{\hat{\sigma}}{\left(\frac{\sigma_\nu - \hat{\sigma}}{\sigma_\nu + \hat{\sigma}}\right)^n}, \quad (32)$$

where $\lambda_n > 0$, $a = \sqrt{\hat{\sigma}/(\sigma_\nu\sigma_B^2)}$, $\hat{\sigma} = (2/\sigma_B^2 + 1/\sigma_\nu^2)^{-1/2}$, and $H_n(x)$ are Hermite polynomials. Eigenfunctions $\psi_n(x)$ form an orthonormal basis in $\mathcal{L}_2(\mathbb{R})$ for $n = 0, 1, \dots$.

3) *Equation (28) has a unique solution $v(x; \tau) \in \mathcal{L}_2(\mathbb{R})$ for $\tau > 0$. This solution and the lower bound on the measure of informational efficiency $J_\infty(\tau)$ are given by*

$$v(x; \tau) = \sum_{n=0}^{\infty} \frac{\hat{f}_n \psi_n(x)}{\tau + \lambda_n}, \quad (33)$$

$$J_\infty(\tau) = \sum_{n=0}^{\infty} \frac{\hat{f}_n^2}{\tau + \lambda_n}, \quad (34)$$

where $\hat{f}_n$ are eigenbasis coefficients of function $f(x)$ in (28) with respect to the orthonormal basis $\psi_n(x)$, given by Equation (A.26) in the Appendix. The series (33) converges with respect to a norm in $\mathcal{L}_2(\mathbb{R})$ and the series (34) converges to $J_\infty(\tau)$, for all $\tau > 0$.

Proposition 5 provides in closed form the solution of Equation (28) and the lower bound on the measure of informational efficiency (34). Next, we take the limit $\tau \rightarrow 0$ in Equation (34) and find conditions under which the limiting value is bounded or unbounded. Lemma 1 shows that unbounded (bounded) $J_\infty(0)$ implies that the equilibrium is fully (partially) revealing. Proposition 6 provides the conditions for determining the nature of equilibrium and derives the limiting value $J_\infty^* = J_\infty(0)$. Proposition I.8 in the Online Appendix proves the convergence $J_M^* \rightarrow J_\infty^*$, when $M \rightarrow \infty$.

Proposition 6 [Informational efficiency]. *Let the mass parameter satisfy $0 < \eta < \infty$. Then, the equilibrium is partially revealing if and only if*

$$\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}. \quad (35)$$

In the fully revealing equilibrium, the measure of informational efficiency is $J_\infty^ = \infty$ and in the partially revealing equilibrium, it is given by:*

$$J_\infty^* = \frac{\eta^2 a}{\sqrt{\pi(1-\rho^2)}\lambda_0(1+a^2\sigma_I^2)} \left[\exp \left\{ -\frac{\mu^2 a^2}{1+a^2\sigma_I^2} \left(1 - \frac{2\rho}{(1+\rho)(1-a^2\sigma_I^2)} \right) \right\} \right. \\ \left. - 2 \exp \left\{ -\frac{1}{2} \frac{\mu^2 a^2}{1+a^2\sigma_I^2} \left(1 + \frac{2\rho^2}{(1-\rho^2)(1-a^2\sigma_I^2)} \right) \right\} + 1 \right], \quad (36)$$

where a and ρ are given by:

$$a = \frac{1}{\sigma_B} \sqrt{\frac{\hat{\sigma}}{\sigma_\nu}}, \quad \rho = \frac{1 - a^2\sigma_I^2}{1 + a^2\sigma_I^2} \cdot \frac{\sigma_\nu + \hat{\sigma}}{\sigma_\nu - \hat{\sigma}}, \quad (37)$$

and $\hat{\sigma} = (2/\sigma_B^2 + 1/\sigma_\nu^2)^{-1/2}$.⁴ Furthermore, J_∞^* is non-monotone in belief dispersion σ_ν , and $J_\infty^* \rightarrow \infty$ as $\sigma_\nu \downarrow \sqrt{(\sigma_I^2 - \sigma_B^2)/2}$ or $\sigma_\nu \rightarrow \infty$, assuming $\sigma_I > \sigma_B$.

Proposition 6 reveals several new economic forces determining informational efficiency. These forces are reflected in the conditions (35) for partially revealing equilibria and the expression for the measure of informational efficiency (36). Figure 1 depicts the parameter regions in which the equilibrium is fully or partially revealing, as characterized by the conditions (35).

We next explain the intuition underlying these two types of equilibria. As shown in Proposition 1, the holdings of Arrow–Debreu securities by informed investors are directly

⁴The apparent singularity in Equation (36) at $a^2\sigma_I^2 = 1$ is removable. Let $q = a^2\sigma_I^2$ and $u = (\sigma_\nu - \hat{\sigma})/(\sigma_\nu + \hat{\sigma})$. Then $\rho = (1 - q)/(u(1 + q))$, and hence the two potentially singular ratios in (36) converge to $1/u$ and 0, respectively, as $q \rightarrow 1$. Equation (36) is interpreted by continuity at this point.

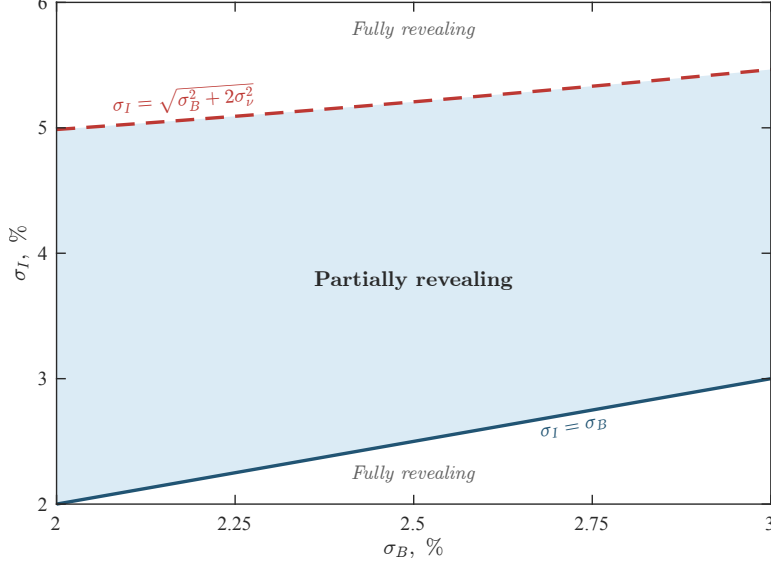


Figure 1

Set of parameters leading to partially revealing equilibria

This Figure shows the shaded region of parameter space in which inequality $\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}$ from Proposition 6 is satisfied, and hence, the equilibrium is partially revealing. The equilibrium is fully revealing outside of this region. We set $\sigma_\nu = 3.23\%$ as in [Atmaz and Basak \(2018\)](#).

proportional to the probabilities of the respective states. When the volatility parameter σ_I is relatively low, the informed investors' PDF in (29) is concentrated around its mean, $\mu\varepsilon$. As a result, the uninformed investors would notice heightened buying activity around states x near $\mu\varepsilon$ compared to other states. With sufficiently many securities traded, these comparisons reveal the signal ε fully.

In the partially revealing region, $\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}$, informed and biased investors trade over broadly the same range of securities. Sufficient statistics therefore contain overlapping information because they reflect similar mixtures of informed and biased demand. This overlap prevents uninformed investors from separating private information from noise.

To provide economic intuition for full revelation when $\sigma_I \geq \sqrt{\sigma_B^2 + 2\sigma_\nu^2}$, note that $\sigma_\nu^2 + 2\sigma_\nu^2 = \sigma_B^2 + \mathbb{E}[(\nu - \tilde{\nu})^2]$, where ν and $\tilde{\nu}$ are the beliefs of two randomly selected biased investors. We interpret $\mathbb{E}[(\nu - \tilde{\nu})^2]$ as a *measure of average disagreement*. This disagreement causes biased investors to concentrate trades in different market segments, because each of them actively trades securities corresponding to states around their belief parameter ν . Accordingly, we define $S_B \equiv \sqrt{\mathbb{E}[(\nu - \tilde{\nu})^2]}/\sigma_B$ as a *measure of market segmentation*. The threshold can then be written as $\sigma_B \sqrt{1 + S_B^2}$ and interpreted as the breadth of biased trading, combining the breadth of an individual strategy, σ_B , with the separation between investors' trading centers. When informed trading is broader than this threshold, the equilibrium is fully revealing because informed investors are active in securities paying in extreme states, where biased trading becomes negligible, allowing private information to be identified.

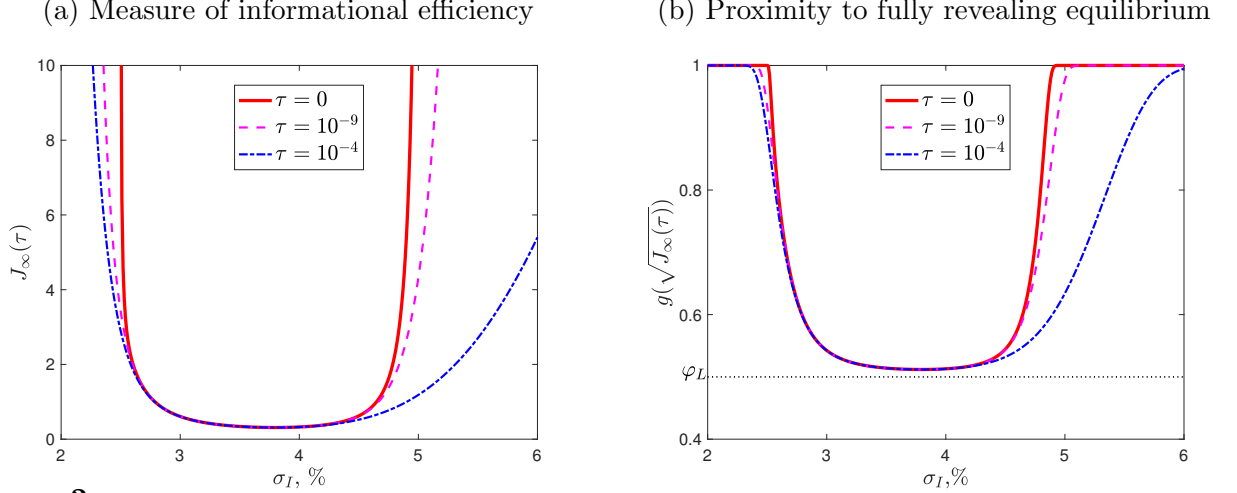


Figure 2

Informational efficiency and proximity to fully revealing equilibrium

Panel (a) shows the measure of informational efficiency $J_\infty^* = J_\infty(0)$ (solid line) and its lower bounds $J_\infty(\tau)$ for $\tau = 10^{-9}$ and $\tau = 10^{-4}$ as functions of log-output volatility σ_I in (29). The equilibrium is partially revealing when the measure J_∞^* is finite and fully revealing otherwise. Panel (b) shows how close the equilibrium is to fully revealing, with proximity measured by $\mathbb{E}[\hat{\pi}_L(\hat{s})|\varepsilon = \varepsilon_L] = g(\sqrt{J_\infty^*})$ (solid line) and its lower bounds are given by $g(\sqrt{J_\infty(\tau)})$ (dash and dash-dot lines), and $g(\sqrt{J_\infty^*}) = 1$ (< 1) indicates a fully (partially) revealing equilibrium. The parameters of PDFs (29) are $\sigma_B = 2.5\%$ and $\mu = 2\%$, belief dispersion is $\sigma_\nu = 3.23\%$, and mass parameter is $\eta = 0.5$.

Figure 2 plots the measure of efficiency J_∞^* and its lower bounds $J_\infty(\tau)$ for different values of the regularization parameter τ as functions of the informed investor's log-output volatility in (29).⁵ When $\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}$ and σ_I is further away from the boundaries, the informational efficiency is low and the equilibrium is partially revealing. In this case, the information overlap across sufficient statistics prevents uninformed investors from perfectly learning ε . The equilibrium is fully revealing outside of the latter interval.

The inequality $\sigma_B < \sigma_I$ means that biased investors assign lower volatility to log-output under their beliefs than informed investors. Following the literature (Daniel, Hirshleifer, and Subrahmanyam, 1998; De Long, Shleifer, Summers, and Waldmann, 1991), we interpret this scenario as overconfidence among biased investors. De Long, Shleifer, Summers, and Waldmann (1991) call the tendency to underestimate variances and to be overconfident one of the best-documented biases and cite extensive empirical evidence in support.

Proposition 6 also shows that J_∞^* is a non-monotone function of the dispersion of noisy beliefs σ_ν . Specifically, J_∞^* tends to infinity both as $\sigma_\nu \downarrow \sqrt{(\sigma_I^2 - \sigma_B^2)/2}$ and as $\sigma_\nu \uparrow \infty$.

⁵The volatility parameter σ_I ranges from 2% to 6%, and the mean growth rate (conditional on a realization of shock ε_H) is set at $\mu = 2\%$, consistent with the evidence that the unconditional volatility and mean consumption growth are around 3.2% and 1.8%, respectively (e.g. Basak and Cuoco, 1998). The standard deviation of biased investors' mean-growth beliefs is $\sigma_\nu = 3.23\%$, as in Atmaz and Basak (2018), who calibrate it from analysts' forecasts about the earnings growth rate of a market portfolio. The consumption growth volatility from the viewpoint of biased investors is set at $\sigma_B = 2.5\%$.

Inequality (35) implies that, for a given $\sigma_I > \sigma_B$, the equilibrium is partially revealing when $\sigma_\nu > \sqrt{(\sigma_I^2 - \sigma_B^2)/2}$. Consequently, J_∞^* decreases with belief dispersion near the left boundary of the partially revealing region, but increases with σ_ν when belief dispersion is sufficiently high. The latter result may appear counterintuitive, because σ_ν measures the dispersion of noisy beliefs. The intuition is that, as σ_ν increases, noise trading becomes more segmented and less concentrated in each particular Arrow–Debreu market. This is because the trading strategy of biased investors, $\pi_B(x; \nu)$ in (29), is concentrated around the security for state $x = \nu$, while other biased investors trade in substantially different markets. Consequently, greater dispersion of noisy beliefs reduces local noise in each market, making informed trading easier to detect and thereby improving informational efficiency.

Similarly, suppose that $\sigma_I^2 > 2\sigma_\nu^2$. Within the partially revealing region, the efficiency measure is non-monotone in the degree of overconfidence of biased investors, measured by their subjective output volatility σ_B . Inequality (35) implies that, for given σ_I and σ_ν , this region can be expressed as $\sqrt{\sigma_I^2 - 2\sigma_\nu^2} < \sigma_B < \sigma_I$. Accordingly, J_∞^* diverges as σ_B approaches either boundary of this interval. Hence, J_∞^* is decreasing in σ_B near the left boundary of the partially revealing region, but increasing in σ_B near the right boundary.

Our findings imply that derivative securities in financial markets reveal valuable information about economic fundamentals, in line with the empirical literature on price discovery in financial markets showing that prices and trading volumes of derivatives reveal information about the payoffs of underlying assets (Chakravarty, Gulen, and Mayhew, 2004; Easley, O’Hara, and Srinivas, 1998; Kacperczyk and Pagnotta, 2019; Pan and Poteshman, 2006). The result is also in line with findings in Bai, Philippon, and Savov (2016), who show that price informativeness is higher for firms with higher levels of option turnover. Our analysis, however, contributes a novel prediction that the amount of information revealed varies with macroeconomic parameters and may be high or low depending on the volatility of fundamentals, investor overconfidence, and belief dispersion.

We now introduce two measures of the economic value of information and the informational rent. The first, Δ_{IU} , measures the difference in expected wealth log-returns of informed and uninformed investors. The second, δ , is the fraction of initial wealth an uninformed investor is willing to give up to become informed before the realizations of the shock ε and beliefs ν_n . Proposition 7 characterizes both measures and establishes their relation.

Proposition 7 [Log wealth growth advantage and value of private information]. *Let $W_{i,1} := C_{1,i}^*(\omega)$ denote the terminal wealth of investor $i \in \{I, U\}$, which equals terminal consumption, and define the informed–uninformed expected wealth log-return differential by*

$$\Delta_{IU} := \mathbb{E} \left[\ln \left(\frac{W_{I,1}}{W_0} \right) \right] - \mathbb{E} \left[\ln \left(\frac{W_{U,1}}{W_0} \right) \right]. \quad (38)$$

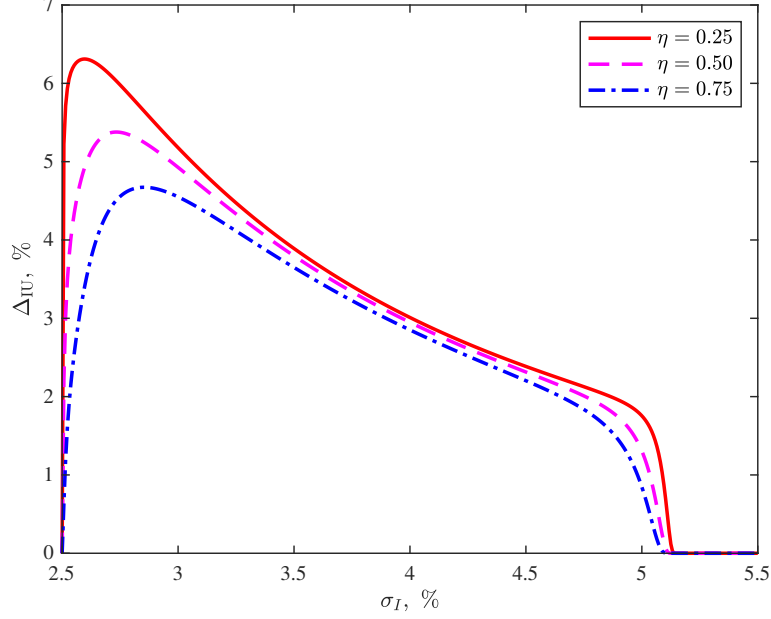


Figure 3
Informed–uninformed expected wealth log-return differential

This Figure shows the difference in the expected wealth log-returns of investors Δ_{IU} given by Equation (38) and equal to the expected Kullback–Leibler divergence in Equation (39). The differential is zero when the equilibrium is fully revealing. The parameters of PDFs (29) are $\sigma_B = 2.5\%$ and $\mu = 2\%$, belief dispersion is $\sigma_\nu = 3.23\%$, and the relative mass parameter takes values $\eta = 0.25$, 0.5 , and 0.75 . The unconditional probabilities of $\varepsilon = \varepsilon_L$ and $\varepsilon = \varepsilon_H$ are $\varphi_L = \varphi_H = 0.5$.

The expectations in (38) are taken under the true ex ante joint distribution, before the productivity shock, beliefs, and terminal state are realized. Then,

$$\Delta_{IU} = \mathbb{E}[D_{KL}(\pi_I || \pi_U)] \geq 0, \quad (39)$$

where $D_{KL}(\pi_I || \pi_U)$ is the Kullback–Leibler divergence of the informed investors’ beliefs π_I from the uninformed investors’ posterior beliefs π_U , given by

$$D_{KL}(\pi_I || \pi_U) = \int_{-\infty}^{\infty} \pi_I(x; \varepsilon) \ln \left(\frac{\pi_I(x; \varepsilon)}{\pi_U(x; \widehat{s})} \right) dx. \quad (40)$$

Furthermore, the fraction of wealth δ that an uninformed investor is willing to give up to become informed before the realization of shocks is given by

$$\delta = 1 - \exp \left(-\frac{\beta}{1 + \beta} \Delta_{IU} \right), \quad (41)$$

where Δ_{IU} can be computed using Equation (A.36) in the Appendix.

Proposition 7 gives the expected Kullback–Leibler divergence Δ_{IU} a direct economic interpretation: it quantifies by how much faster informed investors accumulate wealth compared

to uninformed investors. The fraction δ translates this divergence into a fraction of initial wealth that an uninformed investor is willing to give up to become informed.

Figure 3 plots the expected wealth log-return differential Δ_{IU} as a function of the true output volatility σ_I for three values of the relative mass parameter η . Intuitively, $\Delta_{IU} = 0$ outside the region $\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}$, where the equilibrium is fully revealing. Within the partially revealing region, informed investors' expected log return on wealth exceeds that of uninformed investors by as much as 6.3 percentage points over the investment period. An increase in η lowers this advantage because a larger relative mass of informed investors makes the market more informative, allowing uninformed investors to infer ε more accurately. For small Δ_{IU} , Equation (41) implies the following approximation for the value of information: $\delta \approx \beta/(1 + \beta)\Delta_{IU}$. For $\beta = 0.95$, $\delta \approx 0.487\Delta_{IU}$, and hence, δ ranges from 0% to about 3.0%.

Finally, we explore the interest rates when the investors have normal PDFs (29). Proposition 8 provides the results.

Proposition 8 [Interest rate]. *The risk-free interest rate in the Gaussian-signal economy with binary shock $\varepsilon \in \{\varepsilon_H = 1, \varepsilon_L = 0\}$ is given by*

$$r_f(\hat{s}) = \frac{1/\beta}{\mu_B \exp\{0.5(\sigma_\nu^2 + \sigma_B^2)\} + \mu_U \exp\{0.5\sigma_I^2\} \left[\exp\{-\mu\}(1 - \hat{\pi}_L(\hat{s})) + \hat{\pi}_L(\hat{s}) \right]} - 1, \quad (42)$$

where $\hat{\pi}_L(\hat{s}) = \text{Prob}(\varepsilon = \varepsilon_L | \hat{s})$ is given by Equation (A.19) in the Appendix.

Proposition 8 sheds light on information aggregation by interest rates. Equation (42) reveals that the interest rate is a decreasing function of the bad shock's conditional probability $\hat{\pi}_L(\hat{s}) = \text{Prob}(\varepsilon = \varepsilon_L | \hat{s})$. When $\mu_U > 0$ and the informed investors receive the bad signal ε_L , this probability goes up, leading to lower interest rates, and vice versa for a good signal ε_H . Hence, interest rates are high when expected output growth is high, and low otherwise. This is because the abundance of consumption in the future reduces the value of Arrow–Debreu securities, and hence, also of risk-free bonds, driving interest rates up. The relationship between risk-free rates and macroeconomic conditions is in line with the evidence in Harvey (1988) and Ang, Piazzesi, and Wei (2006) that information in yields predicts GDP growth.

Figure 4 plots conditional expected interest rates $\mathbb{E}[r_f(\hat{s}) | \varepsilon_L]$ and $\mathbb{E}[r_f(\hat{s}) | \varepsilon_H]$ as functions of output volatility σ_I . We observe that, consistent with Proposition 8, interest rates tend to be higher when economic conditions are better (i.e., $\varepsilon = \varepsilon_H$). Figure 4 also illustrates new predictions of the model that the interest rates are nonlinear functions of volatility σ_I and change sharply when σ_I crosses the boundaries separating fully and partially revealing equilibria in the parameter space. The prices of Arrow–Debreu securities given by Equation (10) also depend on the probability $\hat{\pi}_L(\hat{s})$, and hence, experience similar abrupt changes.

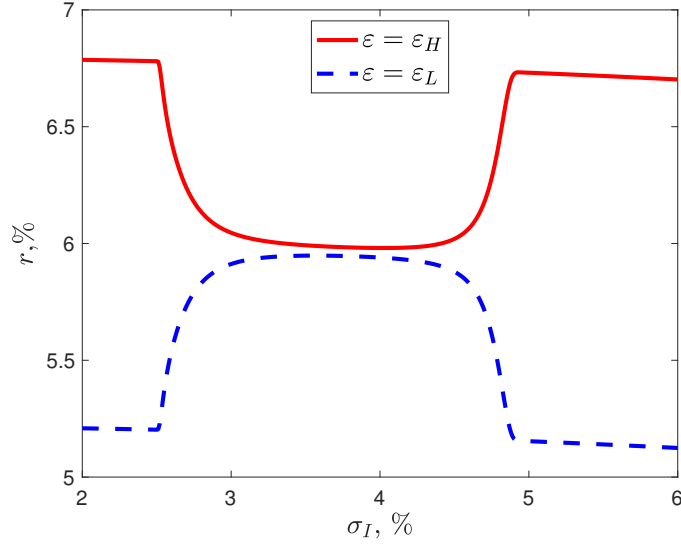


Figure 4

Expected conditional interest rates $\mathbb{E}[r_f(\hat{s})|\varepsilon]$

This Figure shows the expected interest rates conditional on the realization of shock ε . The solid line corresponds to $\varepsilon = \varepsilon_H$ and the dashed line to $\varepsilon = \varepsilon_L$. The model parameters are as for Figure 2.

6. Conclusion

In this paper, we develop a theory of informational efficiency and asset pricing in large markets with asymmetric information. We show that in economies with log investors, equilibria can be either fully or partially revealing, depending on economic parameters such as output volatility, belief dispersion, and investor overconfidence. The results are consistent with the empirical evidence and shed light on the role of derivatives and interest rates in determining the extent of information revelation in financial markets. The analysis highlights that the structure of preferences and probabilities plays a central role. In particular, replacing log investors with CARA investors leads to partially revealing equilibria in which market information is compressed into two scalar statistics. This contrast suggests that further progress in understanding informational efficiency in large markets with contingent claims requires moving beyond the joint CARA/exponential-family benchmark. We provide new tools based on operator theory for analyzing informational efficiency in such markets.

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Appendix: Proofs

Proof of Proposition 1. Due to market completeness, investor i 's budget constraint is:

$$C_{0,i} + \mathbb{E}^i \left[\frac{1}{1 + r_f} \frac{\pi^{\text{RN}}(\omega)}{\pi_i(\omega)} C_{1,i}(\omega) \right] = W_0, \quad (\text{A.1})$$

where $\pi^{\text{RN}}(\omega)$ is a risk-neutral probability. We maximize the expected utility function (1) subject to the budget constraint (A.1). Using the method of Lagrange multipliers, we find:

$$C_{0,i}^* = \frac{1}{\lambda}, \quad C_{1,i}^*(\omega_m) = \frac{\beta(1 + r_f)\pi_{i,m}}{\lambda \frac{\pi_m^{\text{RN}}}{\pi_i(\omega_m)}}, \quad (\text{A.2})$$

where λ is the Lagrange multiplier for the constraint (A.1). Substituting consumptions (A.2) into the budget constraint (A.1), we find $\lambda = (1 + \beta)/W_0$. Substituting this λ and asset prices $p_m = \pi_m^{\text{RN}}/(1 + r_f)$ into (A.2), we obtain consumptions reported in Proposition 1.

Substituting consumptions (7) into budget constraint (3), we obtain:

$$\ell W_0 \frac{(1 + r_f)\pi_{i,m}}{\pi_m^{\text{RN}}} = \frac{W_0 \theta_{i,m}^*}{p_m}. \quad (\text{A.3})$$

Equation (A.3) means that state- m consumption is financed by state- m securities. Noting that $p_m = \pi_m^{\text{RN}}/(1 + r_f)$, cancelling terms in (A.3) yields portfolios (8). ■

Proof of Proposition 2. Substituting date-0 consumptions $C_{0,i}^* = W_0/(1 + \beta)$, derived in Proposition 1, into the market clearing condition (4), we find that $W_0 = (1 + \beta)D_0$. Moreover, the uninformed investors observe the sufficient statistics s , and hence, compute the posterior probability $\pi_{U,m} = \mathbb{E}[\pi_{I,m}(\varepsilon)|s]$. Substituting portfolios (8) and the latter expressions for W_0 and $\pi_{U,m}$ into the market clearing condition (6), we find prices in (10). The risk-free interest rate in terms of prices of Arrow–Debreu securities is given by $1 + r_f = 1/\sum_{m=1}^M p_m$. Substituting prices (10) into the latter equation, we find the risk-free rate (11). ■

Proof of Proposition 3. Suppose there exist two vectors $(\varepsilon, \nu_1, \dots, \nu_N)$ and $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ such that $\varepsilon \neq \tilde{\varepsilon}$ and the corresponding sufficient statistics are the same. Consequently, the system (13) of M equations for $m = 1, \dots, M$ with $2N + 2$ unknowns is satisfied. The equation for $m = M$ holds since probabilities sum up to one. Substituting probabilities $\pi_{I,m}(z)$ and $\pi_{B,m}(z)$ from (14) into the system (13), after cancelling $\exp(a(x_m))$ on both sides, we obtain the system of equations

$$\begin{aligned} \mu_I e^{\varepsilon x - c(\varepsilon)} + (\mu_B/N)(e^{\nu_1 x - c(\nu_1)} + \dots + e^{\nu_N x - c(\nu_N)}) &= \\ \mu_I e^{\tilde{\varepsilon} x - c(\tilde{\varepsilon})} + (\mu_B/N)(e^{\tilde{\nu}_1 x - c(\tilde{\nu}_1)} + \dots + e^{\tilde{\nu}_N x - c(\tilde{\nu}_N)}), \end{aligned} \quad (\text{A.4})$$

where $x = x_1, \dots, x_M$. After fixing $\varepsilon, \tilde{\varepsilon}, \nu_n, \tilde{\nu}_n$, (A.4) can be viewed as one equation with one unknown x that has M solutions $x = x_1, \dots, x_M$. However, Lemma A.1 on zeros of

exponential sums below shows that, unless the left- and right-hand sides of (A.4) are identical for all x , this equation has at most $2N+1$ solutions, contradicting the condition $2N+1 < M$. Hence, system (13) either does not have a solution $(\varepsilon, \nu_1, \dots, \nu_N)$ and $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ or the left- and right-hand sides of (A.4) coincide for all x .

Suppose the exponential sums on the left- and right-hand sides of (A.4) coincide for all x . Because the exponential functions are linearly independent, identical exponents must have identical aggregate coefficients on the two sides. Consider the exponent with parameter ε on the left-hand side. If $\exp\{\varepsilon x - c(\varepsilon)\} = \exp\{\tilde{\varepsilon} x - c(\tilde{\varepsilon})\}$ for all x , then $\varepsilon = \tilde{\varepsilon}$, contradicting the initial assumption that $\varepsilon \neq \tilde{\varepsilon}$. Therefore, this exponent must be matched on the right-hand side by biased-investor exponents. If k biased-investor parameters on the left-hand side are also equal to ε and m biased-investor parameters on the right-hand side are equal to ε , equality of the aggregate coefficients implies $\mu_I + \mu_B k/N = \mu_B m/N$. Since $\mu_I > 0$, we have $m > k$ and hence $\mu_I = \mu_B(m - k)/N$, where $1 \leq m - k \leq N$, contradicting the assumptions of Proposition 3. Hence, no two vectors with $\varepsilon \neq \tilde{\varepsilon}$ can generate the same sufficient statistics.

To identify biased beliefs, consider any two biased-belief vectors that generate the same sufficient statistics. We have shown that $\varepsilon = \tilde{\varepsilon}$. Cancelling the informed terms in (A.4) leaves two exponential sums that agree at the M distinct values $x_1, \dots, x_M$. If these sums were not identical for all x , their difference would have at most $2N - 1$ distinct roots by Lemma A.1, contradicting $2N + 1 < M$. Thus, the sums are identical for all x . For each z , linear independence implies $(\mu_B/N)k(z)e^{-c(z)} = (\mu_B/N)\tilde{k}(z)e^{-c(z)}$, where $k(z)$ and $\tilde{k}(z)$ are the multiplicities of z among $(\nu_1, \dots, \nu_N)$ and $(\tilde{\nu}_1, \dots, \tilde{\nu}_N)$, respectively. Since $\mu_B > 0$, $k(z) = \tilde{k}(z)$ for all z . Hence, the two belief vectors are permutations of one another. ■

Lemma A.1 [Zeros of sums of exponentials]. *Consider the following equation:*

$$A_1 e^{a_1 x} + A_2 e^{a_2 x} + \dots + A_N e^{a_N x} = 0, \quad (\text{A.5})$$

where A_n and a_n are real numbers, at least one of the coefficients A_n is non-zero, and a_n are distinct. Then, Equation (A.5) has at most $N - 1$ distinct real roots.

Proof of Lemma A.1. This Lemma is Corollary 3.2 in Jameson (2006, p. 228). ■

Proposition A.1 [Optimal consumptions and portfolios of CARA investors]. *Consider an investor who maximizes expected CARA utility (15) subject to a self-financing budget constraint and invests in all available Arrow–Debreu securities. Then, the optimal consumptions C_0^* and $C_1^*(\omega_m)$ and numbers of units n_m^* of each security are given by:*

$$C_0^* = \left(W_0 - \frac{\ln(\beta)}{\gamma(1+r_f)} + \frac{\sum_{m=1}^M p_m (\ln(p_m) - \ln(\pi_m))}{\gamma} \right) \frac{1+r_f}{2+r_f}, \quad (\text{A.6})$$

$$C_1^*(\omega_m) = n_m^* = C_0^* + \frac{\ln(\pi_m) - \ln(p_m/\beta)}{\gamma}, \quad (\text{A.7})$$

where p_m is the price of security m and π_m is the physical probability of state ω_m .

Proof of Proposition A.1. The investor's budget constraint is given by (A.1). Using the method of Lagrange multipliers, we find that the first-order conditions for optimization (15) subject to the latter budget constraint are then given by

$$C_0^* = \tilde{\ell}, \quad C_1^*(\omega_m) = \tilde{\ell} + \frac{\ln(\beta)}{\gamma} + \frac{\ln(\pi_m) - \ln(p_m)}{\gamma}, \quad (\text{A.8})$$

where $\tilde{\ell}$ is some function of the Lagrange multiplier. Substituting out $\tilde{\ell}$ gives the expression for the optimal consumption in (A.7). The value of the portfolio of Arrow–Debreu securities in state m at date 1 is equal to the number of state- m securities purchased at date-0, given by n_m^* . The consumption at date 1 then equals $C_1^*(\omega_m) = n_m^*$. Finally, we find the constant $\tilde{\ell}$ by substituting consumptions (A.8) into the static budget constraint (A.1). ■

Proof of Proposition 4. Equation (17) implies that the statistics s and statistics (18)–(19) contain the same information. We next show that shock ε is almost surely not identified from the latter two statistics. Because probabilities (14) sum to one, we normalize $c(z) = \ln(\sum_{m=1}^M \exp\{a(x_m) + zx_m\})$. Direct differentiation of $c(z)$ gives $c'(z) = \sum_{m=1}^M \pi_{I,m}(z)x_m$, $c''(z) = \sum_{m=1}^M \pi_{I,m}(z)(x_m - c'(z))^2 > 0$. The inequality is strict because all probabilities are positive and all x_m are distinct. Hence, $c(z)$ is strictly convex and $c'(z)$ is strictly increasing.

Fix a realization $(\varepsilon, \nu_1, \dots, \nu_N)$ and the corresponding values of $(\tilde{s}_1, \tilde{s}_2)$. Holding $\nu_3, \dots, \nu_N$ fixed, define the continuously differentiable function $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$G(e, u_1, u_2) = \begin{pmatrix} \mu_I e + \frac{\mu_B}{N} \left(u_1 + u_2 + \sum_{n=3}^N \nu_n \right) - \tilde{s}_1 \\ \mu_I c(e) + \frac{\mu_B}{N} \left(c(u_1) + c(u_2) + \sum_{n=3}^N c(\nu_n) \right) - \tilde{s}_2 \end{pmatrix}, \quad (\text{A.9})$$

where the sums from 3 to N are zero when $N = 2$. By construction, $G(\varepsilon, \nu_1, \nu_2) = 0$. The Jacobian of G with respect to (u_1, u_2) , evaluated at the realized parameters, is

$$D_{(u_1, u_2)} G(\varepsilon, \nu_1, \nu_2) = \frac{\mu_B}{N} \begin{pmatrix} 1 & 1 \\ c'(\nu_1) & c'(\nu_2) \end{pmatrix}, \quad \det D_{(u_1, u_2)} G = \left(\frac{\mu_B}{N} \right)^2 (c'(\nu_2) - c'(\nu_1)). \quad (\text{A.10})$$

Because ν_1 and ν_2 are i.i.d. from a distribution with a density, $\nu_1 \neq \nu_2$ almost surely. Strict monotonicity of c' then implies that the determinant in (A.10) is nonzero almost surely.

The implicit function theorem says that if a continuously differentiable function $F : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $F(e_0, u_0) = 0$ and its Jacobian with respect to u is nonsingular at (e_0, u_0) , then, around (e_0, u_0) , there is a unique continuously differentiable function $u = h(e)$ satisfying $u_0 = h(e_0)$ and $F(e, h(e)) = 0$. Applying this to G at $(\varepsilon, \nu_1, \nu_2)$ gives an open neighborhood $\mathcal{O}$ of ε and continuously differentiable functions $u_1(e)$ and $u_2(e)$ such that

$$u_1(\varepsilon) = \nu_1, \quad u_2(\varepsilon) = \nu_2, \quad G(e, u_1(e), u_2(e)) = 0 \quad \text{for every } e \in \mathcal{O}. \quad (\text{A.11})$$

By the interval-support assumption, almost surely, ε is an interior point of the support of the productivity-shock distribution, while ν_1 and ν_2 are interior points of the support of the belief distribution. Because $u_1(e)$ and $u_2(e)$ are continuous at $e = \varepsilon$, we can choose $e \neq \varepsilon$ sufficiently close to ε so that e , $u_1(e)$, and $u_2(e)$ remain in these supports. Replacing ε by e and (ν_1, ν_2) by $(u_1(e), u_2(e))$, while keeping the other beliefs unchanged, leaves both statistics unchanged by (A.11). Hence, the shock is not identified.

The statistics, however, are informative about shock ε . Conditional on $\varepsilon = e$, the statistic $\tilde{s}_1$ has the distribution of $(\mu_B/N) \sum_{n=1}^N \nu_n$ shifted by $\mu_I e$. Because $\mu_I > 0$, these conditional distributions differ across values of e . Thus, the statistics contain information about ε . ■

Lemma A.2. *Consider two linear systems: $\Sigma_\xi \tilde{v} = \tilde{f}$ and $\mathcal{K}v = f$, where $\Sigma_\xi \in \mathbb{R}^{(M-1) \times (M-1)}$ has elements $\overline{\text{cov}}(\pi_{B,m}(\nu), \pi_{B,k}(\nu))$ for $m, k = 1, \dots, M-1$, and $\mathcal{K} \in \mathbb{R}^{M \times M}$ has elements $\overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,k}(\nu)]$ for $m, k = 1, \dots, M$. Both moments are computed with respect to the distribution of ν . Let $f = (f_1, \dots, f_M)^\top \in \mathbb{R}^M$ satisfy $\sum_{m=1}^M f_m = 0$, and let $\tilde{f} = (f_1, \dots, f_{M-1})^\top \in \mathbb{R}^{M-1}$. Then Σ_ξ and $\mathcal{K}$ are symmetric and positive definite, and the systems $\Sigma_\xi \tilde{v} = \tilde{f}$ and $\mathcal{K}v = f$ have unique solutions satisfying*

$$\tilde{v} = (v_1 - v_M, v_2 - v_M, \dots, v_{M-1} - v_M)^\top, \quad (\text{A.12})$$

$$\tilde{v}^\top \tilde{f} = v^\top f. \quad (\text{A.13})$$

Proof of Lemma A.2. Σ_ξ is a symmetric matrix. For any nonzero vector $\tilde{\ell} \in \mathbb{R}^{M-1}$, $\tilde{\ell}^\top \Sigma_\xi \tilde{\ell} = \overline{\text{var}}\left(\sum_{m=1}^{M-1} \tilde{\ell}_m \pi_{B,m}(\nu)\right)$. If $\tilde{\ell}^\top \Sigma_\xi \tilde{\ell} = 0$, then $\sum_{m=1}^{M-1} \tilde{\ell}_m \pi_{B,m}(\nu) = \gamma$ with probability one for some γ . Since the probabilities sum to one, it follows that $\sum_{m=1}^{M-1} (\tilde{\ell}_m - \gamma) \pi_{B,m}(\nu) - \gamma \pi_{B,M}(\nu) = 0$ with probability one. The coefficient vector in this relation is nonzero: if $\gamma = 0$, this follows from $\tilde{\ell} \neq 0$, while if $\gamma \neq 0$, the coefficient on $\pi_{B,M}(\nu)$ is nonzero. Either case contradicts linear independence of $\pi_{B,m}(\nu)$. Hence Σ_ξ is positive definite.

The matrix $\mathcal{K}$ is symmetric, and $\ell^\top \mathcal{K} \ell = \overline{\mathbb{E}}\left[\left(\sum_{m=1}^M \ell_m \pi_{B,m}(\nu)\right)^2\right] > 0$, for any nonzero $\ell \in \mathbb{R}^M$, where the strict inequality follows from linear independence. Thus, $\mathcal{K}$ is positive definite. Hence, both Σ_ξ and $\mathcal{K}$ are invertible, and the two systems have unique solutions.

Next, we establish the relation between these solutions. Write system $\mathcal{K}v = f$ as

$$\overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,1}(\nu)]v_1 + \dots + \overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,M-1}(\nu)]v_{M-1} + \overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,M}(\nu)]v_M = f_m. \quad (\text{A.14})$$

Summing these equations, using $\sum_{m=1}^M \pi_{B,m}(\nu) = 1$, and $\sum_{m=1}^M f_m = 0$, we obtain:

$$\overline{\mathbb{E}}[\pi_{B,1}(\nu)]v_1 + \dots + \overline{\mathbb{E}}[\pi_{B,M-1}(\nu)]v_{M-1} + \overline{\mathbb{E}}[\pi_{B,M}(\nu)]v_M = 0. \quad (\text{A.15})$$

Define $\tilde{v}_m = v_m - v_M$. Substituting $\pi_{B,M}(\nu) = 1 - \sum_{k=1}^{M-1} \pi_{B,k}(\nu)$ into (A.14) and (A.15) yields

$$\overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,1}(\nu)]\tilde{v}_1 + \dots + \overline{\mathbb{E}}[\pi_{B,m}(\nu)\pi_{B,M-1}(\nu)]\tilde{v}_{M-1} + \overline{\mathbb{E}}[\pi_{B,m}(\nu)]v_M = f_m, \quad (\text{A.16})$$

$$-\overline{\mathbb{E}}[\pi_{B,1}(\nu)]\tilde{v}_1 - \dots - \overline{\mathbb{E}}[\pi_{B,M-1}(\nu)]\tilde{v}_{M-1} = v_M, \quad (\text{A.17})$$

where (A.16) is for $m = 1, \dots, M-1$. Substituting v_m from (A.17) into (A.16) gives

$$\overline{\text{cov}}(\pi_{B,m}(\nu), \pi_{B,1}(\nu))\tilde{v}_1 + \dots + \overline{\text{cov}}(\pi_{B,m}(\nu), \pi_{B,M-1}(\nu))\tilde{v}_{M-1} = f_m. \quad (\text{A.18})$$

This is system $\Sigma_\xi \tilde{v} = \tilde{f}$, and so (A.12) follows. From (A.12) and $f_M = -\sum_{m=1}^{M-1} f_m$, we obtain (A.13): $\tilde{v}^\top \tilde{f} = v_1 f_1 + \dots + v_{M-1} f_{M-1} - v_M (f_1 + \dots + f_{M-1}) = v^\top f$. ■

Proof of Lemma 1.

1) Let $\tilde{f} = (f_1, \dots, f_{M-1})^\top \neq 0$. The solution of problem (22) is characterized by the first-order condition $\Sigma_\xi w^* = \lambda \tilde{f}$ and the constraint $\tilde{f}^\top w^* = 1$. Hence, $\lambda = 1/\tilde{f}^\top \Sigma_\xi^{-1} \tilde{f}$, $w^* = \Sigma_\xi^{-1} \tilde{f} / \tilde{f}^\top \Sigma_\xi^{-1} \tilde{f}$, $(w^*)^\top \Sigma_\xi w^* = 1/\tilde{f}^\top \Sigma_\xi^{-1} \tilde{f}$. Consequently, $J_M^* = \tilde{f}^\top \Sigma_\xi^{-1} \tilde{f}$.

The conditional distributions of $\hat{s}$ under the two shocks are normal with the same covariance matrix. Their log-likelihood ratio is therefore

$$\log \frac{p(\hat{s} \mid \varepsilon = \varepsilon_H)}{p(\hat{s} \mid \varepsilon = \varepsilon_L)} = -\frac{1}{2}(\hat{s} - \tilde{f})^\top \Sigma_\xi^{-1} (\hat{s} - \tilde{f}) + \frac{1}{2}\tilde{f}^\top \Sigma_\xi^{-1} \hat{s} = \tilde{f}^\top \Sigma_\xi^{-1} \hat{s} - \frac{J_M^*}{2}.$$

For $\varepsilon \in \{0, 1\}$, the conditional density can be written as $p(\hat{s} \mid \varepsilon) = p(\hat{s} \mid \varepsilon = 0) \exp\{\varepsilon(\tilde{f}^\top \Sigma_\xi^{-1} \hat{s} - 0.5J_M^*)\}$. The first factor does not depend on ε , while the second depends on $\hat{s}$ only through a scalar $\tilde{f}^\top \Sigma_\xi^{-1} \hat{s}$. The Fisher–Neyman factorization criterion then implies that this scalar is sufficient for ε . Because $\tilde{f}^\top \Sigma_\xi^{-1} \hat{s} = J_M^* (w^*)^\top \hat{s}$, the optimal combination $(w^*)^\top \hat{s}$ contains all the information in $\hat{s}$ about ε . Bayes's rule now gives

$$\hat{\pi}_L(\hat{s}) = \frac{\varphi_L p(\hat{s} \mid \varepsilon = \varepsilon_L)}{\varphi_L p(\hat{s} \mid \varepsilon = \varepsilon_L) + \varphi_H p(\hat{s} \mid \varepsilon = \varepsilon_H)} = \frac{1}{1 + (\varphi_H/\varphi_L) \exp\{-J_M^*/2 + \tilde{f}^\top \Sigma_\xi^{-1} \hat{s}\}}. \quad (\text{A.19})$$

Conditional on $\varepsilon = \varepsilon_L = 0$, the last term in the exponent is distributed as $x\sqrt{J_M^*}$, where $x \sim \mathcal{N}(0, 1)$. Taking expectations in (A.19) yields Equation (23), where

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x^2/2)}{1 + (\varphi_H/\varphi_L) \exp(-y^2/2 + xy)} dx. \quad (\text{A.20})$$

Denote the function under the integral in (A.20) by $h(x; y)$. Its derivative is given by:

$$\frac{\partial h(x; y)}{\partial y} = \frac{\varphi_H}{\varphi_L} \frac{\exp(-0.5(y-x)^2)}{(1 + (\varphi_H/\varphi_L) \exp(-0.5(y-x)^2 + 0.5x^2))^2} (y-x). \quad (\text{A.21})$$

Then, we have the following uniform bound on the partial derivative in (A.21) with respect to $y \in [0, d]$, where $d > 0$ is an arbitrary number: $|\partial h(x; y)/\partial y| \leq \varphi_H/\varphi_L \exp(-0.5x^2 + d|x|)(|x| + d)$. The integral $\int_{-\infty}^{\infty} \exp(-0.5x^2 + d|x|)(|x| + d) dx$ converges and is finite, and hence, by the Weierstrass convergence test (Zorich, 2004, p. 420), the integral of $\partial h(x; y)/\partial y$ converges uniformly for all $y \in [0, d]$. Therefore, the derivative of the function $g(y)$ in (A.20) is equal to the integral of the partial derivative $\partial h(x; y)/\partial y$ (Zorich, 2004, p. 426), given by

$$g'(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\varphi_H}{\varphi_L} \frac{\exp(-0.5(y-x)^2)}{(1 + (\varphi_H/\varphi_L) \exp(-0.5(y-x)^2 + 0.5x^2))^2} (y-x) dx.$$

In the latter integral, we change integration from x to $t = y - x$ and split the integral:

$$g'(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\varphi_H}{\varphi_L} \frac{\exp(-0.5t^2)t}{(1 + (\varphi_H/\varphi_L) \exp(-0.5t^2 + 0.5(t-y)^2))^2} dt - \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\varphi_H}{\varphi_L} \frac{\exp(-0.5t^2)t}{(1 + (\varphi_H/\varphi_L) \exp(-0.5t^2 + 0.5(t+y)^2))^2} dt, \quad (\text{A.22})$$

where the last integral is obtained by another change of variable, from t to $-t$. Then, $g'(y) > 0$ for $y > 0$ follows because the difference of functions under the corresponding integrals is positive: for all $t > 0$ and $y > 0$, $\exp(-0.5t^2 + 0.5(t-y)^2) < \exp(-0.5t^2 + 0.5(t+y)^2)$. At $y = 0$, the two integrals in (A.22) are equal, so $g'(0) = 0$. Thus, g is increasing on $[0, \infty)$.

Finally, Equation (A.20) gives $g(0) = \varphi_L$. For a fixed x , $\exp(-y^2/2 + xy) \rightarrow 0$, so $h(x; y) \rightarrow \exp(-x^2/2)$ as $y \rightarrow \infty$. Moreover, $0 \leq h(x; y) \leq \exp(-x^2/2)$, and the latter function is integrable. The dominated convergence theorem applied to (A.20) gives $\lim_{y \rightarrow \infty} g(y) = (1/\sqrt{2\pi}) \int_{-\infty}^\infty \lim_{y \rightarrow \infty} h(x; y) dx = 1$.

2) For statistics (21), let $\tilde{f}$ denote the first $M - 1$ elements of the vector f defined in the lemma. Because the probabilities of informed investors sum to one under either shock, $\sum_{m=1}^M f_m = 0$. Part 1 and Lemma A.2 then imply $J_M^* = \tilde{f}^\top \Sigma_\xi^{-1} \tilde{f} = f^\top \mathcal{K}^{-1} f$. ■

Lemma A.3 [Effective sample size]. Consider exogenous signals $\hat{s}_m = \varepsilon + \tilde{\xi}_m$, for $m = 1, \dots, M - 1$, where $\tilde{\xi}$ is a possibly degenerate jointly Gaussian noise vector, independent of ε and with mean zero. Let J_M^* denote the noise precision of the most informative linear combination normalized to have unit loading on ε . Then, results in Table 1 follow.

Proof of Lemma A.3. A linear combination $\sum_{m=1}^{M-1} a_m \hat{s}_m$ has loading $\sum_{m=1}^{M-1} a_m$ on ε , normalized to $\sum_{m=1}^{M-1} a_m = 1$, and noise $\sum_{m=1}^{M-1} a_m \tilde{\xi}_m$. Cases 1, 3 and 5 in Table 1 follow from standard mean-variance optimization in (22), and their proofs are omitted for brevity.

2) Replication means that, for $m = M - k, \dots, M - 1$, there are scalars c_{mj} such that $\sum_{j=1}^{M-1-k} c_{mj} = 1$, $\tilde{\xi}_m = \sum_{j=1}^{M-1-k} c_{mj} \tilde{\xi}_j$. This implies $\hat{s}_m = \sum_{j=1}^{M-1-k} c_{mj} \hat{s}_j$. For any weights a_m , let $b_j = a_j + \sum_{m=M-k}^{M-1} a_m c_{mj}$, $j = 1, \dots, M - 1 - k$. If $\sum_{m=1}^{M-1} a_m = 1$, then $\sum_{j=1}^{M-1-k} b_j = 1$. The combined noise satisfies $\sum_{m=1}^{M-1} a_m \tilde{\xi}_m = \sum_{j=1}^{M-1-k} b_j \tilde{\xi}_j$. Because the latter noises are independent, the variance is $\sigma^2 \sum_{j=1}^{M-1-k} b_j^2 \geq \sigma^2 / (M - 1 - k)$. The bound is attained by weights $a_m = 1 / (M - 1 - k)$ to the $M - 1 - k$ independent signals and $a_m = 0$ to the replicable signals. Hence, variance minimum is $\sigma^2 / (M - 1 - k)$, and $\text{ESS}_M = M - 1 - k$.

4) Suppose $\text{corr}(\tilde{\xi}_i, \tilde{\xi}_j) = -1$. Then, $\text{var}(\tilde{\xi}_i + \tilde{\xi}_j) = 0$. Setting $a_i = a_j = 1/2$ and other weights to zero gives a signal with unit loading on ε and no noise. Hence, $\text{ESS}_M = +\infty$. ■

Proof of Lemma 2. Matrix $\mathcal{K}$ is positive definite by Lemma A.2. Hence, it has M eigenvalues $\lambda_n > 0$ and eigenvectors ψ_n that form an orthonormal basis in $\mathbb{R}^M$. It can be directly verified that the solution of the regularized equation (25) is given by $v(\tau_M) =$

$\sum_{n=1}^M \hat{f}_n \psi_n / (\lambda_n + \tau_M)$, where $\hat{f}_n = (f, \psi_n)$ are inner products. Moreover, $f = \sum_{n=1}^M \hat{f}_n \psi_n$. Therefore, the regularized measure of efficiency $J_M(\tau_M) = f^\top v(\tau_M)$ is then given by $J_M(\tau_M) = \sum_{n=1}^M \hat{f}_n^2 / (\lambda_n + \tau_M)$. Function $J_M(\tau_M)$ then is decreasing and continuous for all $\tau_M \geq 0$. ■

Lemma A.4 [Convergence of regularized solutions]. *Suppose the probabilities of states are given by (27) and, as $A_M \rightarrow \infty$ and $\Delta x \rightarrow 0$, converge in the usual Riemann-sum sense to the continuous-space PDFs $\pi_I(x; \varepsilon)$ and $\pi_B(x; \nu)$, respectively. For every fixed $\tau > 0$, set $\tau_M = \tau \Delta x$ and let $v^M(\tau_M)$ solve $\mathcal{K}^M v^M(\tau_M) + \tau_M v^M(\tau_M) = f^M$, where $\mathcal{K}_{ml}^M = \mathbb{E}[\pi_{B,m}(\nu) \pi_{B,l}(\nu)]$ and $f_m^M = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L))$. Set $x_0 = -A_M$, and let $\tilde{v}_M(x; \tau) = v_m^M(\tau_M)$ for $x \in (x_{m-1}, x_m]$ and zero otherwise. Then the finite system converges to the Fredholm equation (28) and $\tilde{v}_M(\cdot; \tau) \rightarrow v(\cdot; \tau)$ in $\mathcal{L}_2(\mathbb{R})$, where $v(\cdot; \tau)$ is the unique solution of (28).*

Proof of Lemma A.4. Substituting (27) and $\tau_M = \tau \Delta x$ into the m th equation of the finite system (25) and dividing by Δx gives

$$\sum_{l=1}^M \mathbb{E} \left[\frac{\pi_B(x_m; \nu) \pi_B(x_l; \nu)}{(\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x)^2} \right] v_l^M(\tau_M) \Delta x + \tau v_m^M(\tau_M) = \eta \left[\frac{\pi_I(x_m; \varepsilon_H)}{\sum_{k=1}^M \pi_I(x_k; \varepsilon_H) \Delta x} - \frac{\pi_I(x_m; \varepsilon_L)}{\sum_{k=1}^M \pi_I(x_k; \varepsilon_L) \Delta x} \right].$$

For the normal PDFs (29), Lemma I.5 in the Online Appendix establishes, for every fixed ν and ε , $\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x \rightarrow 1$, $\sum_{k=1}^M \pi_I(x_k; \varepsilon) \Delta x \rightarrow 1$, and, whenever $x_m \rightarrow x$, $\pi_{B,m}/\Delta x \rightarrow \pi_B(x; \cdot)$ in $\mathcal{L}_2(\mathbb{R}, \varphi_\nu(\nu) d\nu)$, $f_m^M/\Delta x \rightarrow f(x)$. Consequently, if also $x_l \rightarrow y$,

$$\frac{\mathcal{K}_{ml}^M}{(\Delta x)^2} = \mathbb{E} \left[\frac{\pi_{B,m} \pi_{B,l}}{\Delta x \Delta x} \right] \rightarrow K(x, y).$$

Thus, the first term on the left is a Riemann-sum approximation to $\int_{-\infty}^{\infty} K(x, y) v(y; \tau) dy$, while the right-hand side converges to $f(x)$. Taking the limit yields (28).

To show convergence of the solutions, let $\mathcal{A}_M$ be the integral operator whose kernel equals $\mathcal{K}_{ml}^M/(\Delta x)^2$ on $(x_{m-1}, x_m] \times (x_{l-1}, x_l]$ and zero outside the grid, and let $\tilde{f}_M(x) = f_m^M/\Delta x$ on $(x_{m-1}, x_m]$, extended by zero otherwise. The same Riemann-sum argument, first on compact intervals and then using the tails of the PDFs, gives convergence of these kernels in $\mathcal{L}_2(\mathbb{R}^2)$ and of $\tilde{f}_M$ in $\mathcal{L}_2(\mathbb{R})$. The Cauchy-Schwarz inequality therefore gives $\|\mathcal{A}_M - \mathcal{A}\|_{\text{op}} \rightarrow 0$, $\|\tilde{f}_M - f\|_2 \rightarrow 0$, where $(\mathcal{A}q)(x) = \int_{-\infty}^{\infty} K(x, y) q(y) dy$. The scaled finite system is exactly $(\mathcal{A}_M + \tau I) \tilde{v}_M = \tilde{f}_M$. Because $\mathcal{A}_M$ and $\mathcal{A}$ are nonnegative, $\|(\mathcal{A}_M + \tau I)^{-1}\|_{\text{op}} \leq 1/\tau$. Hence, with $v = (\mathcal{A} + \tau I)^{-1} f$, $\|\tilde{v}_M - v\|_2 \leq 1/\tau \left(\|\tilde{f}_M - f\|_2 + \|\mathcal{A}_M - \mathcal{A}\|_{\text{op}} \|v\|_2 \right) \rightarrow 0$. This proves convergence of the solutions for every fixed $\tau > 0$. ■

Proof of Proposition 5.

1) The kernel $K(x, y) = \mathbb{E}[\pi_B(x; \nu) \pi_B(y; \nu)]$ of Equation (28) with $\nu \sim \mathcal{N}(0, \sigma_\nu^2)$ is given by:

$$K(x, y) = \frac{1}{2\pi\sigma_B^2 \sqrt{2\pi\sigma_\nu^2}} \int_{-\infty}^{\infty} \exp\left(-0.5(x - \nu)^2/\sigma_B^2 - 0.5(y - \nu)^2/\sigma_B^2 - 0.5\nu^2/\sigma_\nu^2\right) d\nu. \quad (\text{A.23})$$

Integrating (A.23), we find the kernel (30).

2) The change of variables $u = ax$ in the result of Johnston (2014) shows that the functions ψ_n in (31) form an orthonormal basis of $\mathcal{L}_2(\mathbb{R})$.

Next, we find eigenfunctions and the eigenvalues of the integral operator using the following Mehler's formula (Lebedev, 1972, p. 61), slightly rewritten in a convenient form:

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \left(\frac{u}{2}\right)^n = \frac{1}{\sqrt{1-u^2}} \exp \left\{ \frac{u}{1+u}(x^2 + y^2) - \frac{u}{1-u^2}(x-y)^2 \right\}, \quad (\text{A.24})$$

where $H_n(x)$ are Hermite polynomials, and the series on the left-hand side converges iff $|u| < 1$. We rewrite (A.24) for arguments ax and ay , express Hermite polynomials via eigenfunctions (31), multiply both sides by λ_0 , and take $u = (\sigma_\nu - \hat{\sigma})/(\sigma_\nu + \hat{\sigma})$. Because $\hat{\sigma} = (2/\sigma_B^2 + 1/\sigma_\nu^2)^{-1/2} < \sigma_\nu$, we have $0 < u < 1$. Then, using some algebra, we find that

$$\sum_{n=0}^{\infty} \lambda_n \psi_n(x) \psi_n(y) = K(x, y), \quad (\text{A.25})$$

where $K(x, y)$ is given by (30). Due to Cramer's inequality $|\psi_n(x)| < 1.1\sqrt{\pi a}$ (Gradshteyn and Ryzhik, 2007, Eq. 8.954.2) and the fact that λ_n is a decaying exponential function given by (32), the series on the left-hand side of (A.25) converges uniformly, and hence, summation and integration are interchangeable. Multiplying Equation (A.25) by $\psi_k(y)$, integrating, and using the orthogonality of Hermite functions, we find that $\int_{-\infty}^{\infty} K(x, y) \psi_k(y) dy = \lambda_k \psi_k(x)$, and hence, $\psi_n(x)$ and λ_n are the eigenfunctions and eigenvalues of the integral operator $\mathcal{K}$.

3) Operator $\mathcal{K}$ is positive-definite by construction. Moreover, $K(x, y) \leq A \exp(-a(x^2 + y^2))$, where $a > 0$ and $A > 0$ are some constants. Therefore, $K(x, y)$ is square-integrable, and hence, $K(x, y) \in \mathcal{L}_2(\mathbb{R}^2)$. Consequently, operator $\mathcal{K}$ is completely continuous (Kolmogorov and Fomin, 1999, p. 120) and is self-adjoint. Because the operator is positive definite, the operator equation has only zero solution when $f(x) = 0$. Hence, the solution of the integral equation (28) exists and is unique by Fredholm's theorem (Kolmogorov and Fomin, 1999, pp. 117-118). The solution of this equation can then be expanded in series (33), where $\hat{f}_n = \int_{-\infty}^{\infty} f(x) \psi_n(x) dx$ are Fourier coefficients of $f(x) = \eta(\pi_I(x; \varepsilon_H) - \pi_I(x; \varepsilon_L))$. Evaluating the latter integral using Equation 7.374.10 in Gradshteyn and Ryzhik (2007), we find:

$$\hat{f}_n = \frac{\eta\sqrt{a}}{\sqrt{1+a^2\sigma_I^2}} \left(\frac{1-a^2\sigma_I^2}{1+a^2\sigma_I^2}\right)^{n/2} \left(\exp\left(-\frac{0.5\mu^2 a^2}{1+a^2\sigma_I^2}\right) \frac{H_n(a\mu/\sqrt{1-a^4\sigma_I^4})}{(2^n n! \sqrt{\pi})^{1/2}} - \frac{H_n(0)}{(2^n n! \sqrt{\pi})^{1/2}} \right). \quad (\text{A.26})$$

The apparent singularity at $a^2\sigma_I^2 = 1$ in (A.26) is removable because the eigenbasis coefficient $\hat{f}_n$ is defined by a regular convergent integral $\hat{f}_n = \int_{-\infty}^{\infty} f(x) \psi_n(x) dx$. To prove the convergence of series (33) in $\mathcal{L}_2(\mathbb{R})$, using the orthonormality of $\psi_n(x)$, we find that

$$\left\| \sum_{n=m}^{m+k} \frac{\hat{f}_n \psi_n(x)}{\tau + \lambda_n} \right\|_{\mathcal{L}_2(\mathbb{R})}^2 = \sum_{n=m}^{m+k} \frac{\hat{f}_n^2}{(\tau + \lambda_n)^2} < \frac{1}{\tau^2} \sum_{n=m}^{m+k} \hat{f}_n^2.$$

The convergence then follows from Cauchy's convergence test and the fact that $f(x) \in \mathcal{L}_2(\mathbb{R})$, and hence, by Parseval's identity $\sum_{n=0}^{\infty} \hat{f}_n^2 = \|f(x)\|_{\mathcal{L}_2(\mathbb{R})}^2$. Applying Parseval's identity again, we obtain the following expression for the measure of informational efficiency: $J_{\infty}(\tau) := \int_{-\infty}^{\infty} v(x; \tau) f(x) dx = \sum_{n=0}^{\infty} \hat{f}_n^2 / (\tau + \lambda_n)$. The convergence of the latter series follows from the convergence of $\sum_{n=0}^{\infty} \hat{f}_n^2$, and inequality $\sum_{n=0}^{\infty} \hat{f}_n^2 / (\tau + \lambda_n) < (1/\tau) \sum_{n=0}^{\infty} \hat{f}_n^2$. ■

Proof of Proposition 6. We take the limit $\tau \rightarrow 0$ in $J_{\infty}(\tau)$, given by (34), and show that the limit exists and equals $J_{\infty}(0)$ when condition (35) is satisfied. The convergence of $J_{\infty}(\tau)$ to $J_{\infty}(0)$ follows from the monotone convergence theorem for non-negative sequences. Substituting eigenvalues (32) and coefficients (A.26) into (34), and setting $\tau = 0$, we find:

$$\begin{aligned} J_{\infty}(0) &= \frac{\eta^2 a}{\lambda_0 \sqrt{\pi} (1 + a^2 \sigma_I^2)} \sum_{n=0}^{\infty} \rho^n \left(\exp \left(-\frac{0.5 \mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \frac{H_n(y)}{\sqrt{2^n n!}} - \frac{H_n(0)}{\sqrt{2^n n!}} \right)^2 \\ &= \frac{\eta^2 a}{\lambda_0 \sqrt{\pi} (1 + a^2 \sigma_I^2)} \left[\exp \left(-\frac{\mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \sum_{n=0}^{\infty} \frac{\rho^n H_n(y)^2}{2^n n!} \right. \\ &\quad \left. - 2 \exp \left(-\frac{0.5 \mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \sum_{n=0}^{\infty} \frac{\rho^n H_n(y) H_n(0)}{2^n n!} + \sum_{n=0}^{\infty} \frac{\rho^n H_n(0)^2}{2^n n!} \right], \end{aligned} \quad (\text{A.27})$$

where ρ is given by (37) and $y = a\mu/\sqrt{1 - a^4 \sigma_I^4}$. The sums in (A.27) can be computed using formula (A.24) when $|\rho| < 1$. Using some algebra, we verify that $|\rho| < 1$ is equivalent to inequality (35). When the latter inequality is satisfied, the measure of efficiency $J_{\infty}^* = J_{\infty}(0)$ is finite and is given by (36). Hence, the equilibrium is partially revealing.

It then remains to verify that $J_{\infty}(0) = +\infty$ when $|\rho| \geq 1$, and hence, the equilibrium is fully revealing. Using the fact that $H_n(0) = 0$ for odd n (Lebedev, 1972, p. 61, Eq. 4.9.4), from the first equality in (A.27), we obtain:

$$\begin{aligned} J_{\infty}(0) &= A \sum_{n=0}^{\infty} \rho^n \left(\exp \left(-\frac{0.5 \mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \frac{H_n(y)}{\sqrt{2^n n!}} - \frac{H_n(0)}{\sqrt{2^n n!}} \right)^2 \\ &\geq A \exp \left(-\frac{\mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \sum_{n=0}^{\infty} \frac{\rho^{2n+1} H_{2n+1}(y)^2}{2^{2n+1} (2n+1)!}, \end{aligned} \quad (\text{A.28})$$

where $A > 0$ is a constant. We note that $\rho > 0$ and y is real when $1 - a^2 \sigma_I^2 > 0$ and $\rho < 0$ and $y = i\tilde{y}$ is imaginary when $1 - a^2 \sigma_I^2 < 0$, where $\tilde{y} > 0$. We need to encompass both of the latter cases. Note that polynomials $H_n(y)$ have only odd powers y^{2k+1} when n is odd and only even powers y^{2k} when n is even. Consider the case $1 - a^2 \sigma_I^2 < 0$. The closed-form formula for the $(2n+1)$ -th Hermite polynomial (Lebedev, 1972, p. 60, Eq. 4.9.2)

$$H_{2n+1}(y) = (2n+1)! \sum_{l=0}^n \frac{(-1)^l (2y)^{2n+1-2l}}{l! (2n+1-2l)!}$$

implies that for a negative $\rho < 0$ and $y = i\tilde{y}$, we have

$$\rho H_{2n+1}(i\tilde{y})^2 = |\rho| \left((2n+1)! \sum_{l=0}^n \frac{(2|y|)^{2n+1-2l}}{l!(2n+1-2l)!} \right)^2 \geq |\rho| H_{2n+1}(|y|)^2.$$

The latter inequality and inequality (A.28) jointly imply that, for $\rho < 0$,

$$J_\infty(0) \geq A \exp \left(-\frac{\mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \sum_{n=0}^{\infty} \frac{|\rho|^{2n+1} H_{2n+1}(|y|)^2}{2^{2n+1} (2n+1)!}. \quad (\text{A.29})$$

When $\rho \geq 1$, y is real, and inequality (A.29) follows directly from inequality (A.28). Hence, inequality (A.29) holds when $\rho < 0$ or $\rho \geq 1$, and therefore, in particular, whenever $|\rho| \geq 1$. Now, pick an arbitrary $\tilde{\rho}$ such that $|\rho| \geq 1 > \tilde{\rho} > 0$. Inequality (A.29) implies:

$$J_\infty(0) \geq A \exp \left(-\frac{\mu^2 a^2}{1 + a^2 \sigma_I^2} \right) \sum_{n=0}^{\infty} \frac{\tilde{\rho}^{2n+1} H_{2n+1}(|y|)^2}{2^{2n+1} (2n+1)!}. \quad (\text{A.30})$$

Evaluating (A.24) at $u = \tilde{\rho}$ and $u = -\tilde{\rho}$, setting $x = y = |y|$ and taking the difference gives

$$\sum_{n=0}^{\infty} \frac{\tilde{\rho}^{2n+1} H_{2n+1}(|y|)^2}{2^{2n+1} (2n+1)!} = \frac{1}{2\sqrt{1-\tilde{\rho}^2}} \left(\exp \left\{ \frac{2\tilde{\rho}}{1+\tilde{\rho}} |y|^2 \right\} - \exp \left\{ -\frac{2\tilde{\rho}}{1-\tilde{\rho}} |y|^2 \right\} \right). \quad (\text{A.31})$$

The latter identity and inequality (A.30) imply that $J_\infty(0) = +\infty$ because its lower bound converges to infinity when $\tilde{\rho} \uparrow 1$, and hence, the equilibrium is fully revealing when $|\rho| \geq 1$.

Next, we prove that J_∞^* converges to infinity as $\sigma_\nu \downarrow \sqrt{0.5(\sigma_I^2 - \sigma_B^2)}$ or $\sigma_\nu \rightarrow \infty$. Set $\sigma_\nu^0 = \sqrt{0.5(\sigma_I^2 - \sigma_B^2)}$. For fixed $\sigma_I > \sigma_B$, condition (35) is equivalent to $\sigma_\nu > \sigma_\nu^0$. Therefore, $\sigma_\nu \downarrow \sigma_\nu^0$ is the one-sided approach to the boundary from the partially revealing region, where J_∞^* is finite and formula (36) applies; for $\sigma_\nu < \sigma_\nu^0$, we have $J_\infty^* = \infty$.

Let $A = \eta^2 a / [\lambda_0 \sqrt{\pi} (1 + a^2 \sigma_I^2)]$. As $\sigma_\nu \downarrow \sigma_\nu^0$, the definitions of $\hat{\sigma}$, a , ρ , and y give

$$\hat{\sigma} \longrightarrow \frac{\sigma_B \sigma_\nu^0}{\sigma_I}, \quad a^2 \longrightarrow \frac{1}{\sigma_B \sigma_I}, \quad \rho \longrightarrow -1, \quad A \longrightarrow \eta^2 \sqrt{\frac{\sigma_B}{\sigma_I}} > 0, \quad |y| \longrightarrow \frac{\mu \sqrt{\sigma_B / \sigma_I}}{\sqrt{\sigma_I^2 - \sigma_B^2}} > 0.$$

In particular, for $\sigma_\nu > \sigma_\nu^0$ sufficiently close to σ_ν^0 , we have $-1 < \rho < 0$ and $|\rho| \rightarrow 1$ from below. Inequality (A.29) then applies, and its series can be evaluated using (A.31) with $\tilde{\rho} = |\rho|$. In the right-hand side of (A.31), the expression in parentheses converges to a strictly positive finite value, whereas $1/\sqrt{1-|\rho|^2} \rightarrow \infty$. Because the factor $A \exp\{-\mu^2 a^2 / (1 + a^2 \sigma_I^2)\}$ also converges to a strictly positive finite value, inequality (A.29) implies that $J_\infty^* \rightarrow \infty$.

We now investigate J_∞^* when $\sigma_\nu \rightarrow \infty$. First, rewrite Equation (36) as follows: $J_\infty^* = \ell \left(e^{-A_1} - 2e^{-(A_1+A_2)/2} + 1 \right)$, where ℓ (after substituting in λ_0), A_1 , and A_2 are given by:

$$\ell = \frac{\eta^2 (1 + a^2 \sigma_B^2)}{\sqrt{1 - \rho^2} (1 + a^2 \sigma_I^2) a \sigma_B}, \quad A_1 = \frac{\mu^2 a^2}{1 + a^2 \sigma_I^2} \left(1 - \frac{2\rho}{(1+\rho)(1-a^2 \sigma_I^2)} \right), \quad A_2 = \frac{2\rho \mu^2 a^2}{(1 - a^4 \sigma_I^4)(1 - \rho^2)},$$

and a and ρ are given in (37). Taking the limit $\sigma_\nu \rightarrow \infty$, after some algebra, we find that

$$\lim_{\sigma_\nu \rightarrow \infty} \frac{\ell}{\sigma_\nu} = \frac{\eta^2}{\sqrt{2(\sigma_I^2 - \sigma_B^2)}}, \quad \lim_{\sigma_\nu \rightarrow \infty} A_1 = 0, \quad \lim_{\sigma_\nu \rightarrow \infty} A_2 = \frac{\mu^2}{2(\sigma_I^2 - \sigma_B^2)}. \quad (\text{A.32})$$

Therefore, J_∞^* is asymptotically linear in σ_ν , and hence, converges to infinity as $\sigma_\nu \rightarrow \infty$. ■

Proof of Proposition 7. At the terminal date, all wealth is consumed, so that $W_{i,1}(\omega) = C_{1,i}^*(\omega)$. For each state ω_m , optimal consumptions in (7) imply

$$\ln(W_{I,1}(\omega_m)/W_0) - \ln(W_{U,1}(\omega_m)/W_0) = \ln(\pi_{I,m}(\varepsilon)/\pi_{U,m}). \quad (\text{A.33})$$

Conditional on ε , the true probability of state ω_m is $\pi_{I,m}(\varepsilon)$. Therefore, taking expectations under the true ex ante joint distribution yields $\Delta_{\text{IU}} = \mathbb{E}\left[\sum_{m=1}^M \pi_{I,m}(\varepsilon) \ln(\pi_{I,m}(\varepsilon)/\pi_{U,m})\right]$. The expression inside the bracket is the discrete Kullback–Leibler divergence. Its continuous-state limit is (40), which proves (39); nonnegativity follows from Gibbs’ inequality.

Because all investors choose the same date-0 consumption, (38) and the law of iterated expectations imply

$$\mathbb{E}[U_I(W_0)] - \mathbb{E}[U_U(W_0)] = \beta \Delta_{\text{IU}}, \quad (\text{A.34})$$

where $U_I(W_0)$ and $U_U(W_0)$ denote investor utilities. Moreover, logarithmic utility and the linearity of optimal consumption in initial wealth imply $U_I((1-\delta)W_0) = U_I(W_0) + (1+\beta) \ln(1-\delta)$. The latter equation, (A.34), and the indifference condition $\mathbb{E}[U_U(W_0)] = \mathbb{E}[U_I((1-\delta)W_0)]$ give $-(1+\beta)/(\beta) \ln(1-\delta) = \Delta_{\text{IU}} = \mathbb{E}[D_{\text{KL}}(\pi_I \parallel \pi_U)]$. Solving the latter for δ gives (41).

It remains to derive the expected divergence in closed form. Since $\pi_{U,m} = \mathbb{E}[\pi_{I,m}(\varepsilon) | \hat{s}]$, the law of iterated expectations implies $\mathbb{E}\left[\sum_{m=1}^M \pi_{I,m}(\varepsilon) \ln \pi_{U,m}\right] = \mathbb{E}\left[\sum_{m=1}^M \pi_{U,m} \ln \pi_{U,m}\right]$. Consequently, passing to the continuous-state limit gives

$$\mathbb{E}\left[D_{\text{KL}}(\pi_I \parallel \pi_U)\right] = \mathbb{E}\left[\int_{-\infty}^{\infty} \pi_I(x; \varepsilon) \ln \pi_I(x; \varepsilon) dx\right] - \mathbb{E}\left[\int_{-\infty}^{\infty} \pi_U(x; \hat{s}) \ln \pi_U(x; \hat{s}) dx\right]. \quad (\text{A.35})$$

Substituting $\pi_I(x; \varepsilon)$ from (29) into (A.35), we find that $\mathbb{E}\left[\int_{-\infty}^{\infty} \pi_I(x; \varepsilon) \ln \pi_I(x; \varepsilon) dx\right] = -\ln(\sqrt{2\pi}\sigma_I) - 0.5$. Next, we compute the second term on the right-hand side of (A.35). We use a recentered sufficient statistic $\hat{s} = \tilde{f}\varepsilon + \xi$, where $\xi \sim \mathcal{N}(0, \Sigma_\xi)$ and $\tilde{f} \in \mathbb{R}^{M-1}$ is a vector with elements $f_m = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L))$, and ε takes two values, $\varepsilon_L = 0$ and $\varepsilon_H = 1$.

We note that $\pi_U(x; \hat{s}) = \mathbb{E}[\pi_I(x; \varepsilon) | \hat{s}]$. Because $\varepsilon \in \{0, 1\}$, the latter expression can be rewritten as $\pi_U(x; \hat{s}) = \pi_I(x; 1) + (\pi_I(x; 0) - \pi_I(x; 1))\mathbb{E}[1 - \varepsilon | \hat{s}] = \pi_I(x; 1) + (\pi_I(x; 0) - \pi_I(x; 1))\hat{\pi}_L(\hat{s})$, where $\hat{\pi}_L(\hat{s}) = \mathbb{E}[1 - \varepsilon | \hat{s}]$ and is given by equation (A.19). Noting that in the latter equation $\tilde{f}^\top \Sigma_\xi^{-1} \hat{s} = Z\sqrt{J_\infty^*} + \varepsilon J_\infty^*$, where Z is a standard normal random variable, after substituting $\pi_U(x; \hat{s})$ into (A.35) and computing the expectation, we find that:

$$\mathbb{E}\left[D_{\text{KL}}(\pi_I \parallel \pi_U)\right] = -\frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\varphi_L G(z\sqrt{J_\infty^*} - 0.5J_\infty^*) + \varphi_H G(z\sqrt{J_\infty^*} + 0.5J_\infty^*)\right] e^{-z^2/2} dz, \quad (\text{A.36})$$

where the function $G(z)$ is given by:

$$G(z) = \frac{1}{\sqrt{2\pi}\sigma_I} \int_{-\infty}^{+\infty} \left(e^{-\frac{(x-\mu)^2}{2\sigma_I^2}} + \left[e^{-\frac{x^2}{2\sigma_I^2}} - e^{-\frac{(x-\mu)^2}{2\sigma_I^2}} \right] \frac{1}{1 + \frac{\varphi_H}{\varphi_L} e^z} \right) \times \ln \left(e^{-\frac{(x-\mu)^2}{2\sigma_I^2}} + \left[e^{-\frac{x^2}{2\sigma_I^2}} - e^{-\frac{(x-\mu)^2}{2\sigma_I^2}} \right] \frac{1}{1 + \frac{\varphi_H}{\varphi_L} e^z} \right) dx. \quad \blacksquare \quad (\text{A.37})$$

Proof of Proposition 8. Let the state of the economy be $x_m = \ln(D_m/D_0)$. Then, as the number of states converges to infinity, $M \rightarrow \infty$, Equation (11) for the risk-free rate becomes $r_f = (1/\beta)/\left(\int_{-\infty}^{\infty} \bar{\pi}(x; \hat{s}) e^{-x} dx\right) - 1$, where $\bar{\pi}(x; \hat{s}) = \mu_B \bar{\mathbb{E}}[\pi_B(x; \nu)] + \mu_U \mathbb{E}[\pi_I(x; \varepsilon) | \hat{s}]$, and $\pi_B(x; \nu)$ and $\pi_I(x; \varepsilon)$ are densities given by (29). Because ε is binary, $\pi_I(x; \varepsilon) = \pi_I(x; 1)\varepsilon + \pi_I(x; 0)(1 - \varepsilon)$, and hence, $\bar{\pi}(x; \hat{s}) = \mu_B \bar{\mathbb{E}}[\pi_B(x; \nu)] + \mu_U [\pi_I(x; 0)\hat{\pi}_L(\hat{s}) + \pi_I(x; 1)(1 - \hat{\pi}_L(\hat{s}))]$, where $\hat{\pi}_L(\hat{s}) = \text{Prob}(\varepsilon = \varepsilon_L | \hat{s})$ and is given by Equation (A.19). Next, we compute

$$\bar{\mathbb{E}}[\pi_B(x; \nu)] = \frac{1}{2\pi\sigma_B\sigma_\nu} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\nu)^2}{2\sigma_B^2} - \frac{\nu^2}{2\sigma_\nu^2}\right\} d\nu = \frac{1}{\sqrt{2\pi(\sigma_\nu^2 + \sigma_B^2)}} \exp\left\{-\frac{x^2}{2(\sigma_\nu^2 + \sigma_B^2)}\right\}.$$

Consequently, the average density $\bar{\pi}(x; \hat{s})$ can be rewritten as follows:

$$\begin{aligned} \bar{\pi}(x; \hat{s}) &= \frac{\mu_B}{\sqrt{2\pi(\sigma_\nu^2 + \sigma_B^2)}} \exp\left\{-\frac{x^2}{2(\sigma_\nu^2 + \sigma_B^2)}\right\} + \frac{\mu_U}{\sqrt{2\pi\sigma_I^2}} \exp\left\{-\frac{x^2}{2\sigma_I^2}\right\} \hat{\pi}_L(\hat{s}) \\ &\quad + \frac{\mu_U}{\sqrt{2\pi\sigma_I^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma_I^2}\right\} (1 - \hat{\pi}_L(\hat{s})). \end{aligned} \quad (\text{A.38})$$

Next, we compute the following integral:

$$\int_{-\infty}^{\infty} \bar{\pi}(x; \hat{s}) e^{-x} dx = \mu_B \exp\{0.5(\sigma_\nu^2 + \sigma_B^2)\} + \mu_U \left[\exp\{0.5\sigma_I^2\} \hat{\pi}_L(\hat{s}) + \exp\{-\mu + 0.5\sigma_I^2\} (1 - \hat{\pi}_L(\hat{s})) \right].$$

This integral is computed noting that the density $\bar{\pi}(x; \hat{s})$ is a weighted sum of normal densities and that for normally distributed variables x we have $\mathbb{E}[\exp\{-x\}] = \exp\{-\mathbb{E}[x] + 0.5 \text{var}[x]\}$. Substituting the latter integral into the expression for r_f above, we find the rate (42). $\blacksquare$

Online Appendix for

“Informational Efficiency and Asset Prices in Large Markets”

I.1. Incomplete market with an index of securities

Environment and market structure. Consider a two-date economy, $t = 0, 1$, with $\widehat{M}$ states, populated by informed, uninformed, and biased investors. The informed and biased investors have the probabilities of states that have the same structure as in the baseline analysis of Section 2.1. The aggregate output is also given by the same Lucas tree as in Section 2.1. The investors can freely trade securities for states $m = 1, \dots, M$. However, in contrast to the baseline analysis, the market is incomplete in the sense that individual securities for states $m = M + 1, \dots, \widehat{M}$ are not traded, where $1 \leq M < \widehat{M} - 1$. Instead, the investors can trade a dividend-weighted index of those securities with weights $w_m = D_m / \sum_{k=M+1}^{\widehat{M}} D_k$. The payoff of the index at the final date is then given by $w_{M+1} 1_{\{\omega=\omega_{M+1}\}} + \dots + w_{\widehat{M}} 1_{\{\omega=\omega_{\widehat{M}}\}}$. The index is in net supply of $(D_{M+1} + \dots + D_{\widehat{M}})$ units. All investors are endowed with identical quantities of Arrow–Debreu securities for states $m = 1, \dots, M$ and with identical quantities of the index, so that endowments sum up to the aggregate supply of traded securities. The investors maximize the intertemporal logarithmic utility function (1) subject to the following budget constraints:

$$C_{0,i} = (1 - \theta_{i,1} - \dots - \theta_{i,M} - \theta_{i,\text{Ind}})W_0, \quad (\text{I.1})$$

$$\begin{aligned} C_{1,i}(\omega) &= \frac{\theta_{i,1}W_0}{p_1} 1_{\{\omega=\omega_1\}} + \dots + \frac{\theta_{i,M}W_0}{p_M} 1_{\{\omega=\omega_M\}} \\ &\quad + \frac{\theta_{i,\text{Ind}}W_0}{p_{\text{Ind}}} \left(w_{M+1} 1_{\{\omega=\omega_{M+1}\}} + \dots + w_{\widehat{M}} 1_{\{\omega=\omega_{\widehat{M}}\}} \right), \end{aligned} \quad (\text{I.2})$$

where $1_{\{\omega=\omega_m\}}$ is an indicator, i indexes an investor, and $\theta_{i,\text{Ind}}$ is the fraction of wealth invested in the index, p_m are prices of Arrow–Debreu securities, and p_{Ind} is the price of the index. All states $m > M$ can be aggregated into a single traded direction indexed by their total probability mass, which allows solving investors’ optimization problems in closed form.

Proposition I.1 *Consider an investor with state probabilities $\pi_m(z)$ that can depend on a parameter z . Optimal date-0 consumptions of all investors are given by $C_{0,i} = W_0/(1 + \beta)$. Optimal date-1 consumption in state ω_m is given by*

$$C_m(z) = \begin{cases} \ell \frac{\pi_m(z)}{p_m} W_0, & m = 1, \dots, M, \\ \ell \frac{\pi_{\text{Ind}}(z)}{p_{\text{Ind}}} W_0 w_m, & m = M + 1, \dots, \widehat{M}, \end{cases}$$

and the optimal shares of wealth invested in state- m security and the index are given by

$$\theta_m = \ell \pi_m(z), m = 1, \dots, M, \quad \theta_{\text{Ind}} = \ell \pi_{\text{Ind}}(z), \quad (\text{I.3})$$

where $w_m = D_m/(D_{M+1} + \dots + D_{\widehat{M}})$, $\ell = \beta/(1 + \beta)$, and $\pi_{\text{Ind}}(z) = \sum_{m=M+1}^{\widehat{M}} \pi_m(z)$ is the probability that the index payoff is non-zero.

Using Equations (I.3) and the fact that the uninformed investors observe prices and combined demands of informed and biased investors, we obtain the following sufficient statistics:

$$s_m = \mu_I \pi_{I,m}(\varepsilon) + \mu_B \frac{\pi_{B,m}(\nu_1) + \dots + \pi_{B,m}(\nu_N)}{N}, \quad (\text{I.4})$$

where $m = 1, \dots, M$, and the sufficient statistic corresponding to the index is given by $s_{\text{Ind}} = \mu_I + \mu_B - \sum_{m=1}^M s_m$. The latter index statistic does not contain any new information relative to statistics (I.4). We denote the vector of relevant statistics by $s = (s_1, \dots, s_M)^\top$.

Proposition I.2 [Equilibrium: incomplete market with an index]. *Let $D_{\text{Ind}} = \sum_{k=M+1}^{\widehat{M}} D_k$ and $\bar{\pi}_{\text{Ind}}(s) = \sum_{m=M+1}^{\widehat{M}} \bar{\pi}_m(s)$. The prices of securities are then given by*

$$p_m = \beta \frac{\bar{\pi}_m(s)}{D_m/D_0}, \quad m = 1, \dots, M, \quad (\text{I.5})$$

$$p_{\text{Ind}} = \beta \frac{\bar{\pi}_{\text{Ind}}(s)}{D_{\text{Ind}}/D_0}, \quad (\text{I.6})$$

where $\bar{\pi}_m(s)$ is the average probability across investors as a function of statistics s , given by

$$\bar{\pi}_m(s) = s_m + \mu_U \mathbb{E}[\pi_{I,m}(\varepsilon)|s], \quad m = 1, \dots, M,$$

and $\bar{\pi}_{\text{Ind}}(s) \equiv 1 - \sum_{m=1}^M \bar{\pi}_m(s)$. Because the market is incomplete, state prices and risk-neutral probabilities over primitive states are not unique. The prices of traded securities pin down state prices for states $m = 1, \dots, M$ and one weighted average of state prices for states $m = M + 1, \dots, \widehat{M}$, but not the individual state prices within the index.

Proof of Proposition I.1. Given portfolio shares $(\theta_1, \dots, \theta_M, \theta_{\text{Ind}})$, consumption is

$$C_0 = W_0 \left(1 - \sum_{m=1}^M \theta_m - \theta_{\text{Ind}} \right), \quad C_m = \begin{cases} \frac{\theta_m W_0}{p_m}, & m = 1, \dots, M, \\ \frac{\theta_{\text{Ind}} W_0}{p_{\text{Ind}}} w_m, & m = M + 1, \dots, \widehat{M}. \end{cases}$$

Substituting into expected utility and collecting terms yields

$$\ln \left(1 - \sum_{m=1}^M \theta_m - \theta_{\text{Ind}} \right) + \beta \sum_{m=1}^M \pi_m(z) \ln \theta_m + \beta \left(\sum_{m=M+1}^{\widehat{M}} \pi_m(z) \right) \ln \theta_{\text{Ind}} + \text{const.}$$

The objective is strictly concave, and the first-order conditions imply

$$\frac{1}{1 - \sum_{m=1}^M \theta_m - \theta_{\text{Ind}}} = \frac{\beta \pi_m(z)}{\theta_m} \quad m = 1, \dots, M, \quad \frac{1}{1 - \sum_{m=1}^M \theta_m - \theta_{\text{Ind}}} = \frac{\beta \pi_{\text{Ind}}(z)}{\theta_{\text{Ind}}}.$$

The latter first-order conditions in turn imply that $\theta_m = \ell \pi_m(z)$, $m = 1, \dots, M$, $\theta_{\text{Ind}} = \ell \pi_{\text{Ind}}(z)$, for some $\ell > 0$. Using $\sum_{m=1}^M \pi_m(z) + \pi_{\text{Ind}}(z) = 1$, we obtain $\sum_{m=1}^M \theta_m + \theta_{\text{Ind}} = \ell$. Then the F.O.C. yields $1/(1 - \ell) = \beta/\ell$, and hence, $\ell = \beta/(1 + \beta)$, completing the proof. ■

Proof of Proposition I.2. By Proposition I.1, an investor with beliefs $\pi_m(z)$ invests the fraction $\theta_m(z) = \ell \pi_m(z)$ in Arrow–Debreu (AD) security $m = 1, \dots, M$ and the fraction $\theta_{\text{Ind}}(z) = \ell \pi_{\text{Ind}}(z)$ in the index, where $\ell = \beta/(1 + \beta)$ and $\pi_{\text{Ind}}(z) = \sum_{m=M+1}^{\widehat{M}} \pi_m(z)$. Hence, the quantity of AD security m held by this investor is $\ell \pi_m(z) W_0/p_m$, and the number of index shares held is $\ell \pi_{\text{Ind}}(z) W_0/p_{\text{Ind}}$.

Market clearing for AD security $m = 1, \dots, M$ implies $\ell (W_0/p_m) \bar{\pi}_m(s) = D_m$, where $\bar{\pi}_m(s)$ is the average probability across investors. Using date-0 market clearing, $W_0 = (1 + \beta)D_0$, and $\ell = \beta/(1 + \beta)$ yields $p_m = \beta \bar{\pi}_m(s) D_0/D_m$, $m = 1, \dots, M$.

The index delivers $w_m = D_m/D_{\text{Ind}}$ in state $m > M$ and is in net supply D_{Ind} . Thus its aggregate payoff coincides with the aggregate output in all states $m > M$, and market clearing implies $\ell (W_0/p_{\text{Ind}}) \bar{\pi}_{\text{Ind}}(s) = D_{\text{Ind}}$, which gives $p_{\text{Ind}} = \beta \bar{\pi}_{\text{Ind}}(s) D_0/D_{\text{Ind}}$.

Finally, the market is incomplete because individual Arrow–Debreu securities for states $m = M+1, \dots, \widehat{M}$ are not traded. Therefore, state prices and risk-neutral probabilities over primitive states are not unique. The prices of traded Arrow–Debreu securities pin down state prices $q_m = p_m$ for $m = 1, \dots, M$, while state prices for states $m > M$ only have to price the index: $\sum_{m=M+1}^{\widehat{M}} w_m q_m = p_{\text{Ind}}$. Thus, the traded securities impose one pricing restriction on the state prices for the bundled states, but do not pin down these state prices separately. ■

I.2. Economy with CRRA preferences

We here study the economy in which all investors have constant relative risk aversion preferences (CRRA) with risk aversion parameter $\gamma \neq 1$. The results for log preferences are obtained as a special case by taking a limit $\gamma \rightarrow 1$. The rest of the details are as in the baseline analysis. In particular, the informed investors have beliefs $\pi_m(\varepsilon)$ and group- n biased investors have beliefs $\pi_m(\nu_n)$. The masses of these groups of investors are μ_I and μ_B/N , and there are N groups of biased investors. All investors have equal endowments of date-0 output and the Arrow–Debreu securities. First, we solve the portfolio choice problem of investors, and then find conditions for a fully revealing equilibrium. We first retain the baseline information set, in which uninformed investors observe asset prices and the combined demands of informed and biased investors. We then show that, under a separate mass condition, the combined demands are redundant because equilibrium prices alone reveal the information.

Consider a representative investor with CRRA preferences over date-0 consumption C_0 and date-1 state-contingent consumptions $\{C_m\}_{m=1}^M$, and subjective beliefs $\pi_m(z)$, where z is a belief parameter. The investor solves the following portfolio choice problem:

$$\max_{C_0, \{C_m\}_{m=1}^M} \frac{C_0^{1-\gamma}}{1-\gamma} + \beta \sum_{m=1}^M \pi_m(z) \frac{C_m^{1-\gamma}}{1-\gamma} \quad \text{s.t.} \quad C_0 + \frac{1}{1+r_f} \sum_{m=1}^M \pi_m(z) \frac{\pi_m^{\text{RN}}}{\pi_m(z)} C_m = W_0, \quad (\text{I.7})$$

where $\gamma > 0$ is risk aversion, $\beta \in (0, 1)$ is discounting, π_m^{RN} are the risk-neutral probabilities, r_f is the risk-free rate, and W_0 is initial wealth.

A full-information equilibrium exists by the standard excess-demand result in ?, Proposition 17.C.1. At strictly positive complete prices, the CRRA first-order conditions give unique continuous demands. Aggregate excess demand is therefore continuous and homogeneous of degree zero, satisfies Walras' law, and is bounded below by minus aggregate endowments. Because all utility weights and endowment components are strictly positive, at least one demand component diverges whenever normalized prices approach a nonzero boundary vector. These are the excess-demand properties listed in Proposition 17.B.2 of the same reference, so the cited result yields a Walrasian equilibrium. This establishes existence only; uniqueness is proved below.

Proposition I.3 (Optimal consumptions and trading strategies) *Let π_m^{RN} denote risk-neutral probabilities. The optimal date-0 and date-1 consumptions in states $m = 1, \dots, M$ are given by*

$$C_m = C_0 (\beta(1+r_f))^{1/\gamma} \left(\frac{\pi_m(z)}{\pi_m^{\text{RN}}} \right)^{1/\gamma}, \quad (\text{I.8})$$

$$C_0 = \frac{W_0}{1 + \frac{(\beta(1+r_f))^{1/\gamma}}{1+r_f} \sum_{m=1}^M (\pi_m^{\text{RN}})^{1-\frac{1}{\gamma}} \pi_m(z)^{\frac{1}{\gamma}}}. \quad (\text{I.9})$$

The optimal fraction of wealth invested in state- m Arrow–Debreu security is given by:

$$\theta_m = \frac{C_0 \beta}{W_0} \left(\frac{\beta(1+r_f)}{\pi_m^{\text{RN}}} \right)^{1/\gamma-1} \pi_m(z)^{1/\gamma}. \quad (\text{I.10})$$

Proof of Proposition I.3. Form the Lagrangian

$$\mathcal{L} = \frac{C_0^{1-\gamma}}{1-\gamma} + \beta \sum_{m=1}^M \pi_m(z) \frac{C_m^{1-\gamma}}{1-\gamma} - \ell \left(C_0 + \frac{1}{1+r_f} \sum_{m=1}^M \pi_m(z) \frac{\pi_m^{\text{RN}}}{\pi_m(z)} C_m - W_0 \right),$$

with multiplier $\ell > 0$. The first-order conditions are

$$C_0^{-\gamma} = \ell, \quad (\text{I.11})$$

$$\beta \pi_m(z) C_m^{-\gamma} = \ell \cdot \frac{1}{1+r_f} \pi_m(z) \frac{\pi_m^{\text{RN}}}{\pi_m(z)} = \ell \cdot \frac{1}{1+r_f} \pi_m^{\text{RN}}, \quad m = 1, \dots, M. \quad (\text{I.12})$$

Substituting $\ell = C_0^{-\gamma}$ into (I.12) gives $\beta \pi_m(z) C_m^{-\gamma} = C_0^{-\gamma} \cdot \pi_m^{\text{RN}} / (1 + r_f)$, which yields (I.8).

Substituting (I.8) into the budget constraint in (I.7), we obtain

$$\begin{aligned} W_0 &= C_0 + \frac{1}{1 + r_f} \sum_{m=1}^M \pi_m^{\text{RN}} C_m = C_0 + \frac{1}{1 + r_f} \sum_{m=1}^M \pi_m^{\text{RN}} \left[C_0 [\beta(1 + r_f)]^{1/\gamma} \left(\frac{\pi_m(z)}{\pi_m^{\text{RN}}} \right)^{1/\gamma} \right] \\ &= C_0 \left[1 + \frac{[\beta(1 + r_f)]^{1/\gamma}}{1 + r_f} \sum_{m=1}^M (\pi_m^{\text{RN}})^{1 - \frac{1}{\gamma}} \pi_m(z)^{\frac{1}{\gamma}} \right], \end{aligned}$$

and solving for C_0 gives (I.9). The fraction of wealth invested in state- m security is

$$\theta_m = \frac{1}{W_0} \frac{\pi_m^{\text{RN}}}{1 + r_f} C_m = \frac{C_0 \beta}{W_0} \left(\frac{\beta(1 + r_f)}{\pi_m^{\text{RN}}} \right)^{1/\gamma - 1} \pi_m(z)^{1/\gamma},$$

which gives (I.10). This completes the proof. $\blacksquare$

Proposition I.4 [Almost-sure full revelation and price invertibility]. *Suppose all investors have CRRA preferences with risk aversion $\gamma > 0$, $\gamma \neq 1$, and the informed and biased investors have probabilities from the same exponential family, given by (14), where all x_m are distinct, albeit with different parameters z across investors ($z = \varepsilon$ for the informed and $z = \nu_n$ for the n -th biased investor). Suppose also that $\varepsilon, \nu_1, \dots, \nu_N$ are independent and have continuous probability density functions, and $\mu_I, \mu_B > 0$. If $2N + 1 < M$, then the following statements hold.*

- 1) *If $\mu_I \neq \mu_B/N$, then, for almost every realization $(\varepsilon, \nu_1, \dots, \nu_N)$, every alternative vector $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ that generates the same asset prices and combined demands of informed and biased investors satisfies $\tilde{\varepsilon} = \varepsilon$, and $(\tilde{\nu}_1, \dots, \tilde{\nu}_N)$ is a permutation of $(\nu_1, \dots, \nu_N)$.*
- 2) *If $\mu_I + \mu_U \neq \mu_B/N$, then the full-information equilibrium price vector $P(\varepsilon, \nu_1, \dots, \nu_N)$ is unique and almost surely invertible in ε and the cross-section of biased beliefs, up to permutation. Specifically, for almost every realization $(\varepsilon, \nu_1, \dots, \nu_N)$, $P(\varepsilon, \nu_1, \dots, \nu_N) = P(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ implies that $\tilde{\varepsilon} = \varepsilon$, and $(\tilde{\nu}_1, \dots, \tilde{\nu}_N)$ is a permutation of $(\nu_1, \dots, \nu_N)$. Consequently, there exists a fully revealing rational expectations equilibrium in which uninformed investors learn from asset prices alone.*

In both cases, the cross-section of biased beliefs is identified up to a permutation.

Proof of Proposition I.4.

Part 1). Given optimal portfolios (I.10), the uninformed investors who observe asset prices and combined demands of informed and biased investors can recover the following statistics:

$$s_m = \mu_I C_{0,I} \pi_{I,m}(\varepsilon)^{1/\gamma} + \frac{\mu_B}{N} \left(C_{0,1} \pi_{B,m}(\nu_1)^{1/\gamma} + \dots + C_{0,N} \pi_{B,m}(\nu_N)^{1/\gamma} \right), \quad (\text{I.13})$$

where $C_{0,I}$ and $C_{0,n}$ are optimal date-0 consumptions, given by (I.9). These date-0 consumptions depend on prices, ε , and $\nu_1, \dots, \nu_N$ through risk-neutral and physical probabilities.

Because the shock and beliefs are independent and have continuous probability density functions, $\varepsilon, \nu_1, \dots, \nu_N$ are pairwise distinct with probability one. Fix a realization in this probability-one set, and let $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$ be an arbitrary alternative vector that generates the same asset prices and combined demands of informed and biased investors. The equality of these observables implies that the two vectors satisfy the following system of M equations. We first show that $\tilde{\varepsilon} = \varepsilon$. Suppose, to the contrary, that $\tilde{\varepsilon} \neq \varepsilon$:

$$\begin{aligned} \mu_I C_{0,I} \pi_{I,m}(\varepsilon)^{1/\gamma} + \frac{\mu_B}{N} \left(C_{0,1} \pi_{B,m}(\nu_1)^{1/\gamma} + \dots + C_{0,N} \pi_{B,m}(\nu_N)^{1/\gamma} \right) &= \\ \mu_I \tilde{C}_{0,I} \pi_{I,m}(\tilde{\varepsilon})^{1/\gamma} + \frac{\mu_B}{N} \left(\tilde{C}_{0,1} \pi_{B,m}(\tilde{\nu}_1)^{1/\gamma} + \dots + \tilde{C}_{0,N} \pi_{B,m}(\tilde{\nu}_N)^{1/\gamma} \right), \end{aligned} \quad (\text{I.14})$$

for $m = 1, \dots, M$. Substituting probabilities $\pi_{I,m}(z)$ and $\pi_{B,m}(z)$ from (14) into the system (I.14), after cancelling $\exp(a(x_m)/\gamma)$ on both sides, we obtain the system of equations

$$\begin{aligned} \mu_I C_{0,I} e^{\varepsilon x/\gamma - c(\varepsilon)/\gamma} + (\mu_B/N) (C_{0,1} e^{\nu_1 x/\gamma - c(\nu_1)/\gamma} + \dots + C_{0,N} e^{\nu_N x/\gamma - c(\nu_N)/\gamma}) &= \\ \mu_I \tilde{C}_{0,I} e^{\tilde{\varepsilon} x/\gamma - c(\tilde{\varepsilon})/\gamma} + (\mu_B/N) (\tilde{C}_{0,1} e^{\tilde{\nu}_1 x/\gamma - c(\tilde{\nu}_1)/\gamma} + \dots + \tilde{C}_{0,N} e^{\tilde{\nu}_N x/\gamma - c(\tilde{\nu}_N)/\gamma}), \end{aligned} \quad (\text{I.15})$$

where $x \in \{x_1, \dots, x_M\}$ and all x_m are distinct. After moving all terms in (I.15) to one side and combining identical exponents, the resulting expression contains at most $2N + 2$ distinct exponential terms. If it were not identically zero, Lemma A.1 would imply that it has at most $2N + 1$ distinct roots. Equation (I.15), however, holds at the $M > 2N + 1$ distinct values $x_1, \dots, x_M$. Therefore, the left- and right-hand sides of (I.15) coincide for all real x .

Because the fixed realization $\varepsilon, \nu_1, \dots, \nu_N$ is pairwise distinct and $\tilde{\varepsilon} \neq \varepsilon$, linear independence of the exponential functions implies that the exponent with parameter $\tilde{\varepsilon}$ on the right-hand side of (I.15) must match the unique exponent with parameter ν_k on the left-hand side for some k . Hence, $\tilde{\varepsilon} = \nu_k$. Let r be the number of exponents on the right-hand side with parameters $\tilde{\nu}_n = \tilde{\varepsilon}$. Equality of the coefficients on the two sides then implies

$$\frac{\mu_B}{N} C_{0,k} = \mu_I \tilde{C}_{0,I} + \frac{\mu_B}{N} \sum_{\{n: \tilde{\nu}_n = \tilde{\varepsilon}\}} \tilde{C}_{0,n}.$$

Equation (I.9), equality of all investors' endowments, equality of prices under the two parameter vectors, $(\varepsilon, \nu_1, \dots, \nu_N)$ and $(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N)$, and $\tilde{\varepsilon} = \nu_k$ imply that $\tilde{C}_{0,I} = C_{0,k}$. Moreover, if $\tilde{\nu}_n = \tilde{\varepsilon}$, then $\tilde{C}_{0,n} = \tilde{C}_{0,I} = C_{0,k}$. Therefore, the coefficient equality becomes

$$\frac{\mu_B}{N} C_{0,k} = \left(\mu_I + r \frac{\mu_B}{N} \right) C_{0,k}.$$

If $r = 0$, this equality implies $\mu_I = \mu_B/N$, contradicting the assumption of the proposition. If $r \geq 1$, the right-hand side is strictly larger than the left-hand side because $\mu_I > 0$ and $C_{0,k} > 0$. Thus, both cases yield a contradiction, proving that every observationally equivalent vector satisfies $\tilde{\varepsilon} = \varepsilon$.

It remains to identify the cross-section of biased beliefs. For the common price vector P , let $C_0(z; P)$ denote the date-0 consumption in Equation (I.9) of an investor with belief parameter z . Because the two vectors generate the same prices and all investors have equal endowments, W_0 , r_f , and the risk-neutral probabilities are the same for both vectors. Hence, $C_0(z; P) > 0$ is the same function of z for both vectors. Since $\tilde{\varepsilon} = \varepsilon$, the informed-investor terms in (I.15) are identical. Subtracting them and cancelling the positive factor μ_B/N gives

$$\sum_{n=1}^N C_0(\nu_n; P) e^{\nu_n x / \gamma - c(\nu_n) / \gamma} = \sum_{n=1}^N C_0(\tilde{\nu}_n; P) e^{\tilde{\nu}_n x / \gamma - c(\tilde{\nu}_n) / \gamma}, \quad x \in \{x_1, \dots, x_M\}. \quad (\text{I.16})$$

If the two exponential sums in (I.16) were not identical for every real x , their difference would contain at most $2N$ distinct exponential terms and, by Lemma A.1, would have at most $2N - 1$ distinct roots. This contradicts the fact that the equality holds at the $M > 2N + 1$ distinct values $x_1, \dots, x_M$. Thus, the two sums are identical for every real x .

For all z , let $k(z)$ and $\tilde{k}(z)$ denote its multiplicities among $\nu_1, \dots, \nu_N$ and $\tilde{\nu}_1, \dots, \tilde{\nu}_N$, respectively. Linear independence of the exponentials implies $k(z)C_0(z; P) \exp(-c(z)/\gamma) = \tilde{k}(z)C_0(z; P) \exp(-c(z)/\gamma)$ for every z in either vector. Because $C_0(z; P) \exp(-c(z)/\gamma) > 0$, we have $k(z) = \tilde{k}(z)$ for every such z . Hence, $(\tilde{\nu}_1, \dots, \tilde{\nu}_N)$ is a permutation of $(\nu_1, \dots, \nu_N)$. This proves statement 1).

Part 2). Existence does not imply uniqueness, which we establish directly from individual optimality and market clearing. For belief parameter z , use the following strictly increasing transformation of the CRRA utility in (I.7):

$$f_z(C) = \frac{1}{1-\gamma} \ln \left(C_0^{1-\gamma} + \beta \sum_{m=1}^M \pi_m(z) C_m^{1-\gamma} \right), \quad C = (C_0, C_1, \dots, C_M)^\top. \quad (\text{I.17})$$

On the strictly positive orthant, f_z is twice continuously differentiable and strictly increasing in every component. Moreover, $f_z(tC) = f_z(C) + \ln t$, so differentiation with respect to t at $t = 1$ gives $C^\top \nabla_C f_z(C) = 1$. For every nonzero vector $h \in \mathbb{R}^{M+1}$, direct differentiation gives

$$h^\top \nabla_C^2 f_z(C) h = -\gamma \sum_{m=0}^M \frac{\partial f_z(C)}{\partial C_m} \frac{h_m^2}{C_m} - (1-\gamma) [\nabla_C f_z(C)^\top h]^2 < 0. \quad (\text{I.18})$$

Differentiating, $C_0 \partial f_z(C) / \partial C_0$ and $C_m \partial f_z(C) / \partial C_m$, $m \geq 1$, are $C_0^{1-\gamma}$ and $\beta \pi_m(z) C_m^{1-\gamma}$ times $[C_0^{1-\gamma} + \beta \sum_{j=1}^M \pi_j(z) C_j^{1-\gamma}]^{-1} > 0$, respectively. Hence, they are positive and sum to one.

For $0 < \gamma < 1$, (I.18)'s two terms are strictly negative and nonpositive, respectively. If $\gamma > 1$, weighted Cauchy-Schwarz gives

$$[\nabla_C f_z(C)^\top h]^2 = \left[\sum_{m=0}^M \left(C_m \frac{\partial f_z(C)}{\partial C_m} \right) \frac{h_m}{C_m} \right]^2 \leq \sum_{m=0}^M \frac{\partial f_z(C)}{\partial C_m} \frac{h_m^2}{C_m}.$$

Substitution into (I.18) bounds its right-hand side above by minus the final sum, which is strictly negative for $h \neq 0$. Hence, $\nabla_C^2 f_z(C)$ is negative definite for every $\gamma > 0$, $\gamma \neq 1$.

Suppose that $P = (p_1, \dots, p_M)^\top$ and $\tilde{P} = (\tilde{p}_1, \dots, \tilde{p}_M)^\top$ are two full-information equilibrium price vectors for the same realization $(\varepsilon, \nu_1, \dots, \nu_N)$. Let $C(z; P)$ and $C(z; \tilde{P})$ denote the corresponding optimal consumptions of an investor with belief parameter z ; Equations (I.8)–(I.9) imply that every component of these vectors is strictly positive. Because date-0 consumption is the numeraire, the complete commodity-price vectors are $(1, P^\top)^\top$ and $(1, \tilde{P}^\top)^\top$. Since every investor has endowment D , wealth at P is $W_0(P) = D_0 + \sum_{m=1}^M p_m D_m$, and analogously at $\tilde{P}$. Strictly increasing utility implies that the two budget constraints bind: $(1, P^\top)C(z; P) = W_0(P)$ and $(1, \tilde{P}^\top)C(z; \tilde{P}) = W_0(\tilde{P})$. The individual first-order conditions initially give $\nabla_C f_z(C(z; P)) = \lambda_z(P)(1, P^\top)^\top$ and the analogous equality at $\tilde{P}$. Premultiplying these conditions by the corresponding consumption vectors, and using both the binding budget constraints and $C^\top \nabla_C f_z(C) = 1$, gives $\lambda_z(P) = 1/W_0(P)$, and hence

$$\nabla_C f_z(C(z; P)) = \frac{(1, P^\top)^\top}{W_0(P)}, \quad \nabla_C f_z(C(z; \tilde{P})) = \frac{(1, \tilde{P}^\top)^\top}{W_0(\tilde{P})}.$$

For a fixed belief parameter z , define $C_t(z) = C(z; \tilde{P}) + t[C(z; P) - C(z; \tilde{P})]$, $g_z(t) = f_z(C_t(z))$, $t \in [0, 1]$. This function remains in the strictly positive orthant, and differentiation with respect to t gives $C'_t(z) = C(z; P) - C(z; \tilde{P})$ and $C''_t(z) = 0$. The chain rule gives

$$g'_z(t) = C'_t(z)^\top \nabla_C f_z(C_t(z)), \quad g''_z(t) = C'_t(z)^\top \nabla_C^2 f_z(C_t(z)) C'_t(z).$$

Applying the fundamental theorem of calculus to the scalar function g'_z on $[0, 1]$ gives

$$\begin{aligned} & [C(z; P) - C(z; \tilde{P})]^\top [\nabla_C f_z(C(z; P)) - \nabla_C f_z(C(z; \tilde{P}))] \\ &= g'_z(1) - g'_z(0) = \int_0^1 \underbrace{C'_t(z)^\top \nabla_C^2 f_z(C_t(z)) C'_t(z)}_{g''_z(t)} dt \leq 0. \end{aligned}$$

If $C(z; P) \neq C(z; \tilde{P})$, then $C'_t(z) \neq 0$, so negative definiteness in (I.18) makes the integrand strictly negative for every $t \in [0, 1]$. If the two consumption vectors are equal, the integrand is zero. Thus, equality holds if and only if $C(z; P) = C(z; \tilde{P})$. Substituting the first-order conditions into the first line of the preceding display gives

$$\left[\frac{(1, P^\top)^\top}{W_0(P)} - \frac{(1, \tilde{P}^\top)^\top}{W_0(\tilde{P})} \right]^\top [C(z; P) - C(z; \tilde{P})] \leq 0,$$

again with equality if and only if $C(z; P) = C(z; \tilde{P})$.

In a full-information equilibrium, informed and uninformed investors both have belief parameter ε . Thus, subtracting market clearing at the two price vectors gives

$$(\mu_I + \mu_U) [C(\varepsilon; P) - C(\varepsilon; \tilde{P})] + \frac{\mu_B}{N} \sum_{n=1}^N [C(\nu_n; P) - C(\nu_n; \tilde{P})] = 0.$$

Premultiplying this equality by the bracketed normalized price difference in the preceding inequality gives a positive weighted sum of nonpositive scalar products equal to zero. If the consumption vector of any investor type differed across the two equilibria, the corresponding scalar product would be strictly negative, while all the others would remain nonpositive. Since $\mu_I + \mu_U > 0$ and $\mu_B/N > 0$, their weighted sum would then be strictly negative, a contradiction. Therefore, $C(\varepsilon; P) = C(\varepsilon; \tilde{P})$, $C(\nu_n; P) = C(\nu_n; \tilde{P})$, $n = 1, \dots, N$. Using the first-order conditions of the investors with belief parameter ε , we obtain

$$\frac{(1, P^\top)^\top}{W_0(P)} = \nabla_C f_\varepsilon(C(\varepsilon; P)) = \nabla_C f_\varepsilon(C(\varepsilon; \tilde{P})) = \frac{(1, \tilde{P}^\top)^\top}{W_0(\tilde{P})}.$$

Equality of the first components implies $W_0(P) = W_0(\tilde{P})$; equality of the remaining components then implies $P = \tilde{P}$. Thus, the full-information equilibrium price vector is unique.

We now establish invertibility. Because $p_m = \pi_m^{\text{RN}}/(1 + r_f)$, Equation (I.8) becomes $C_m(z; P) = C_0(z; P)[\beta\pi_m(z)/p_m]^{1/\gamma}$. Moreover, Equation (I.9) simplifies to

$$C_0(z; P) = \frac{W_0(P)}{1 + \beta^{1/\gamma} \sum_{m=1}^M p_m^{1-1/\gamma} \pi_m(z)^{1/\gamma}}. \quad (\text{I.19})$$

Because all endowments are equal, common initial wealth is $W_0(P) = D_0 + \sum_{m=1}^M p_m D_m$. Hence, $C_0(z; P) > 0$ is the same function of z in any two full-information equilibria with the same price vector. In a full-information equilibrium, the beliefs of the uninformed and informed investors coincide. Therefore, consumption clearing can be written as

$$D_m \left(\frac{p_m}{\beta} \right)^{1/\gamma} = (\mu_I + \mu_U) C_0(\varepsilon; P) \pi_m(\varepsilon)^{1/\gamma} + \frac{\mu_B}{N} \sum_{n=1}^N C_0(\nu_n; P) \pi_m(\nu_n)^{1/\gamma}. \quad (\text{I.20})$$

Because $\varepsilon, \nu_1, \dots, \nu_N$ are independent and have continuous probability density functions, they are pairwise distinct with probability one. Fix a realization in this probability-one set and suppose that $P(\varepsilon, \nu_1, \dots, \nu_N) = P(\tilde{\varepsilon}, \tilde{\nu}_1, \dots, \tilde{\nu}_N) = P$. The left-hand side of Equation (I.20) is then the same for the two vectors. Substituting the exponential-family probabilities (14) into the two versions of Equation (I.20) and cancelling $\exp(a(x_m)/\gamma)$ gives

$$\begin{aligned} & (\mu_I + \mu_U) C_0(\varepsilon; P) e^{\varepsilon x/\gamma - c(\varepsilon)/\gamma} + \frac{\mu_B}{N} \sum_{n=1}^N C_0(\nu_n; P) e^{\nu_n x/\gamma - c(\nu_n)/\gamma} \\ &= (\mu_I + \mu_U) C_0(\tilde{\varepsilon}; P) e^{\tilde{\varepsilon} x/\gamma - c(\tilde{\varepsilon})/\gamma} + \frac{\mu_B}{N} \sum_{n=1}^N C_0(\tilde{\nu}_n; P) e^{\tilde{\nu}_n x/\gamma - c(\tilde{\nu}_n)/\gamma}, \end{aligned} \quad (\text{I.21})$$

for $x \in \{x_1, \dots, x_M\}$. Equation (I.21) has the same exponential-sum form as (I.15), with μ_I replaced by $\mu_I + \mu_U$ and date-0 consumption written as $C_0(z; P)$. The rest of the proof is analogous to that of part 1). Because excess demand is jointly continuous in normalized

prices, the shock, and beliefs, and market clearing rules out limits in which any normalized price is zero, price uniqueness implies that the full information price vector is continuous in $(\varepsilon, \nu_1, \dots, \nu_N)$. Its almost sure invertibility therefore makes ε and the cross section of biased beliefs measurable functions of prices. Consequently, the full information equilibrium is a fully revealing REE in which uninformed investors learn from prices alone. ■

I.3. Partial revelation with many biased investor types

This section establishes the finite- N partial-revelation result used in Section 5.1. We retain the baseline economy and the sufficient statistics in Equation (9), holding the aggregate masses of informed, uninformed, and biased investors fixed.

Proposition I.5 [Partial revelation with many biased investor types]. *Suppose $M \geq 2$, the values $x_1, \dots, x_M$ are distinct, and the probabilities of informed and biased investors belong to possibly different exponential families,*

$$\pi_{i,m}(z) = \exp\{a_i(x_m) + zx_m - c_i(z)\}, \quad i = I, B, \quad m = 1, \dots, M, \quad (\text{I.22})$$

where the functions a_i and c_i may differ across investor types, and c_i normalizes the corresponding probabilities to sum to one. Suppose also that $\mu_I, \mu_B > 0$ and that the supports of ε and ν are nondegenerate intervals. For every finite integer $N \geq M - 1$, the sufficient statistics in Equation (9) do not fully reveal ε , but are informative about ε . Hence, the equilibrium is partially revealing.

Economically, changes in $M - 1$ biased beliefs can locally offset a change in the informed shock in all $M - 1$ nonredundant sufficient statistics. Nevertheless, the statistics remain informative because their conditional means depend on the informed shock. For the proof, we will also need the following standard implicit function theorem.

Theorem I.1 [Implicit function theorem]. *Let $H : \mathcal{O} \rightarrow \mathbb{R}^{M-1}$ be continuously differentiable, where $\mathcal{O}$ is an open subset of $\mathbb{R} \times \mathbb{R}^{M-1}$. Suppose that $H(e_0, u_0) = 0$ and that the $(M-1) \times (M-1)$ derivative matrix $D_u H(e_0, u_0)$ is nonsingular. Then, on some open interval containing e_0 , there is a continuously differentiable function $u(e) \in \mathbb{R}^{M-1}$, with $u(e_0) = u_0$, such that $H(e, u(e)) = 0$ throughout that interval.*

Proof of Proposition I.5. Let $\pi_I(z)$ and $\pi_B(z)$ denote the vectors of the first $M - 1$ informed and biased probabilities, respectively. We first show that, on every open interval contained in the interior of the support of ν , the derivative vectors $\pi'_B(z)$ span $\mathbb{R}^{M-1}$. Suppose otherwise on one such interval. Then, there is a nonzero vector $b \in \mathbb{R}^{M-1}$ such that $b^\top \pi'_B(z) = 0$ throughout that interval. Hence, $b^\top \pi_B(z) = C$ there for some constant C . Multiplying this

identity by $\exp\{c_B(z)\}$, we obtain $\sum_{m=1}^{M-1} b_m \exp\{a_B(x_m) + zx_m\} = C \sum_{m=1}^M \exp\{a_B(x_m) + zx_m\}$. Because the values x_m are distinct, the exponential functions in the latter equation are linearly independent by Lemma A.1. The coefficient of the final exponential implies $C = 0$, and the remaining coefficients then imply $b = 0$, a contradiction. Thus, the derivative vectors span $\mathbb{R}^{M-1}$. Therefore, there exist $z_1, \dots, z_{M-1}$ in the interior of the support of ν such that the matrix $[\pi'_B(z_1) \cdots \pi'_B(z_{M-1})]$ is nonsingular.

Choose e_0 in the interior of the support of ε , write $u = (u_1, \dots, u_{M-1})$, and, on the product of the interiors of the relevant supports, define

$$H(e, u) = \mu_I[\pi_I(e) - \pi_I(e_0)] + \frac{\mu_B}{N} \sum_{j=1}^{M-1} [\pi_B(u_j) - \pi_B(z_j)]. \quad (\text{I.23})$$

The map H is continuously differentiable. The j -th column of the Jacobian $D_u H(e, u)$ is $(\mu_B/N)\pi'_B(u_j)$. At $u_0 = (z_1, \dots, z_{M-1})$, we have $H(e_0, u_0) = 0$, and

$$D_u H(e_0, u_0) = \frac{\mu_B}{N} [\pi'_B(z_1) \cdots \pi'_B(z_{M-1})]. \quad (\text{I.24})$$

This matrix is nonsingular by construction because $\mu_B/N > 0$. Theorem I.1 therefore provides a continuously differentiable function $u(e)$, defined on some open interval containing e_0 , with $u(e_0) = u_0$, such that $H(e, u(e)) = 0$ throughout that interval.

After shrinking the interval if necessary, every coordinate of $u(e)$ lies in the interior of the support of ν . Choose any $e \neq e_0$ in this interval. Compare the vector with shock e and first $M - 1$ biased beliefs $u_1(e), \dots, u_{M-1}(e)$ with the vector with shock e_0 and first $M - 1$ biased beliefs $z_1, \dots, z_{M-1}$. Assign identical values to all remaining beliefs $\nu_M, \dots, \nu_N$; this common tail is omitted when $N = M - 1$. The equality $H(e, u(e)) = 0$ means that these two vectors generate identical sufficient statistics, even though $e \neq e_0$. Thus, the statistics do not fully reveal the shock.

However, the statistics are informative. Independence implies, for every finite N , that

$$\mathbb{E}[s \mid \varepsilon] = \mu_I \pi_I(\varepsilon) + \frac{\mu_B}{N} \sum_{n=1}^N \mathbb{E}[\pi_B(\nu_n)]. \quad (\text{I.25})$$

Equation (I.25) concerns only the conditional mean; the noise term in s remains random. If s and ε were independent, $\mathbb{E}[s \mid \varepsilon]$ would be constant. Since $\pi_I(\varepsilon)$ is nonconstant, the conditional mean in (I.25) depends on ε . Thus, s and ε are not independent, so the statistics are informative about ε . Hence, the equilibrium is partially revealing. ■

I.4. Pointwise convergence of prices

In this section, we study conditions under which prices in an economy with a finite number N of groups of biased investors converge to those in the Gaussian-signal economy studied in

Section 5. We retain the structure of the economy in the baseline model in Section 2.1, and following Section 5, we assume that the mass of informed investors is given by $\mu_I = \hat{\mu}_I/\sqrt{N}$. We retain the arbitrary PDF φ_ε from the baseline model.

For tractability, we assume that biased investors' probabilities have the following form:

$$\pi_{B,m}(\nu_n, \zeta_n; t) = \frac{\exp\{a(x_m) + \nu_n x_m / \sigma_B^2 + t \zeta_{n,m}\}}{\sum_{k=1}^M \exp\{a(x_k) + \nu_n x_k / \sigma_B^2 + t \zeta_{n,k}\}}, \quad m = 1, \dots, M, \quad (\text{I.26})$$

where x_m is a state variable (e.g., $x_m = \ln(D_m/D_0)$), $\nu_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\nu^2)$, $\zeta_{n,m} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$, $\sigma_B > 0$, $\sigma_\nu > 0$, and $t > 0$ is a fixed, arbitrarily small parameter. The two families of random variables are mutually independent and independent of ε . We denote $\zeta_n = (\zeta_{n,1}, \dots, \zeta_{n,M})^\top$.

Probabilities (I.26) generalize the exponential-family probabilities in (14) by adding state-specific errors $t\zeta_{n,m}$ to beliefs. The errors may capture small imperfections in the assessment of probabilities by biased investors. The perturbation parameter $t > 0$ makes it possible to apply the local limit theorem needed to establish price convergence.

Following the same steps as in Section 5, we find the finite- N sufficient statistics

$$\hat{s}_{N,m} = \eta \pi_{I,m}(\varepsilon) + \frac{(\pi_{B,m}(\nu_1, \zeta_1; t) - \bar{\pi}_{B,m}(t)) + \dots + (\pi_{B,m}(\nu_N, \zeta_N; t) - \bar{\pi}_{B,m}(t))}{\sqrt{N}}, \quad (\text{I.27})$$

where $m = 1, \dots, M$, $\eta = \hat{\mu}_I/\mu_B$ is the relative mass parameter of informed investors, and $\bar{\pi}_{B,m}(t) = \mathbb{E}_{\nu, \zeta}[\pi_{B,m}(\nu, \zeta; t)]$. The M transformed sufficient statistics sum to η , so the last statistic is redundant and we retain the first $M - 1$ statistics. The centered random vectors

$$X_n = (\pi_{B,1}(\nu_n, \zeta_n; t) - \bar{\pi}_{B,1}(t), \dots, \pi_{B,M-1}(\nu_n, \zeta_n; t) - \bar{\pi}_{B,M-1}(t))^\top \quad (\text{I.28})$$

are i.i.d. across groups of biased investors. Proposition I.7 below implies that the joint PDF of the noise terms in (I.27) converges uniformly to the $\mathcal{N}(0, \Sigma_\xi(t))$ PDF. This gives the following limiting sufficient statistics:

$$\hat{s}_m = \eta \pi_{I,m}(\varepsilon) + \xi_m, \quad \xi = (\xi_1, \dots, \xi_{M-1})^\top \sim \mathcal{N}(0, \Sigma_\xi(t)), \quad (\text{I.29})$$

where $m = 1, \dots, M - 1$, ξ is independent of ε , and $\Sigma_\xi(t) \in \mathbb{R}^{(M-1) \times (M-1)}$ is a symmetric, positive-definite cross-sectional covariance matrix with elements equal to covariances $\overline{\text{cov}}(\pi_{B,m}(\nu, \zeta; t), \pi_{B,k}(\nu, \zeta; t))$ with respect to beliefs ν and shocks ζ .

Proposition 2 for the equilibrium prices in terms of sufficient statistics remains valid in our economy. We denote the price of the state- m Arrow–Debreu security in the finite- N economy by $p_{N,m}(y; t)$, where y is a realization of the vector of sufficient statistics. Similarly, by $p_m(y; t)$ we denote the price in the limiting Gaussian-signal economy. Proposition I.6 establishes the pointwise convergence of prices to those in the Gaussian-signal economy.

Proposition I.6 [Pointwise convergence of prices]. Assume $0 < \mu_B < 1$ and $0 \leq \hat{\mu}_I < \infty$. Let the probabilities of states of biased investors be given by Equation (I.26), where $a(x_1), \dots, a(x_M)$ are finite and M and $t > 0$ are fixed. Then, for every $y \in \mathbb{R}^{M-1}$ and every state $m = 1, \dots, M$, the equilibrium price $p_{N,m}(y; t)$ in the finite- N economy is well-defined for all sufficiently large N and the following pointwise convergence holds:

$$p_{N,m}(y; t) \longrightarrow p_m(y; t), \quad N \longrightarrow \infty, \quad (\text{I.30})$$

where $p_m(y; t)$ is the corresponding price in the Gaussian-signal economy.

The proof uses the following multidimensional local limit theorem, which is the i.i.d. specialization of Theorem 1 in ? needed here.

Proposition I.7 [Shervashidze's local limit theorem]. Let $Z_1, Z_2, \dots$ be i.i.d. centered random vectors in $\mathbb{R}^{M-1}$. Suppose that Z_1 has a Lebesgue PDF f_Z satisfying $\|f_Z\|_\infty < \infty$, a positive-definite covariance matrix Σ , and a finite third moment

$$m_3 = \mathbb{E}[\|Z_1\|^3] < \infty. \quad (\text{I.31})$$

Let f_N be the PDF of

$$\frac{Z_1 + \dots + Z_N}{\sqrt{N}}, \quad (\text{I.32})$$

and let ϕ_Σ denote the $\mathcal{N}(0, \Sigma)$ PDF. There is a finite constant C_Z , depending only on M and the distribution of Z_1 , such that, for all sufficiently large N ,

$$\sup_{u \in \mathbb{R}^{M-1}} |f_N(u) - \phi_\Sigma(u)| \leq \frac{C_Z}{\sqrt{N}}. \quad (\text{I.33})$$

Remark (binary shocks). Proposition I.6 also applies when the productivity shock has finite support with strictly positive prior probabilities, including the binary specification in Section 5.1. In this case, posterior densities are replaced by posterior probabilities, and the integrals over the productivity shock in the proof of Lemma I.3 are replaced by finite sums. Uniform convergence of the noise densities implies convergence of the posterior probabilities because the limiting Gaussian density is strictly positive. Posterior expectations, and hence equilibrium prices, therefore converge as in Equation (I.30).

I.4.1. Proof of the convergence of prices

In this subsection, we provide the proof of Proposition I.6, which is preceded by the proof of the local limit theorem in Proposition I.7 and some auxiliary lemmas.

Proof of Proposition I.7. Proposition I.7 is the i.i.d. specialization of Theorem 1 in ?. In the notation of that theorem, set

$$A_i = A = \|f_Z\|_\infty, \quad \beta_i = m_3, \quad R = m_3 A^{3/(M-1)}, \quad \sigma_i^2 = \mathbb{E}[\|Z_i\|^2] = \text{tr}(\Sigma). \quad (\text{I.34})$$

Thus, Shervashidze's condition on the third moment and the PDF bound holds with the finite constant R . Moreover,

$$D_N = \sum_{i=1}^N \mathbb{E}[Z_i Z_i^\top] = N\Sigma, \quad B_N^2 = \text{tr}(D_N) = \sum_{i=1}^N \sigma_i^2 = N \text{tr}(\Sigma), \quad \max_{1 \leq i \leq N} \frac{\sigma_i}{B_N} = \frac{1}{\sqrt{N}}. \quad (\text{I.35})$$

Because Σ is positive definite, the covariance matrix of every Z_i is nondegenerate, as required by Shervashidze's theorem. Theorem 1 of Shervashidze contains a freely chosen constant $\epsilon \in (0, 1/2)$ in its sample-size condition (5) in ?. This constant is unrelated to the productivity shock ε . Choose $\epsilon = 1/4$. By (I.35), condition (5) becomes

$$\max_{1 \leq i \leq N} \frac{\sigma_i}{B_N} = \frac{1}{\sqrt{N}} \leq \sqrt{0.5 - \epsilon} = 0.5, \quad (\text{I.36})$$

which holds whenever $N \geq 4$. We next explain the normalization used in Shervashidze's theorem. Let $S_N = Z_1 + \dots + Z_N$. By (I.35), the covariance matrix of S_N is $D_N = N\Sigma$. Define $K_N = N^{-1/2}\Sigma^{-1/2}$. Then, $K_N D_N K_N^\top = I_{M-1}$, $K_N(Z_1 + \dots + Z_N) = \Sigma^{-1/2}(Z_1 + \dots + Z_N)/\sqrt{N}$. Thus, $K_N S_N$ has identity covariance matrix.

Let u_N be the PDF of $K_N S_N$, and let $\phi_{I_{M-1}}$ be the standard normal PDF on $\mathbb{R}^{M-1}$. Shervashidze's theorem now gives

$$\sup_{w \in \mathbb{R}^{M-1}} |u_N(w) - \phi_{I_{M-1}}(w)| \leq \frac{C(M-1, R, 1/4)}{\sqrt{N}}. \quad (\text{I.37})$$

The result we need concerns the density f_N of $S_N/\sqrt{N}$, whose covariance matrix is Σ , rather than the density of the standardized vector $K_N S_N$. Since $K_N S_N = \Sigma^{-1/2} S_N/\sqrt{N}$, the change-of-variables formula gives

$$f_N(u) = (\det \Sigma)^{-1/2} u_N(\Sigma^{-1/2} u), \quad \phi_\Sigma(u) = (\det \Sigma)^{-1/2} \phi_{I_{M-1}}(\Sigma^{-1/2} u). \quad (\text{I.38})$$

Combining (I.37) and (I.38) therefore gives

$$\sup_{u \in \mathbb{R}^{M-1}} |f_N(u) - \phi_\Sigma(u)| \leq \frac{(\det \Sigma)^{-1/2} C(M-1, R, 1/4)}{\sqrt{N}}. \quad (\text{I.39})$$

This completes the proof. ■

Lemma I.1 [Regularity of the noise vector]. *For fixed M , $t > 0$, and finite values $a(x_1), \dots, a(x_M)$, the centered vector X_1 in (I.28) has a bounded Lebesgue PDF on $\mathbb{R}^{M-1}$, its covariance matrix $\Sigma_\xi(t)$ is positive definite, and*

$$\mathbb{E}[\|X_1\|^3] \leq 2^{3/2}. \quad (\text{I.40})$$

Proof of Lemma I.1. Use state M as the reference state and define the log odds

$$Y_m = \ln \left(\frac{\pi_{B,m}(\nu, \zeta; t)}{\pi_{B,M}(\nu, \zeta; t)} \right) = a(x_m) - a(x_M) + \frac{\nu(x_m - x_M)}{\sigma_B^2} + t(\zeta_m - \zeta_M), \quad (\text{I.41})$$

where $m = 1, \dots, M-1$. Denote $Y = (Y_1, \dots, Y_{M-1})^\top$, $\mu_Y = (a(x_1) - a(x_M), \dots, a(x_{M-1}) - a(x_M))^\top$, $\tilde{x} = (x_1 - x_M, \dots, x_{M-1} - x_M)^\top$, $e_{M-1} = (1, \dots, 1)^\top \in \mathbb{R}^{M-1}$. Then, Y is normal with mean μ_Y and covariance matrix

$$\Omega(t) = \frac{\sigma_\nu^2}{\sigma_B^4} \tilde{x} \tilde{x}^\top + t^2 (I_{M-1} + e_{M-1} e_{M-1}^\top) \succeq t^2 I_{M-1}. \quad (\text{I.42})$$

Therefore, $\det \Omega(t) \geq t^{2(M-1)} > 0$. Thus, the values of a affect only the finite mean μ_Y , while $t > 0$ makes $\Omega(t)$ positive definite.

We next derive and bound the PDF of the probability vector. Represent the interior of the M -state probability simplex by its first $M-1$ coordinates:

$$\text{int}(\Delta_M) = \left\{ \rho \in (0, 1)^{M-1} : \sum_{m=1}^{M-1} \rho_m < 1 \right\}, \quad \rho_M = 1 - \sum_{m=1}^{M-1} \rho_m. \quad (\text{I.43})$$

The multinomial-logit map from $\mathbb{R}^{M-1}$ onto $\text{int}(\Delta_M)$ and its inverse are

$$\rho_m(q) = \frac{e^{q_m}}{1 + \sum_{k=1}^{M-1} e^{q_k}}, \quad \rho_M(q) = \frac{1}{1 + \sum_{k=1}^{M-1} e^{q_k}}, \quad q_m(\rho) = \ln \left(\frac{\rho_m}{\rho_M} \right). \quad (\text{I.44})$$

Since $q_i(\rho) = \ln \rho_i - \ln \rho_M$ and $\partial \rho_M / \partial \rho_j = -1$, direct differentiation gives

$$\frac{\partial q_i(\rho)}{\partial \rho_j} = \frac{\mathbf{1}_{\{i=j\}}}{\rho_i} + \frac{1}{\rho_M}, \quad i, j = 1, \dots, M-1. \quad (\text{I.45})$$

Let $H = \text{diag}(\rho_1^{-1}, \dots, \rho_{M-1}^{-1})$, $c = e_{M-1}/\rho_M$, and $d = e_{M-1}$. The matrix determinant lemma implies that $\det(H + cd^\top) = \det(H)(1 + d^\top H^{-1}c)$, and hence,

$$\det \left(\frac{\partial q}{\partial \rho^\top} \right) = \frac{1}{\prod_{m=1}^{M-1} \rho_m} \left(1 + \frac{1}{\rho_M} \sum_{m=1}^{M-1} \rho_m \right) = \frac{1}{\prod_{m=1}^{M-1} \rho_m} \frac{1}{\rho_M} = \frac{1}{\prod_{m=1}^M \rho_m}. \quad (\text{I.46})$$

The second equality uses $\rho_M + \sum_{m=1}^{M-1} \rho_m = 1$. Hence, the multivariate change-of-variables theorem gives the PDF of the first $M-1$ probabilities in (I.26):

$$f_B(\rho; t) = \frac{\phi_{\Omega(t)}(q(\rho) - \mu_Y)}{\prod_{m=1}^M \rho_m}, \quad \rho \in \text{int}(\Delta_M), \quad (\text{I.47})$$

where $\phi_{\Omega(t)}$ is the centered normal PDF with covariance matrix $\Omega(t)$. The PDF in (I.47) is strictly positive throughout the simplex interior.

It remains to show that the Jacobian factor in (I.47) does not make the PDF unbounded near the simplex boundary. Substituting (I.44) into (I.47) gives

$$\begin{aligned} \ln f_B(\rho(q); t) = & -\frac{M-1}{2} \ln(2\pi) - \frac{1}{2} \ln(\det \Omega(t)) \\ & - \frac{1}{2} (q - \mu_Y)^\top \Omega(t)^{-1} (q - \mu_Y) + M \ln \left(1 + \sum_{m=1}^{M-1} e^{q_m} \right) - \sum_{m=1}^{M-1} q_m. \end{aligned} \quad (\text{I.48})$$

Let $\lambda_{\max}(t)$ be the largest eigenvalue of $\Omega(t)$. For every $q \in \mathbb{R}^{M-1}$ we have:

$$\begin{aligned} \ln \left(1 + \sum_{m=1}^{M-1} e^{q_m} \right) & \leq \ln M + \|q\|, \quad \left| \sum_{m=1}^{M-1} q_m \right| \leq \sqrt{M-1} \|q\|, \\ (q - \mu_Y)^\top \Omega(t)^{-1} (q - \mu_Y) & \geq \frac{\|q\|^2}{2\lambda_{\max}(t)} - \frac{\|\mu_Y\|^2}{\lambda_{\max}(t)}. \end{aligned} \quad (\text{I.49})$$

The first inequality follows because each of the M terms inside the logarithm is at most $e^{\|q\|}$, and the second follows from the Cauchy–Schwarz inequality. For the third inequality, use

$$\|q - \mu_Y\|^2 \geq \frac{1}{2} \|q\|^2 - \|\mu_Y\|^2, \quad (\text{I.50})$$

and the smallest eigenvalue of $\Omega(t)^{-1}$ given by $1/\lambda_{\max}(t)$. Then, (I.50) follows by expanding $\|q - \mu_Y\|^2$ and applying $2q^\top \mu_Y \leq \frac{1}{2} \|q\|^2 + 2\|\mu_Y\|^2$, which follows from $0 \leq \|q/\sqrt{2} - \sqrt{2}\mu_Y\|^2$.

Combining (I.48)–(I.50) and completing the square gives

$$\begin{aligned} \ln f_B(\rho(q); t) & \leq -\frac{M-1}{2} \ln(2\pi) - \frac{1}{2} \ln(\det \Omega(t)) + M \ln M + \frac{\|\mu_Y\|^2}{2\lambda_{\max}(t)} \\ & \quad - \frac{\|q\|^2}{4\lambda_{\max}(t)} + (M + \sqrt{M-1}) \|q\| \\ & \leq -\frac{M-1}{2} \ln(2\pi) - \frac{1}{2} \ln(\det \Omega(t)) + M \ln M + \frac{\|\mu_Y\|^2}{2\lambda_{\max}(t)} \\ & \quad + \lambda_{\max}(t) (M + \sqrt{M-1})^2 < \infty. \end{aligned} \quad (\text{I.51})$$

Every point in the simplex interior equals $\rho(q)$ for a unique q . Thus, $\|f_B(\cdot; t)\|_\infty < \infty$. Moreover, if ρ approaches the simplex boundary from the interior, then $\|q(\rho)\| \rightarrow \infty$ by (I.44); otherwise, a bounded subsequence of $q(\rho)$ would have a limit whose image under the softmax map lies in the simplex interior. The negative quadratic term in the first bound in (I.51) then dominates its positive linear term, and hence

$$\lim_{\substack{\rho \rightarrow \rho^0 \\ \rho \in \text{int}(\Delta_M)}} f_B(\rho; t) = 0, \quad \rho^0 \in \partial \Delta_M. \quad (\text{I.52})$$

We may consequently set $f_B(\rho; t) = 0$, $\rho \in \mathbb{R}^{M-1} \setminus \text{int}(\Delta_M)$. The latter boundary condition and Equation (I.52) then define a continuous PDF on $\mathbb{R}^{M-1}$.

The local limit theorem applies to the centered vector X_1 . For compactness, let $\bar{\pi}_B(t) = (\bar{\pi}_{B,1}(t), \dots, \bar{\pi}_{B,M-1}(t))^\top$. By (I.28), the PDF of X_1 is

$$f_{X_1}(u; t) = f_B(u + \bar{\pi}_B(t); t), \quad \|f_{X_1}(\cdot; t)\|_\infty = \|f_B(\cdot; t)\|_\infty < \infty. \quad (\text{I.53})$$

Thus, centering translates the PDF without changing its supremum.

We next prove that $\Sigma_\xi(t)$ is positive definite. For any $u \in \mathbb{R}^{M-1}$,

$$u^\top \Sigma_\xi(t) u = \mathbb{E}[(u^\top X_1)^2] = \overline{\text{var}} \left(\sum_{m=1}^{M-1} u_m \pi_{B,m}(\nu, \zeta; t) \right). \quad (\text{I.54})$$

If (I.54) were zero, the displayed linear function of the first $M-1$ probabilities would be constant almost surely. Because the PDF in (I.47) is strictly positive on the simplex interior and a nonzero linear function cannot be constant on an open set, this is possible only if $u = 0$. Therefore, $u^\top \Sigma_\xi(t) u > 0$ for every $u \neq 0$, which proves positive definiteness. The state-specific perturbations therefore make a separate linear-independence condition on the unperturbed probabilities unnecessary.

Finally, any two M -state probability vectors r and s satisfy

$$\|r - s\|^2 = \|r\|^2 + \|s\|^2 - 2r^\top s \leq \|r\|_1^2 + \|s\|_1^2 = 2. \quad (\text{I.55})$$

Indeed, their coordinates are nonnegative, so $r^\top s \geq 0$, while $\|r\| \leq \|r\|_1 = 1$ and $\|s\| \leq \|s\|_1 = 1$. Applying (I.55) to the full probability vector and its mean gives

$$\|X_1\|^2 = \sum_{m=1}^{M-1} \{\pi_{B,m}(\nu, \zeta; t) - \bar{\pi}_{B,m}(t)\}^2 \leq \sum_{m=1}^M \{\pi_{B,m}(\nu, \zeta; t) - \bar{\pi}_{B,m}(t)\}^2 \leq 2. \quad (\text{I.56})$$

Hence, $\|X_1\|^3 \leq (\sqrt{2})^3 = 2^{3/2}$, $\mathbb{E}[\|X_1\|^3] \leq 2^{3/2}$, which completes the proof. $\blacksquare$

Lemma I.2 [Uniform convergence of noise PDFs]. *Let $\phi_N(u; t)$ be the joint PDF of the noise terms in (I.27), and let $\phi(u; t)$ be the $\mathcal{N}(0, \Sigma_\xi(t))$ PDF. For fixed M , $t > 0$, and finite $a(x_1), \dots, a(x_M)$, there is a finite constant C such that, for all sufficiently large N ,*

$$\Delta_N(t) := \sup_{u \in \mathbb{R}^{M-1}} |\phi_N(u; t) - \phi(u; t)| \leq \frac{C}{\sqrt{N}}. \quad (\text{I.57})$$

Proof of Lemma I.2. Lemma I.1 shows that the i.i.d. centered summands X_n have a bounded Lebesgue PDF, positive-definite covariance matrix $\Sigma_\xi(t)$, and a finite third moment. Thus, every assumption of Proposition I.7 is satisfied. Applying that proposition to the noise term $(X_1 + \dots + X_N)/\sqrt{N}$ in (I.27) gives (I.57). $\blacksquare$

Lemma I.3 [Convergence of posterior probabilities]. Fix $y \in \mathbb{R}^{M-1}$, and let $\mathbb{E}_N$ denote expectation in the finite- N economy. Let $\phi_{N,\varepsilon|\hat{s}}(z | y; t)$ and $\phi_{\varepsilon|\hat{s}}(z | y; t)$ be the posterior PDFs of ε computed from the finite- N and Gaussian-signal statistics, respectively. The finite- N posterior PDF is well-defined for all sufficiently large N , and

$$\int_{\mathbb{R}} |\phi_{N,\varepsilon|\hat{s}}(z | y; t) - \phi_{\varepsilon|\hat{s}}(z | y; t)| dz \longrightarrow 0. \quad (\text{I.58})$$

Consequently, for every state $m = 1, \dots, M$,

$$\mathbb{E}_N[\pi_{I,m}(\varepsilon) | \hat{s}_N = y] \longrightarrow \mathbb{E}[\pi_{I,m}(\varepsilon) | \hat{s} = y]. \quad (\text{I.59})$$

Proof of Lemma I.3. Whenever the denominator is positive, Bayes' rule gives

$$\phi_{N,\varepsilon|\hat{s}}(z | y; t) = \frac{\phi_N(y - \eta(\pi_{I,1}(z), \dots, \pi_{I,M-1}(z)))^\top; t) \varphi_\varepsilon(z)}{I_N(y; t)}, \quad (\text{I.60})$$

where

$$I_N(y; t) = \int_{\mathbb{R}} \phi_N(y - \eta(\pi_{I,1}(z), \dots, \pi_{I,M-1}(z)))^\top; t) \varphi_\varepsilon(z) dz. \quad (\text{I.61})$$

The Gaussian-signal posterior has the analogous PDF

$$\phi_{\varepsilon|\hat{s}}(z | y; t) = \frac{\phi(y - \eta(\pi_{I,1}(z), \dots, \pi_{I,M-1}(z)))^\top; t) \varphi_\varepsilon(z)}{I(y; t)}, \quad (\text{I.62})$$

where

$$I(y; t) = \int_{\mathbb{R}} \phi(y - \eta(\pi_{I,1}(z), \dots, \pi_{I,M-1}(z)))^\top; t) \varphi_\varepsilon(z) dz. \quad (\text{I.63})$$

Because $0 \leq \pi_{I,m}(z) \leq 1$, for fixed y every argument of the Gaussian PDF in (I.63) lies in the compact set $y - \eta[0, 1]^{M-1}$. The Gaussian PDF is continuous and strictly positive. Hence,

$$\underline{\phi}(y; t) := \min_{r \in [0, 1]^{M-1}} \phi(y - \eta r; t) > 0. \quad (\text{I.64})$$

Since φ_ε is a PDF, $I(y; t) \geq \underline{\phi}(y; t)$. Let $h_y(z) = y - \eta(\pi_{I,1}(z), \dots, \pi_{I,M-1}(z))^\top$. Equation (I.57) then implies $|I_N(y; t) - I(y; t)| \leq \int_{\mathbb{R}} |\phi_N(h_y(z); t) - \phi(h_y(z); t)| \varphi_\varepsilon(z) dz \leq \Delta_N(t)$. Consequently, for all sufficiently large N , $I_N(y; t) \geq 0.5 \underline{\phi}(y; t) > 0$. Thus, the finite- N posterior is well-defined at the fixed value y . Next, we obtain:

$$\begin{aligned} & \int_{\mathbb{R}} |\phi_{N,\varepsilon|\hat{s}}(z | y; t) - \phi_{\varepsilon|\hat{s}}(z | y; t)| dz \\ &= \int_{\mathbb{R}} \left| \frac{\phi_N(h_y(z); t) - \phi(h_y(z); t)}{I_N(y; t)} + \phi(h_y(z); t) \left(\frac{1}{I_N(y; t)} - \frac{1}{I(y; t)} \right) \right| \varphi_\varepsilon(z) dz \\ &\leq \frac{1}{I_N(y; t)} \int_{\mathbb{R}} |\phi_N(h_y(z); t) - \phi(h_y(z); t)| \varphi_\varepsilon(z) dz + \frac{|I_N(y; t) - I(y; t)|}{I_N(y; t)} \\ &\leq \frac{2\Delta_N(t)}{I_N(y; t)} \leq \frac{4\Delta_N(t)}{\underline{\phi}(y; t)} \longrightarrow 0, \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (\text{I.65})$$

This proves (I.58). Finally, since $0 \leq \pi_{I,m}(z) \leq 1$, $|\mathbb{E}_N[\pi_{I,m}(\varepsilon) \mid \hat{s}_N = y] - \mathbb{E}[\pi_{I,m}(\varepsilon) \mid \hat{s} = y]| \leq \int_{\mathbb{R}} \pi_{I,m}(z) |\phi_{N,\varepsilon|\hat{s}}(z \mid y; t) - \phi_{\varepsilon|\hat{s}}(z \mid y; t)| dz \rightarrow 0$, which proves (I.59). ■

Proof of Proposition I.6. Because the M transformed sufficient statistics sum to η , define the omitted coordinate of y by $y_M = \eta - \sum_{k=1}^{M-1} y_k$. Thus, every occurrence of y_M below refers to this value. Let s_m denote the original sufficient statistic in Proposition 2. At $\hat{s}_N = y$, its relation to the transformed statistic is

$$s_m = \frac{\hat{\mu}_I}{\sqrt{N}} \pi_{I,m}(\varepsilon) + \frac{\mu_B}{N} \sum_{n=1}^N \pi_{B,m}(\nu_n, \zeta_n; t) = \mu_B \bar{\pi}_{B,m}(t) + \frac{\mu_B}{\sqrt{N}} y_m, \quad (\text{I.66})$$

where $m = 1, \dots, M$. Indeed, the first equality is the definition of s_m , and the second follows by multiplying (I.27) by $\mu_B/\sqrt{N}$ and using $\eta = \hat{\mu}_I/\mu_B$. For the first $M-1$ coordinates, this is an affine one-to-one transformation, so conditioning on s is equivalent to conditioning on $\hat{s}_N$. The remaining mass of uninformed investors is $1 - \mu_B - \hat{\mu}_I/\sqrt{N}$. Substituting (I.66) into Proposition 2 gives the following price in the finite- N economy:

$$p_{N,m}(y; t) = \frac{\beta D_0}{D_m} \left\{ \mu_B \bar{\pi}_{B,m}(t) + \frac{\mu_B}{\sqrt{N}} y_m + \left(1 - \mu_B - \frac{\hat{\mu}_I}{\sqrt{N}} \right) \mathbb{E}_N[\pi_{I,m}(\varepsilon) \mid \hat{s}_N = y] \right\}. \quad (\text{I.67})$$

Lemma I.3 shows that the conditional expectation in (I.67) is well-defined for all sufficiently large N and belongs to $[0, 1]$. The population-share restriction implies that $1 - \mu_B - \hat{\mu}_I/\sqrt{N}$ is nonnegative in every finite- N economy under consideration. Moreover, $\bar{\pi}_{B,m}(t) > 0$ and, for fixed y , $y_m/\sqrt{N} \rightarrow 0$, including when $m = M$. Hence, $\bar{\pi}_{B,m}(t) + y_m/\sqrt{N} > 0$ for all sufficiently large N . The first two terms in braces in (I.67) therefore have a strictly positive sum, while the last term is nonnegative. Since $\beta, D_0, D_m > 0$, the finite- N price is well-defined and positive.

The same lemma shows that the conditional expectation in (I.67) converges to its Gaussian-signal counterpart. In addition, $y_m/\sqrt{N} \rightarrow 0$ and $1 - \mu_B - \hat{\mu}_I/\sqrt{N} \rightarrow 1 - \mu_B$. Taking the limit in (I.67) gives $p_m(y; t) = \frac{\beta}{D_m/D_0} \{ \mu_B \bar{\pi}_{B,m}(t) + (1 - \mu_B) \mathbb{E}[\pi_{I,m}(\varepsilon) \mid \hat{s} = y] \}$, which is the equilibrium price in the Gaussian-signal economy and proves (I.30). ■

I.5. Convergence of informational efficiency

This section establishes convergence of the unregularized finite-state measure of informational efficiency $J_M^* = J_M(0)$ to its continuous-state counterpart J_∞^* . The result complements the regularized convergence analysis in Section 5.2 and is proved directly, without interchanging the limits $M \rightarrow \infty$ and $\tau \downarrow 0$.

Throughout this section, the informed- and biased-investor PDFs are the normal PDFs in (29), the exact discrete probabilities are given by (27), and $\nu \sim \mathcal{N}(0, \sigma_\nu^2)$. To display the

dependence on M explicitly, we write

$$f_m^M = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L)), \quad \mathcal{K}_{m\ell}^M = \mathbb{E}[\pi_{B,m}(\nu)\pi_{B,\ell}(\nu)], \quad J_M^* = (f^M)^\top (\mathcal{K}^M)^{-1} f^M. \quad (\text{I.68})$$

where $m, \ell = 1, \dots, M$, and $f^M = (f_1^M, \dots, f_M^M)^\top$. These are the finite-dimensional objects in Lemma 1, with their dependence on M made explicit. By Proposition 5 and monotone convergence applied to (34), the corresponding unregularized continuous-state value is

$$J_\infty^* := \lim_{\tau \downarrow 0} J_\infty(\tau) = \sum_{n=0}^{\infty} \frac{\widehat{f}_n^2}{\lambda_n} \in [0, +\infty]. \quad (\text{I.69})$$

Proposition I.8 [Convergence of informational efficiency]. *Suppose that the probabilities are given by (29) and (27), $A_M \rightarrow \infty$, $\Delta x \rightarrow 0$, and $0 < \eta < \infty$. Then,*

$$J_M^* \longrightarrow J_\infty^*, \quad \text{as } M \rightarrow \infty. \quad (\text{I.70})$$

In particular, Proposition 6 implies that J_M^ converges to a finite value when*

$$\sigma_B < \sigma_I < \sqrt{\sigma_B^2 + 2\sigma_\nu^2}, \quad (\text{I.71})$$

and $J_M^ \rightarrow +\infty$ otherwise.*

The proof uses three auxiliary lemmas. The first gives a minimum-norm representation of the finite-dimensional quadratic form in (I.68). The second establishes the normal-grid limits needed to pass the finite constraints to the continuous state space. The third identifies (I.69) with the corresponding continuous minimum-norm problem.

Define the following Hilbert space, which is used throughout this section:

$$\mathcal{H} = \mathcal{L}_2(\mathbb{R}, \varphi_\nu(\nu) d\nu), \quad \langle q, r \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} q(\nu) r(\nu) \varphi_\nu(\nu) d\nu. \quad (\text{I.72})$$

Lemma I.4 [Finite minimum-norm representation]. *For every M ,*

$$J_M^* = \min_{q \in \mathcal{H}} \{ \|q\|_{\mathcal{H}}^2 : \langle \pi_{B,m}, q \rangle_{\mathcal{H}} = f_m^M, \quad m = 1, \dots, M \}. \quad (\text{I.73})$$

Proof of Lemma I.4. We first verify linear independence of the functions $\pi_{B,1}, \dots, \pi_{B,M}$, which is the hypothesis needed to apply Lemma A.2. Suppose that $\sum_{m=1}^M c_m \pi_{B,m} = 0$ in $\mathcal{H}$. After multiplying by the common strictly positive normalizing denominator in (27) and cancelling the common normal factors, the resulting analytic function vanishes almost everywhere, and the relation becomes $\sum_{m=1}^M c_m \exp(-0.5x_m^2/\sigma_B^2) \exp(x_m\nu/\sigma_B^2) = 0$, $\nu \in \mathbb{R}$. The exponential functions in the latter equation have distinct exponents x_m/σ_B^2 and are therefore linearly independent. Hence every c_m is zero. Lemma A.2 therefore implies that $\mathcal{K}^M$ is positive definite and invertible.

Let $v^M = (\mathcal{K}^M)^{-1} f^M$ and define $q_M^0 = \sum_{m=1}^M v_m^M \pi_{B,m}$. Because $\mathcal{K}^M v^M = f^M$, q_M^0 satisfies

$$\langle \pi_{B,\ell}, q_M^0 \rangle_{\mathcal{H}} = \sum_{m=1}^M \mathcal{K}_{\ell m}^M v_m^M = f_{\ell}^M, \quad \ell = 1, \dots, M. \quad (\text{I.74})$$

Moreover, $\|q_M^0\|_{\mathcal{H}}^2 = (v^M)^\top \mathcal{K}^M v^M = (f^M)^\top (\mathcal{K}^M)^{-1} f^M = J_M^*$. If q is any other feasible element, then $q - q_M^0$ is orthogonal to every $\pi_{B,m}$, and hence to q_M^0 . Therefore, $\|q\|_{\mathcal{H}}^2 = \|q_M^0\|_{\mathcal{H}}^2 + \|q - q_M^0\|_{\mathcal{H}}^2 \geq \|q_M^0\|_{\mathcal{H}}^2$, which proves (I.73). $\blacksquare$

Lemma I.5 [Normal-grid convergence]. *Suppose that $A_M \rightarrow \infty$ and $\Delta x \rightarrow 0$ as $M \rightarrow \infty$. All limits below are taken as $M \rightarrow \infty$. For every fixed $\nu \in \mathbb{R}$ and every fixed $\varepsilon \in \mathbb{R}$,*

$$\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x \longrightarrow 1, \quad \sum_{k=1}^M \pi_I(x_k; \varepsilon) \Delta x \longrightarrow 1. \quad (\text{I.75})$$

The biased-investor normalizing sums also satisfy

$$\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x \leq 1 + \frac{\sqrt{2/\pi}}{\sigma_B} \Delta x, \quad \nu \in \mathbb{R}. \quad (\text{I.76})$$

Finally, if on each grid an index m is chosen so that $x_m \rightarrow x$ as $M \rightarrow \infty$, with its dependence on M suppressed, then

$$\frac{\pi_{B,m}}{\Delta x} \longrightarrow \pi_B(x; \cdot) \quad \text{in } \mathcal{H}, \quad (\text{I.77})$$

$$\frac{f_m^M}{\Delta x} \longrightarrow f(x). \quad (\text{I.78})$$

Proof of Lemma I.5. For $a \in \mathbb{R}$ and $\sigma > 0$, let $\phi = \phi_{a,\sigma}$ denote the translated normal PDF $\phi_{a,\sigma}(w) = (\sqrt{2\pi}\sigma)^{-1} \exp\{-(w-a)^2/(2\sigma^2)\}$. Taking $(a, \sigma) = (\nu, \sigma_B)$ gives $\phi(w) = \pi_B(w; \nu)$, while taking $(a, \sigma) = (\mu\varepsilon, \sigma_I)$ gives $\phi(w) = \pi_I(w; \varepsilon)$. Thus $\sum_{k=1}^M \phi(x_k) \Delta x$ is precisely the corresponding normalizing sum in the denominator of the discrete probabilities (27). Set $x_0 = -A_M$. On each grid cell, $\phi(x_k) - \phi(w) = \int_w^{x_k} \phi'(u) du$. Therefore,

$$\begin{aligned} & \left| \sum_{k=1}^M \phi(x_k) \Delta x - \int_{-A_M}^{A_M} \phi(w) dw \right| = \left| \sum_{k=1}^M \int_{x_{k-1}}^{x_k} (\phi(x_k) - \phi(w)) dw \right| \\ &= \left| \sum_{k=1}^M \int_{x_{k-1}}^{x_k} \int_w^{x_k} \phi'(u) du dw \right| \leq \sum_{k=1}^M \int_{x_{k-1}}^{x_k} \int_w^{x_k} |\phi'(u)| du dw \\ &\leq \Delta x \int_{-A_M}^{A_M} |\phi'(u)| du \leq \Delta x \int_{-\infty}^{\infty} |\phi'(u)| du = \frac{\Delta x}{\sigma} \int_{-\infty}^{\infty} |z| \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \frac{\sqrt{2/\pi}}{\sigma} \Delta x. \end{aligned} \quad (\text{I.79})$$

For a fixed mean, the truncated integral in (I.79) tends to one as $A_M \rightarrow \infty$, proving (I.75). Since that integral is at most one, (I.79) with $\sigma = \sigma_B$ also proves (I.76).

It remains to obtain a bound that is uniform in ν and relative to the numerator. Let $d = 1$ when $\nu \geq x_m$ and $d = -1$ otherwise, and let $\mathcal{I}_M(\nu)$ contain the grid indices on the segment of length σ_B starting at x_m in direction d . From the normal PDF, for $k \in \mathcal{I}_M(\nu)$,

$$\frac{\pi_B(x_k; \nu)}{\pi_B(x_m; \nu)} = \exp \left\{ -\frac{(x_k - \nu)^2 - (x_m - \nu)^2}{2\sigma_B^2} \right\} = \exp \left\{ -\frac{(x_k - x_m)^2 + 2(x_k - x_m)(x_m - \nu)}{2\sigma_B^2} \right\}. \quad (\text{I.80a})$$

Let $r = |\nu - x_m|$ and $t = d(x_k - x_m) \in [0, \sigma_B]$. By the definition of d , $x_k - x_m = dt$ and $x_m - \nu = -dr$. Substitution into (I.80a) gives

$$\frac{\pi_B(x_k; \nu)}{\pi_B(x_m; \nu)} = \exp \left\{ -\frac{t^2 - 2rt}{2\sigma_B^2} \right\} \geq e^{-1/2}, \quad k \in \mathcal{I}_M(\nu), \quad (\text{I.80b})$$

because $r, t \geq 0$ and $t^2 - 2rt \leq t^2 \leq \sigma_B^2$. Although the index m may be near an endpoint on a particular grid, the assumption $x_m \rightarrow x \in \mathbb{R}$ rules this out asymptotically. Indeed, since $x_1 = -A_M + \Delta x$ and $x_M = A_M$, we have $x_m - x_1 = x_m + A_M - \Delta x \rightarrow \infty$ and $x_M - x_m = A_M - x_m \rightarrow \infty$. Consequently, the rightward segment $[x_m, x_m + \sigma_B]$, used when $\nu \geq x_m$, and the leftward segment $[x_m - \sigma_B, x_m]$, used when $\nu < x_m$, are both contained in the grid interval $[x_1, x_M]$ for all sufficiently large M . Because adjacent grid points are Δx apart, at most one terminal piece of the segment, of length less than Δx , is not covered by the rectangle weights of the selected grid points. Hence,

$$\#\mathcal{I}_M(\nu) \Delta x \geq \sigma_B - \Delta x \geq \frac{\sigma_B}{2} \quad \text{once } \Delta x \leq \frac{\sigma_B}{2}. \quad (\text{I.81})$$

Equation (I.80b) bounds the density at each selected grid point, while (I.81) bounds the total mesh weight of all those points. Thus, for every ν and all sufficiently large M ,

$$\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x \geq \sum_{k \in \mathcal{I}_M(\nu)} \pi_B(x_k; \nu) \Delta x \underset{\text{by (I.80b)}}{\geq} e^{-1/2} \pi_B(x_m; \nu) \#\mathcal{I}_M(\nu) \Delta x \underset{\text{by (I.81)}}{\geq} \frac{\sigma_B}{2\sqrt{e}} \pi_B(x_m; \nu). \quad (\text{I.82})$$

In words, (I.81) ensures that the combined weight of the grid points to which the pointwise estimate (I.80b) is applied is bounded away from zero. Using (I.82) in the denominator and the fact that $\pi_B(x_m; \nu) > 0$ gives

$$0 \leq \frac{\pi_{B,m}(\nu)}{\Delta x} = \frac{\pi_B(x_m; \nu)}{\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x} \leq \frac{\pi_B(x_m; \nu)}{\{\sigma_B/(2\sqrt{e})\} \pi_B(x_m; \nu)} = \frac{2\sqrt{e}}{\sigma_B}, \quad \nu \in \mathbb{R}. \quad (\text{I.83})$$

For each fixed ν , the middle expression in (I.83) converges to $\pi_B(x; \nu)$ by (I.75) and $x_m \rightarrow x$. The bound in (I.83), as well as the limiting normal density, is bounded by a constant and is therefore square-integrable under the probability measure with PDF φ_ν . The dominated con-

vergence theorem⁶ gives $\int_{-\infty}^{\infty} |\pi_{B,m}(\nu)/\Delta x - \pi_B(x; \nu)|^2 \varphi_\nu(\nu) d\nu \rightarrow 0$, which proves (I.77). Finally, (I.78) follows from (27) by applying the limit in (I.75) at ε_H and ε_L . ■

Lemma I.6 [Continuous minimum-norm representation]. Define $T : \mathcal{H} \rightarrow \mathcal{L}_2(\mathbb{R})$ by

$$(Tq)(x) = \langle \pi_B(x; \cdot), q \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} \pi_B(x; \nu) q(\nu) \varphi_\nu(\nu) d\nu. \quad (\text{I.84})$$

The operator T is bounded and injective, and the integral operator with kernel K in (28) is induced by TT^* . Moreover,

$$J_\infty^* = \min_{q \in \mathcal{H}} \{ \|q\|_{\mathcal{H}}^2 : Tq = f \}, \quad (\text{I.85})$$

where the minimum equals $+\infty$ when $Tq = f$ has no solution in $\mathcal{H}$.

Proof of Lemma I.6. Proposition 5 provides the positive eigenvalues λ_n and the complete orthonormal basis ψ_n of the integral operator with kernel K , while Proposition 6 characterizes the parameter region in which the series in (I.69) is finite.

The limit in (I.69) does not mean that one can simply set $\tau = 0$ in the Fredholm solution (33). An $\mathcal{L}_2(\mathbb{R})$ solution of the first-kind equation $\mathcal{K}v = f$ requires the stronger condition $\sum_{n=0}^{\infty} \hat{f}_n^2 / \lambda_n^2 < \infty$, whereas finiteness of J_∞^* requires only the series in (I.69) to be finite.

The operator T is a bounded Hilbert–Schmidt operator. Indeed, the integrand is non-negative, so Tonelli’s theorem permits reversing the order of integration, and hence,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi_B(x; \nu)^2 \varphi_\nu(\nu) d\nu dx &= \int_{-\infty}^{\infty} \varphi_\nu(\nu) \left\{ \frac{1}{2\pi\sigma_B^2} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\nu)^2}{\sigma_B^2}\right) dx \right\} d\nu \\ &= \frac{1}{2\sqrt{\pi}\sigma_B} \int_{-\infty}^{\infty} \varphi_\nu(\nu) d\nu = \frac{1}{2\sqrt{\pi}\sigma_B} < \infty. \end{aligned} \quad (\text{I.86})$$

Thus the squared Hilbert–Schmidt norm of T is the finite quantity in (I.86). For $q \in \mathcal{H}$, the Cauchy–Schwarz inequality gives $\int_{-\infty}^{\infty} |q(\nu)| \varphi_\nu(\nu) d\nu \leq \|q\|_{\mathcal{H}} < \infty$. Therefore $q\varphi_\nu \in \mathcal{L}_1(\mathbb{R})$, and Tq is its convolution with a normal PDF. If $Tq = 0$, Fourier transformation gives $0 = \widehat{Tq}(t) = \exp(-0.5\sigma_B^2 t^2) \widehat{q\varphi_\nu}(t)$, $t \in \mathbb{R}$. Uniqueness of the Fourier transform on $\mathcal{L}_1(\mathbb{R})$ gives $q\varphi_\nu = 0$. Since $\varphi_\nu > 0$, this gives $q = 0$, and hence T is injective.

For $g \in \mathcal{L}_2(\mathbb{R})$, the adjoint is $(T^*g)(\nu) = \int_{-\infty}^{\infty} \pi_B(y; \nu) g(y) dy$. To justify the interchange of y - and ν -integration below, fix $x \in \mathbb{R}$ and $g \in \mathcal{L}_2(\mathbb{R})$. The Cauchy–Schwarz inequality gives $\int_{-\infty}^{\infty} |\pi_B(y; \nu) g(y)| dy \leq (2\sqrt{\pi}\sigma_B)^{-1/2} \|g\|_{\mathcal{L}_2(\mathbb{R})}$, uniformly in ν , because the $\mathcal{L}_2(\mathbb{R})$ norm of the translated normal PDF is independent of ν . Hence, the absolute double integral is

⁶The dominated convergence theorem states that if $u_j \rightarrow u$ almost everywhere and $|u_j| \leq G$ for an integrable function G , then u is integrable and $\int u_j \rightarrow \int u$. Here it is applied to the squared difference with respect to the probability measure $\varphi_\nu(\nu) d\nu$; see ?.

at most $(2\sqrt{\pi}\sigma_B)^{-1/2}\|g\|_{\mathcal{L}_2(\mathbb{R})}\int_{-\infty}^{\infty}\pi_B(x;\nu)\varphi_\nu(\nu)d\nu < \infty$, because $\pi_B(x;\nu) \leq (\sqrt{2\pi}\sigma_B)^{-1}$ and φ_ν is a PDF. Fubini's theorem therefore permits the interchange. Consequently,

$$(TT^*g)(x) = \int_{-\infty}^{\infty}\pi_B(x;\nu)\left(\int_{-\infty}^{\infty}\pi_B(y;\nu)g(y)dy\right)\varphi_\nu(\nu)d\nu = \int_{-\infty}^{\infty}K(x,y)g(y)dy.$$

This proves that the kernel K in (28) is induced by TT^* .

Since $TT^*\psi_n = \lambda_n\psi_n$, we have

$$\left\langle \frac{T^*\psi_n}{\sqrt{\lambda_n}}, \frac{T^*\psi_k}{\sqrt{\lambda_k}} \right\rangle_{\mathcal{H}} = 1_{\{n=k\}}, \quad n, k \geq 0, \quad (\text{I.87})$$

$$T\left(\frac{T^*\psi_n}{\sqrt{\lambda_n}}\right) = \sqrt{\lambda_n}\psi_n, \quad n \geq 0. \quad (\text{I.88})$$

If $Tq = f$, then $\langle q, T^*\psi_n/\sqrt{\lambda_n} \rangle_{\mathcal{H}} = \widehat{f}_n/\sqrt{\lambda_n}$, $n \geq 0$. Bessel's inequality therefore gives

$$\|q\|_{\mathcal{H}}^2 \geq \sum_{n=0}^{\infty} \frac{\widehat{f}_n^2}{\lambda_n} = J_{\infty}^*. \quad (\text{I.89})$$

If the series in (I.89) diverges, no $q \in \mathcal{H}$ can satisfy $Tq = f$. Conversely, if $J_{\infty}^* < \infty$, then

$$q^{\dagger} = \sum_{n=0}^{\infty} \frac{\widehat{f}_n}{\lambda_n} T^*\psi_n \quad (\text{I.90})$$

converges in $\mathcal{H}$: by (I.87), its squared coefficient sum in the orthonormal family $T^*\psi_n/\sqrt{\lambda_n}$ is $\sum_n \widehat{f}_n^2/\lambda_n < \infty$. Let $q_N^{\dagger}$ denote the partial sum through $n = N$. Boundedness of T gives

$$\|Tq_N^{\dagger} - Tq^{\dagger}\|_{\mathcal{L}_2(\mathbb{R})} \leq \|T\| \|q_N^{\dagger} - q^{\dagger}\|_{\mathcal{H}} \longrightarrow 0. \quad (\text{I.91})$$

For every finite N , linearity permits T to pass through the partial sum, and $TT^*\psi_n = \lambda_n\psi_n$. Therefore, by (I.91), taking limits in $\mathcal{L}_2(\mathbb{R})$ and using completeness of the basis $\{\psi_n\}$,

$$Tq^{\dagger} = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{\widehat{f}_n}{\lambda_n} TT^*\psi_n = \sum_{n=0}^{\infty} \widehat{f}_n \psi_n = f. \quad (\text{I.92})$$

Thus the interchange of T and the infinite sum in (I.92) follows from continuity of the bounded operator T , rather than from a pointwise interchange. Moreover,

$$\|q^{\dagger}\|_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} \frac{\widehat{f}_n^2}{\lambda_n} = J_{\infty}^*. \quad (\text{I.93})$$

Together with (I.89), this proves (I.85). ■

Proof of Proposition I.8. To prove (I.70), it is enough to establish

$$J_\infty^* \leq \liminf_{M \rightarrow \infty} J_M^*, \quad (\text{I.94a})$$

$$\limsup_{M \rightarrow \infty} J_M^* \leq J_\infty^* \quad \text{when } J_\infty^* < \infty. \quad (\text{I.94b})$$

We begin with (I.94a). Suppose that $\liminf_{M \rightarrow \infty} J_M^* < \infty$ and pass to a subsequence, not relabeled, along which J_M^* converges to this liminf. By (I.73), let q_M denote the corresponding unique minimizer, which satisfies

$$\|q_M\|_{\mathcal{H}}^2 = J_M^*, \quad (\text{I.95})$$

so, the sequence $\{q_M\}$ is bounded in $\mathcal{H}$. Closed bounded balls in a Hilbert space are weakly compact (?). Hence, after passing to a further subsequence, again not relabeled, there is a $q \in \mathcal{H}$ such that

$$q_M \rightharpoonup q \quad \text{weakly in } \mathcal{H}. \quad (\text{I.96})$$

Fix $x \in \mathbb{R}$. On each grid choose an index m such that $x_m \rightarrow x$; its dependence on M is suppressed. Dividing the corresponding constraint in (I.73) by Δx gives

$$\left\langle \frac{\pi_{B,m}}{\Delta x}, q_M \right\rangle_{\mathcal{H}} = \frac{f_m^M}{\Delta x}. \quad (\text{I.97})$$

Both entries in the inner product in (I.97) vary with M , so weak convergence cannot be invoked without separating the two variations. Put $a_M = \pi_{B,m}/\Delta x$ and $a = \pi_B(x; \cdot)$. Lemma I.5 gives $a_M \rightarrow a$ strongly in $\mathcal{H}$, while (I.95) gives $\sup_M \|q_M\|_{\mathcal{H}} < \infty$ along the chosen subsequence. Consequently, $|\langle a_M, q_M \rangle_{\mathcal{H}} - \langle a, q \rangle_{\mathcal{H}}| \leq \|a_M - a\|_{\mathcal{H}} \|q_M\|_{\mathcal{H}} + |\langle a, q_M - q \rangle_{\mathcal{H}}| \rightarrow 0$. In the latter limit, the first term vanishes by strong convergence and boundedness; the second vanishes by weak convergence in (I.96), because a is now fixed. The right-hand side of (I.97) converges to $f(x)$ by Lemma I.5. Therefore, $(Tq)(x) = f(x)$, $x \in \mathbb{R}$. If $J_\infty^* = +\infty$, this contradicts Lemma I.6; hence a finite liminf is impossible in that case. If $J_\infty^* < \infty$, the same lemma and weak lower semicontinuity of the norm (?) give

$$J_\infty^* \leq \|q\|_{\mathcal{H}}^2 \leq \liminf_{M \rightarrow \infty} J_M^*. \quad (\text{I.98})$$

Thus (I.94a) holds in all cases. In particular, $J_M^* \rightarrow +\infty$ whenever $J_\infty^* = +\infty$.

It remains to prove (I.94b). Here $J_\infty^* < \infty$, so Proposition 6 implies the parameter restriction (35). Let $s^2 = \sigma_I^2 - \sigma_B^2 > 0$ and let ϕ_ε be the $\mathcal{N}(\mu\varepsilon, s^2)$ PDF for $\varepsilon \in \{\varepsilon_L, \varepsilon_H\}$. Define $q_\varepsilon(\nu) = \phi_\varepsilon(\nu)/\varphi_\nu(\nu)$. Normal convolution gives

$$(Tq_\varepsilon)(x) = \int_{-\infty}^{\infty} \pi_B(x; \nu) \phi_\varepsilon(\nu) d\nu = \pi_I(x; \varepsilon). \quad (\text{I.99})$$

To determine whether $q_\varepsilon \in \mathcal{H}$, observe that $\|q_\varepsilon\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} \phi_\varepsilon(\nu)^2 / \varphi_\nu(\nu) d\nu$. The integrand is

$$\frac{\phi_\varepsilon(\nu)^2}{\varphi_\nu(\nu)} = \frac{\sigma_\nu}{\sqrt{2\pi} s^2} \exp \left\{ -\frac{(\nu - \mu\varepsilon)^2}{s^2} + \frac{\nu^2}{2\sigma_\nu^2} \right\}. \quad (\text{I.100})$$

The expression in (I.100) is integrable precisely when $1/s^2 - 0.5/\sigma_\nu^2 > 0$, or equivalently $s^2 < 2\sigma_\nu^2$, which is the right-hand inequality in (35). Hence $q_{\varepsilon_L}, q_{\varepsilon_H} \in \mathcal{H}$. By (I.99),

$$q^* = \eta(q_{\varepsilon_H} - q_{\varepsilon_L}) \quad \text{satisfies} \quad Tq^* = f. \quad (\text{I.101})$$

Injectivity of T and Lemma I.6 imply $\|q^*\|_{\mathcal{H}}^2 = J_\infty^*$.

To reproduce the exact finite-state probabilities, define

$$\widehat{q}_M(\nu) = \eta \left(\sum_{k=1}^M \pi_B(x_k; \nu) \Delta x \right) \left\{ \frac{q_{\varepsilon_H}(\nu)}{\sum_{k=1}^M \pi_I(x_k; \varepsilon_H) \Delta x} - \frac{q_{\varepsilon_L}(\nu)}{\sum_{k=1}^M \pi_I(x_k; \varepsilon_L) \Delta x} \right\}. \quad (\text{I.102})$$

The normalizing sum in (I.102) cancels the denominator in $\pi_{B,m}$. Equation (I.99) then gives

$$\langle \pi_{B,m}, \widehat{q}_M \rangle_{\mathcal{H}} = \eta(\pi_{I,m}(\varepsilon_H) - \pi_{I,m}(\varepsilon_L)) = f_m^M, \quad m = 1, \dots, M. \quad (\text{I.103})$$

Thus $\widehat{q}_M$ is feasible in (I.73). Therefore,

$$J_M^* \leq \|\widehat{q}_M\|_{\mathcal{H}}^2. \quad (\text{I.104})$$

Lemma I.5 shows that all three normalizing sums in (I.102) converge pointwise to one and that the biased-investor sum is uniformly bounded above. Since the two informed normalizing sums converge to one, they are bounded away from zero for all sufficiently large M . Hence, for a constant C independent of M , $|\widehat{q}_M| \leq C(|q_{\varepsilon_H}| + |q_{\varepsilon_L}|)$. Since $q_{\varepsilon_L}, q_{\varepsilon_H} \in \mathcal{H}$, the squared difference $|\widehat{q}_M - q^*|^2$ is dominated by an integrable function under $\varphi_\nu(\nu) d\nu$. The dominated convergence theorem (?) therefore gives $\widehat{q}_M \rightarrow q^*$ in $\mathcal{H}$. Combining the latter limit, inequality (I.104), and $\|q^*\|_{\mathcal{H}}^2 = J_\infty^*$, we obtain

$$\limsup_{M \rightarrow \infty} J_M^* \leq \|q^*\|_{\mathcal{H}}^2 = J_\infty^*. \quad (\text{I.105})$$

When $J_\infty^* < \infty$, (I.98) and (I.105) imply $J_M^* \rightarrow J_\infty^*$. When $J_\infty^* = +\infty$, the argument establishing (I.94a) gives $J_M^* \rightarrow +\infty$. Proposition 6 supplies the finite region (I.71). ■