

Financing Small Investments*

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Abstract

How does the amount of external funding required by a firm affect its optimal financing contract? We consider a canonical moral hazard model with entrepreneurial risk aversion. For small financing needs, the optimal contract pays the entrepreneur first, and levered equity financing dominates unlevered equity and debt. Moreover, among these contracts, only levered equity is approximately optimal. With debt financing, the entrepreneur bears the firm's entire risk; while this maximizes effort incentives, this benefit is second-order if the amount raised is small. However, when financing needs are sufficiently large, levered equity financing is dominated by unlevered equity and debt, and the optimal contract involves debt financing. These results rationalize levered equity financing for small investments, as in early-stage start-ups, and debt financing for larger investments.

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1 Introduction

How should a firm raise external financing? The canonical principal-agent model highlights a fundamental tension.¹ Giving investors a claim that is insensitive to output maintains the entrepreneur’s incentives to exert effort. However, giving them an output-sensitive claim improves risk sharing. This trade-off is central to the choice between debt and equity: debt leaves the entrepreneur as the residual claimant at high outputs, strengthening incentives, while equity shares more risk with investors.

Empirically, financing choices vary substantially over the life cycle of a firm. Young firms rely heavily on debt: Robb and Robinson (2014) find that external debt, particularly bank borrowing, is a major source of financing for firms in their first year. Firms obtain greater access to outside equity as they grow and mature (Berger and Udell, 1998), and equity issuance is pervasive among established public firms rather than merely an instrument of last resort (Fama and French, 2005).

However, these patterns could arise either from the size of the firm or the size of the financing need. This paper studies how the firm’s choice of financing depends on the amount of financing raised, holding constant the firm’s technology. We consider a risk-averse entrepreneur who requires external financing for a project and raises it from a risk-neutral investor. Project output depends on the entrepreneur’s unobservable effort, and both parties are protected by limited liability. Our first insight is that, when the financing need is small, the resulting adjustment in effort has only a second-order effect on the entrepreneur’s expected utility. Because the entrepreneur chooses effort optimally, his objective is locally insensitive to changes in effort. Since limited liability precludes payments that are negative or exceed the total output, the entrepreneur remains close to the residual claimant and his effort remains close to the level he would choose without external financing. As a result, for small financing needs, risk sharing rather than effort incentives is the first-order consideration.

This observation leads to a general characterization of the optimal contract. When the financing need is sufficiently small, the investor receives zero at low outputs, so the entrepreneur retains the entire output in these states. Thus, the entrepreneur is paid first: he holds the senior claim, while the investor holds a junior claim. This result is surprising from an incentive perspective. High-output states are more informative about effort, so incentives would be stronger if the entrepreneur were paid primarily in those states. Risk

¹Moral hazard is widely viewed as a central friction behind financing choices: Tirole (2006) argues that “the premise behind modern corporate finance ... is that corporate insiders need not act in the best interests of the providers of the funds.”

sharing points in the opposite direction: because the entrepreneur is risk averse, he values output most in low-output states, where his marginal utility is highest. Since the value effect of the effort adjustment is only second order when the financing need is small, risk sharing dominates and the optimal contract pays the entrepreneur first. Importantly, the zero payment at low outputs is exact, rather than merely converging to zero as the financing need approaches zero.

We next compare three standard financing contracts: levered equity, unlevered equity, and debt. When the financing need is small, levered equity Pareto dominates unlevered equity, which in turn Pareto dominates debt. The ranking is driven by risk sharing, whose benefits are first order, while the value effects of differences in effort are only second order. Levered equity pays the investor only when output is high, when the risk-averse entrepreneur values additional income least. Debt does the opposite: it pays the investor first and leaves the entrepreneur exposed to the riskier residual claim. Unlevered equity lies between these two extremes.

Beyond being the best among these three securities, levered equity is approximately optimal among all contracts satisfying limited liability. Its loss relative to the optimal contract is second order in the financing need, whereas the losses from unlevered equity and debt are first order. Thus, when the financing need is small, levered equity comes close to the optimal financing contract, while unlevered equity and debt do not. More broadly, our analysis provides a way to assess the importance of restrictions on the contracts being considered. For theorists, it shows how much is lost by focusing on a particular class of contracts rather than taking an optimal contracting approach. For practitioners, it shows how much is lost by using familiar contracts rather than the theoretical optimum. In our setting, restricting attention to levered equity is nearly costless when financing needs are small, while restricting attention to unlevered equity or debt is not.

These findings have broader implications for external finance. First, when financing needs are small, the agency cost of moral hazard is negligible relative to the amount raised, so moral hazard has little effect on whether worthwhile projects can obtain financing. Second, external finance can be preferable even for an entrepreneur wealthy enough to self-finance. Although self-financing preserves his incentives, it forces him to bear all of the project's risk; external finance improves risk sharing, and this benefit outweighs the small incentive cost. Indeed, levered equity itself strictly dominates self-financing.

To highlight the importance of small financing needs for our results, we next consider large financing needs. Under an additional monotonicity condition, the priority structure reverses at the financing limit – the largest amount that can be raised. Raising as much as possible requires preserving the entrepreneur's incentives, so his compensation is concen-

trated in high-output states, where it is most effective at encouraging effort. The investor therefore receives the entire output when output is low, while the entrepreneur is rewarded when output is high. Thus, the investor is now paid first: she holds a senior, debt-like claim, while the entrepreneur holds a junior, equity-like claim. This is the mirror image of the small-financing result, where risk sharing dominates and the entrepreneur is paid first.

The ranking of the three standard securities also changes. When the financing need approaches the maximum amount that levered equity can support, levered equity provides very weak incentives because the entrepreneur’s payoff barely responds to high output. Debt and unlevered equity preserve greater exposure to high-output states and therefore stronger incentives. Under the conditions of our result, both therefore dominate levered equity for financing needs sufficiently close to this boundary.

Our results highlight how the balance between risk sharing and incentives changes with the amount of financing required. Innes (1990) obtains debt when the entrepreneur is risk neutral, so risk sharing is irrelevant and the contract is designed to provide incentives. With a risk-averse entrepreneur, risk sharing becomes important. When financing needs are small, risk sharing dominates the incentive effect, so the entrepreneur is paid first and levered equity is approximately optimal; at the financing limit, incentive provision dominates and debt-like priority reemerges.

Taken together, our results predict equity-like financing when financing needs are small and debt-like priority at the financing limit. This runs counter to the broad empirical pattern in which young firms rely heavily on debt while established firms make greater use of outside equity. Because our model varies financing needs while holding other project characteristics fixed, this is not a direct empirical test. Instead, it shows that the canonical moral-hazard mechanism pushes financing choices in the opposite direction, suggesting an important role for other frictions such as collateral constraints or asymmetric information.

Other frictions can also generate equity financing. Models of asymmetric information predict equity when investors have superior information about firm value or growth opportunities, or incentives to acquire such information (Boot and Thakor (1993), Fulghieri and Lukin (2001), Axelson (2007), Fulghieri, García, and Hackbarth (2020)). Such mechanisms may be less applicable to some of the small-financing settings that motivate our analysis. Angel investors, for example, may be relatively unsophisticated and conduct limited due diligence (Edelman, Manolova, and Brush (2017), Nanda and Phillips (2023), Denes et al. (2023), Huang et al. (2023), Yimfor (2025)). Our mechanism instead requires no informational advantage on the part of investors: equity-like financing arises from risk sharing between a risk-averse entrepreneur and a risk-neutral investor.

Our paper is also related to research showing when optimal contracts under moral haz-

and depart from debt. Equity components can arise when the entrepreneur or investor is ambiguity averse (Lee and Rajan (2020), Malenko and Tsoy (2023)).

The small-financing regime that drives many of these results is economically important in practice. Many start-ups require little external funding at early stages, and technological change has reduced these needs further. Ewens et al. (2018) show that cloud computing allowed entrepreneurs to replace large fixed investments in hardware with variable cloud expenses; Nanda and Phillips (2023) argue that this enabled entrepreneurs to start firms with substantially less capital. Small financing needs are also common among micro, small, and medium enterprises in developing countries: the average financing amount reported in the IFC’s MSME Finance Gap study is \$22,840, while the average loan on Kiva is below \$1,000.

2 The Model

We consider a canonical principal-agent problem. A risk-averse entrepreneur requires an amount $I > 0$ of financing for a risky project and raises it from an investor. Because the investor holds a large, well-diversified portfolio, she is risk neutral. The output from the project is $q \in [0, \bar{q}]$ and depends on the entrepreneur’s unobservable effort $e \in [0, \bar{e}]$. As is standard, effort entails any action that improves output but is costly to the entrepreneur, such as working rather than shirking, choosing projects that generate cash flows rather than private benefits, or not extracting rents.

Conditional on effort, output is distributed according to a strictly positive density

$$f(q|e) > 0 \quad \forall q, e.$$

We assume that the density is continuously differentiable in q , twice continuously differentiable in e , and satisfies the monotone likelihood ratio property (MLRP):²

$$\frac{f_e(q|e)}{f(q|e)} \text{ is strictly increasing in } q.$$

The entrepreneur has outside wealth $\bar{W} > 0$. He is risk averse, with a thrice continuously differentiable, strictly increasing, and strictly concave utility function $u(\cdot)$. His effort cost function $C(\cdot)$ is strictly increasing and twice continuously differentiable. Without loss of generality, we normalize $C(0) = 0$. To ensure that the first-order approach holds, we assume

²Since f is strictly positive and is continuous on a compact set, there exists $\delta > 0$ such that $f(q|e) \geq \delta$ for all q, e .

that the cost of effort is sufficiently convex:

$$u'(\bar{W})K_e < C''(e) \quad \text{for all } e \in [0, \bar{e}], \quad (1)$$

where $K_e := \int_0^{\bar{q}} q \max\left\{\frac{\partial^2 f}{\partial e^2}(q|e), 0\right\} dq$. We also assume that the marginal cost of effort is high enough at the highest effort level:

$$C'(\bar{e}) > (u(\bar{W} + \bar{q}) - u(\bar{W})) \max_{e \in [0, \bar{e}]} \int_0^{\bar{q}} \max\{f_e(q|e), 0\} dq. \quad (2)$$

which simplifies the proofs by ruling out corner solutions at $\bar{e}$.

We use as our benchmark the effort chosen when no external financing is required and the entrepreneur retains the entire output:

$$e^* \in \arg \max_{e \in [0, \bar{e}]} \int_0^{\bar{q}} u(\bar{W} + q) f(q|e) dq - C(e).$$

As will be shown in Lemma 1 below, e^* is unique. We assume it is interior: $e^* \in (0, \bar{e})$.³

A contract is a Borel-measurable function $s : [0, \bar{q}] \rightarrow \mathbb{R}_+$ specifying the investor's payment for each output. The entrepreneur retains the residual $q - s(q)$. Let $\mathcal{S}_{LL}$ denote the space of contracts satisfying bilateral limited liability:

$$0 \leq s(q) \leq q \quad \forall q. \quad (3)$$

For any given contract s , the entrepreneur's expected utility is

$$U(s, e) := \int_0^{\bar{q}} u(\bar{W} + q - s(q)) f(q|e) dq - C(e).$$

The entrepreneur chooses effort to maximize his expected utility:

$$e(s) \in \arg \max_{e \in [0, \bar{e}]} U(s, e). \quad (4)$$

Let $\mathbf{0}$ denote the contract that always repays investors zero: $s(q) \equiv 0$. By construction, $e(\mathbf{0}) = e^*$.

The investor's expected payment given contract s is

$$\Pi(s) := \int_0^{\bar{q}} s(q) f(q|e(s)) dq.$$

³Given the assumption in equation (2), a sufficient condition is $C'(0) = 0$. It is straightforward to generalize our results to the case in which the zero-financing benchmark is attained at the upper effort boundary: $e^* = \bar{e}$. If instead the lowest effort is first-best optimal, $e^* = 0$, there is no incentive problem to start with.

In a competitive capital market, $\Pi(s)$ is pinned down by the zero-profit condition, so it equals the amount of funds I provided to the entrepreneur. If the investor has bargaining power, $\Pi(s)$ may also incorporate an additional markup.

As mentioned above, (1) implies the validity of the first-order approach. Lemma 1 states this result formally:

Lemma 1 (First-order approach). *For any $s \in \mathcal{S}_{LL}$, $U(s, \cdot)$ is strictly concave in e .*

Lemma 1 also implies that the entrepreneur's effort $e(s)$ is unique for every contract satisfying limited liability.

3 Small Financing Needs

We first establish a uniform bound on the effort distortion. When the investor's expected payment is small, the entrepreneur's effort remains close to the zero-financing benchmark, and the resulting cost of the adjustment in effort is second order.

Lemma 2 (Effort distortion is second order). *There exist constants $\bar{I}, K_1, K_2 > 0$ such that, for every $I \in (0, \bar{I})$ and every $s \in \mathcal{S}_{LL}$ with $\Pi(s) = I$, the optimal effort is interior, $0 < e(s) < \bar{e}$, and*

$$|e(s) - e^*| \leq K_1 I, \quad (5)$$

$$0 \leq U(s, e(s)) - U(s, e^*) \leq K_2 I^2. \quad (6)$$

Inequality (5) bounds the change in effort, relative to $I = 0$, by a first-order term in I . Inequality (6) bounds the resulting gain from reoptimizing effort, relative to holding effort fixed at e^* , by a second-order term. This allows us to decompose the change in the entrepreneur's value relative to the zero-financing benchmark into a direct effect of the contract, holding effort fixed, and an indirect effect through the entrepreneur's adjustment of effort:

$$U(\mathbf{0}, e^*) - U(s, e(s)) = U(\mathbf{0}, e^*) - U(s, e^*) - \underbrace{(U(s, e(s)) - U(s, e^*))}_{O(I^2)}.$$

The last term is second order. Thus, although effort itself may respond at the first order, the contribution of that response to the entrepreneur's value is only second order.

It is not automatic that small I will have similar effects to the case of $I = 0$. A small expected payment to the investor need not imply small payments in every state: a contract could give the investor a large payment for some output realizations and a negative payment

for others, potentially having a significant effect on the entrepreneur's effort incentives. The key to our result is limited liability. Because the investor's payment cannot be negative, large positive payments cannot be offset by negative payments in other states. Thus, when her expected payment is small, she cannot receive large payments in many states. As a result, there is little change to the entrepreneur's stake in output, and therefore on his effort incentives.⁴ The entrepreneur then chooses effort optimally under the contract. At the optimum, his objective is locally insensitive to effort. By the envelope theorem, the small adjustment in effort therefore has an even smaller effect on his value.

3.1 Levered Equity, Unlevered Equity, and Debt

In this subsection, we compare three classes of contracts often used in practice.

- A *levered-equity* contract is indexed by a threshold $D_\ell \in [0, \bar{q}]$ and pays the investor $s_\ell(q) = \max\{q - D_\ell, 0\}$. The investor receives zero for outputs below D_ℓ and the residual $q - D_\ell$ for outputs above D_ℓ .
- An *unlevered-equity* contract is indexed by an equity share $\alpha \in [0, 1]$ and pays the investor $s_u(q) = \alpha q$. The investor receives a fraction α of output in every state.
- A *debt* contract is indexed by a face value $D_d \in [0, \bar{q}]$ and pays the investor $s_d(q) = \min\{q, D_d\}$. The investor receives total output if it falls short of D_d , and D_d otherwise.

All three contracts belong to $\mathcal{S}_{LL}$. To compare them at a common financing need, we first establish that each family contains a unique contract that gives the investor an expected payment of I . The following lemma establishes this result for sufficiently small I .

Lemma 3 (Existence and uniqueness of break-even parameters). *There exists $\bar{I} > 0$ such that, for every $I \in (0, \bar{I})$, there exist unique parameters $D_\ell^I \in (0, \bar{q})$, $\alpha^I \in (0, 1)$, and $D_d^I \in (0, \bar{q})$ such that*

$$\Pi(s_\ell^{D_\ell^I}) = \Pi(s_u^{\alpha^I}) = \Pi(s_d^{D_d^I}) = I.$$

Moreover, $\lim_{I \downarrow 0} D_\ell^I = \bar{q}$, $\lim_{I \downarrow 0} \alpha^I = 0$, and $\lim_{I \downarrow 0} D_d^I = 0$.

⁴Without the investor's limited-liability constraint $s(q) \geq 0$, this conclusion can fail. For sufficiently small I , let

$$c = \int_0^{\bar{q}} q f(q|0) dq - I$$

and consider $s(q) = q - c$. The entrepreneur then retains the constant payoff c , chooses zero effort, and the investor receives expected payment I . Thus, the effort distortion need not vanish as $I \downarrow 0$.

For every $I \in (0, \bar{I})$, let $s_\ell^I := s_\ell^{D_\ell^I}$, $s_u^I := s_u^{\alpha^I}$, and $s_d^I := s_d^{D_d^I}$. By construction,

$$\Pi(s_j^I) = I, \quad j \in \{\ell, u, d\}.$$

Define the entrepreneur's value under contract s as:

$$V(s) := U(s, e(s)).$$

We now show that, when I is small, levered equity Pareto dominates unlevered equity, which in turn Pareto dominates debt.

Theorem 1 (Pareto ranking for small financing needs). *There exists $\bar{I} > 0$ such that for every $I \in (0, \bar{I})$,*

$$V(s_\ell^I) > V(s_u^I) > V(s_d^I).$$

By construction, the three contracts provide the same expected payoff to the risk-neutral investor. As a result, the entrepreneur's utility ranking also gives a Pareto ranking.

For small financing needs, the value effect of the contracts' different effort responses is only second order. Their first-order ranking is consequently determined by risk sharing. The entrepreneur's marginal utility is highest in low-output states, so he prefers to retain output in those states and transfer output to the investor in high-output states. Levered equity does exactly this: the investor is paid only from the upper tail of output, leaving the entrepreneur a capped claim. Debt does the opposite, giving the investor the senior, relatively safe claim and leaving the entrepreneur exposed to the volatile residual. Unlevered equity lies between these two extremes: it transfers the same fraction of output to the investor in every state, rather than concentrating her payment in either the upper or lower tail. Hence, for sufficiently small I , levered equity Pareto dominates unlevered equity, which in turn Pareto dominates debt.

3.2 Approximate Optimality

Theorem 1 ranks the three standard securities, but it does not reveal how close any of them comes to the optimal contract. The second-best contract maximizes the entrepreneur's utility while giving an expected payment of I to the investor. The entrepreneur's utility in the second-best optimal contract equals:

$$V^* := \sup_{s \in \mathcal{S}_{LL}} \{V(s) : \Pi(s) = I\}. \quad (7)$$

We now measure each security's utility loss relative to the optimal value V^* defined above.

Theorem 2 (Approximate optimality of levered equity). *There exist constants $\bar{I}, \kappa, K > 0$ such that for every $I \in (0, \bar{I})$,*

$$0 \leq V^* - V(s_\ell^I) \leq KI^2 \quad \text{and} \quad \min \{V^* - V(s_u^I), V^* - V(s_d^I)\} \geq \kappa I.$$

Theorem 2 strengthens the pairwise ranking in Theorem 1. Levered equity is not only the best of these three standard securities: it attains approximately the entire utility that can be attained by any limited liability contract. By contrast, unlevered equity and debt incur first-order losses.

With levered equity financing, the entrepreneur gets all the firm's output in low-output states. Holding effort fixed, this is the least costly way to transfer an expected payment to the investor under limited liability: it concentrates her claim in high-output states, where the entrepreneur's marginal utility is lowest, while leaving him the entire output in low states, where marginal utility is highest. The investor's limited-liability constraint prevents her from providing full insurance by making payments to the entrepreneur in the worst states.

The remaining question is whether allowing effort to respond to the contract overturns this risk-sharing advantage. When I is small, it does not: the resulting effect on the entrepreneur's value is only second order. Its risk-sharing advantage therefore survives the endogenous effort response, yielding only a second-order loss relative to the optimal contract. Unlevered equity and debt, by contrast, lead to inefficient risk sharing. Both require the entrepreneur to surrender output in states in which his marginal utility is higher, generating a first-order loss that cannot be offset by the second-order value effects of their different effort responses.

The conclusions from Theorem 2 automatically hold on any subclass of contracts satisfying limited liability that includes levered equity. For example, they hold if one imposes under additional monotonicity conditions (see Section 4).

3.3 Agency Costs and Self-Financing

We have so far asked which financing contract maximises the entrepreneur's value when effort is unobservable. We now ask a different question: how costly is moral hazard itself? The optimal value V^* studied above is the second-best value where effort is unobservable and has to be induced by the contract. At the first best, effort is contractible and can be chosen jointly with the financing contract. The difference between the first-best and second-best values measures the agency cost of moral hazard.

This comparison matters because moral hazard can prevent external financing even when a project is worth undertaking. A project may generate sufficient value at the first best, yet leave the entrepreneur below his reservation utility once the incentive cost of unobservable effort is taken into account. The magnitude of the agency cost therefore determines the wedge between the projects that are efficient and the projects that can be financed.

Formally, in the first-best benchmark, the contract can specify the entrepreneur's effort. The entrepreneur's first-best expected utility is:

$$V^{FB} := \sup \left\{ U(s, e) : s \in \mathcal{S}_{LL}, e \in [0, \bar{e}], \int_0^{\bar{q}} s(q) f(q|e) dq = I \right\}.$$

Proposition 1 states that, when the financing need is small, the agency cost of moral hazard is second order in the amount of financing required.

Proposition 1 (Agency cost is second order). *There exist constants $\bar{I}, \hat{K} > 0$ such that $V^{FB} - V^* \leq \hat{K}I^2$ for every $I \in (0, \bar{I})$. In particular, the agency cost is negligible relative to the financing need:*

$$\lim_{I \searrow 0} \frac{V^{FB} - V^*}{I} = 0.$$

Let $\underline{V}$ denote the entrepreneur's reservation utility. A project requiring financing I is worth undertaking at the first best if $V^{FB} \geq \underline{V}$, and it can be financed under moral hazard if $V^* \geq \underline{V}$. Proposition 1 therefore implies that every project satisfying

$$V^{FB} \geq \underline{V} + \hat{K}I^2$$

can be financed externally. Only projects whose first-best value lies in the interval

$$\left[\underline{V}, \underline{V} + \hat{K}I^2 \right)$$

may be worth undertaking yet unfinanceable. Thus, for small financing needs, moral hazard can prevent financing only for projects whose first-best surplus is itself of second order.

The result distinguishes the cost of external finance from the agency cost of moral hazard. In both the first best and the second best, the entrepreneur must give the investor a claim with expected payment I . The resulting risk-sharing cost is therefore present even when effort is contractible and there are no agency costs.

At the first best, levered equity is exactly optimal because it minimizes the risk-sharing cost of giving the investor her required payment. When effort is unobservable, levered equity remains approximately optimal, and the value effect of the resulting effort adjustment is only second order. The inability to specify effort therefore creates only a second-order gap between the first-best and second-best values.

Even though the agency cost of moral hazard is second order, the Modigliani and Miller (1958) capital structure irrelevance theorem is still violated. The reason is that the small agency cost is achieved by choosing the optimal financing contract. Levered equity is exactly optimal at the first best when effort is contractible (see Proposition 5 in the Online Appendix) and approximately optimal when effort is unobservable (Theorem 2). Thus, the inability to contract on effort generates a second-order loss only when the financing contract is chosen optimally. Capital structure itself nevertheless remains important. If unlevered equity or debt are instead used, Theorem 2 shows that first-order losses arise even when I is small. Hence, moral hazard is approximately costless under the optimal capital structure, but choosing the optimal capital structure remains first-order.

The first-best/second-best comparison holds external financing fixed and asks how much is lost because effort cannot be specified in the contract. We end this section with a different question: if the entrepreneur has sufficient wealth to contribute I himself, should he use external finance at all? Self-financing eliminates the moral-hazard distortion, but it also prevents the entrepreneur from sharing project risk with the risk-neutral investor.⁵

Corollary 1 (External finance dominates self-financing). *If u is C^3 on a neighborhood of $[\bar{W}, \bar{W} + \bar{q}]$, there exists $\bar{I} > 0$ such that for every $I \in (0, \bar{I})$, the entrepreneur is strictly better off under optimal external financing than under self-financing.*

Corollary 1 establishes that optimal external financing strictly dominates self-financing. Combined with Theorem 2, it also implies that levered equity itself strictly dominates self-financing for sufficiently small I .

Intuition might suggest the opposite. Self-financing allows the entrepreneur to remain the residual claimant on project output, thereby avoiding the moral-hazard distortion created by external finance. One might therefore expect an entrepreneur with sufficient wealth to finance the project himself. However, self-financing also requires the risk-averse entrepreneur to bear all of the project's output risk. External finance allows him instead to sell a claim concentrated in high-output states to a risk-neutral investor, thereby improving risk sharing. When I is small, this risk-sharing benefit is first order, whereas the value loss from weakened effort incentives is only second order. The benefit therefore outweighs the cost, and the entrepreneur is strictly better off using external finance.

Together, Theorem 1 and Corollary 1 invert the conventional pecking order for small financing needs. External equity, normally viewed as a financing instrument of last resort

⁵The assumption that u is three times continuously differentiable on a neighborhood of $[\bar{W}, \bar{W} + \bar{q}]$ ensures that self-financing a small project is well-defined – we consider this hypothetical case in Corollary 1.

(Myers and Majluf, 1984), is the entrepreneur's first choice, preferred to both debt and self-financing. The next section examines whether the priority structure of the optimal contract and the ranking of the three standard securities continue to hold as the financing need becomes large.

3.4 Optimal Contracts

We now characterize a feature shared by every optimal contract: which party is paid first when output is low. Let s_I^* denote a second-best optimal contract, which can be shown to exist. Proposition 2 studies whether the entrepreneur or investor is paid first, i.e. receives a positive payoff when output is low.

Proposition 2 (Entrepreneur is paid first when I is small). *For every $q_0 < \bar{q}$ there exists $\bar{I}(q_0) > 0$ such that, for every $I \in (0, \bar{I}(q_0))$ and every optimal contract,*

$$s_I^*(q) = 0 \quad \text{for Lebesgue almost every } q \in [0, q_0].$$

Proposition 2 states that, for sufficiently small I , the optimal contract is such that the investor's payoff is zero when output is below a threshold – equivalently, the entrepreneur's payoff below the threshold is q . Thus, the entrepreneur is paid first. This result is surprising from an incentive perspective. Under MLRP, high-output states are more informative about effort, so incentives would be stronger if the entrepreneur were paid primarily in those states rather than at low outputs. However, as Lemma 2 established, when the financing need is small, differences in effort across contracts have only a second-order effect on the entrepreneur's value. Risk sharing is therefore the first-order consideration. Because the entrepreneur is risk averse, it is optimal for him to retain output in low-output states, where his marginal utility is highest, and to give the risk-neutral investor a claim on higher outputs. This is why the optimal contract pays the entrepreneur first.

The small financing need is also important for implementing this risk-sharing improvement. Given that the investor's overall claim is small, there are higher-output states in which her payment can be increased without violating the entrepreneur's limited liability. Thus, if the investor were paid in low-output states, her claim could be shifted from these states to higher-output states while keeping her expected payment unchanged. The resulting risk-sharing gain is first order, while the effect through effort is only second order. Such a reallocation therefore increases the entrepreneur's value when I is small, so an optimal contract cannot pay the investor at low outputs.

This conclusion is exact, not merely asymptotic. For every $q_0 < \bar{q}$, once the financing need is sufficiently small, every optimal contract pays the investor zero for almost every

output in $[0, q_0]$. The investor's payment does not merely converge to zero in these states as $I \downarrow 0$; it is exactly zero for each sufficiently small positive I .

4 Large Financing Needs

Section 3 delivers two central conclusions. First, among the three standard securities, levered equity dominates unlevered equity, which in turn dominates debt, and levered equity is approximately optimal. Second, every optimal contract pays the entrepreneur first: the investor receives zero in low-output states. To highlight the role of a small I in delivering the above results, we now turn to the case of large financing needs. We study two complementary large-financing benchmarks. First, we compare the three standard securities for financing needs close to levered equity's own financing capacity, denoted below by $\bar{\mu}$. Second, we characterize the optimal contract at the financing limit, defined as the largest amount that can be raised over the admissible contract space. These two exercises respectively address the ranking of standard securities and the shape of the unrestricted optimal contract.

Unlike in the small- I case, the investor's expected payment need not be monotone in the debt face value or the unlevered-equity share. A given financing need may therefore be supported by more than one parameter. We select the smallest break-even parameter. Among contracts that give the investor the same expected payment, this choice leaves the entrepreneur weakly more output in every state and is therefore Pareto dominant. Let

$$\bar{\mu} := \int_0^{\bar{q}} q f(q|0) dq$$

denote expected output at zero effort. Under levered equity, setting $D_\ell = 0$ gives the investor the entire output, leaves the entrepreneur with no project payoff, and therefore induces zero effort; the investor then receives $\bar{\mu}$. We study financing needs approaching this benchmark from below.

To compare all three securities near $\bar{\mu}$, we require two conditions. The first ensures the investor's expected payment under levered equity decreases monotonically with D_ℓ , so its maximum financing capacity is $\bar{\mu}$. Differentiating $\Pi(s_\ell)$ using the entrepreneur's first-order condition, which is valid by Lemma 1, yields the following sufficient condition for $\Pi(s_\ell)$ to be strictly decreasing in D_ℓ :

$$\begin{aligned} & u'(\bar{W} + D_\ell) \int_{D_\ell}^{\bar{q}} f_e(q|e) dq \int_{D_\ell}^{\bar{q}} (q - D_\ell) f_e(q|e) dq \\ & < \int_{D_\ell}^{\bar{q}} f(q|e) dq \left(C''(e) - \int_0^{\bar{q}} u(\bar{W} + \min\{q, D_\ell\}) f_{ee}(q|e) dq \right) \end{aligned} \quad (8)$$

The two sides of condition 8 capture the opposing effects of increasing D_ℓ on the investor's expected payment. The left-hand side (LHS) captures the positive indirect effect: a higher D_ℓ gives the entrepreneur a larger stake in output, which may induce greater effort and thereby increase the investor's expected payment. The right-hand side (RHS) captures the negative direct effect, adjusted for the curvature of the entrepreneur's effort problem: holding effort fixed, a higher D_ℓ reduces the investor's claim. The condition requires the latter effect to dominate the former. Thus, increasing D_ℓ lowers the investor's expected payment.

Second, we require debt and unlevered equity each to have financing capacity strictly greater than $\bar{\mu}$. This ensures that both securities remain feasible at $I = \bar{\mu}$ while leaving the entrepreneur a positive stake in output. Let

$$L(q|e) := \frac{f_e(q|e)}{f(q|e)}$$

denote the likelihood ratio, and let covariance below be taken under the output distribution $f(q|0)$. Lemma 4 in the Online Appendix shows that debt and unlevered equity can each generate pledgeable income strictly greater than $\bar{\mu}$ if $C'(0) = 0$ and:

$$u'(\bar{W}) \operatorname{cov}(q, L(q|0)) \min \left\{ L(\bar{q}|0), \frac{\operatorname{cov}(q, L(q|0))}{\bar{\mu}} \right\} > C'''(0), \quad (9)$$

Given $C'(0) = 0$, condition (9) is automatically satisfied when $C'''(0) = 0$.

Proposition 3 shows that levered equity, which is the best of the three standard securities and approximately optimal for small financing needs, becomes strictly dominated when the financing need is large.

Proposition 3 (Levered equity is dominated at large I). *If $C'(0) = 0$ and the conditions in (8) and (9) are satisfied, then there exists $\underline{I} < \bar{\mu}$ such that for every $I \in (\underline{I}, \bar{\mu})$,*

$$V(s_d) > V(s_\ell) \quad \text{and} \quad V(s_u) > V(s_\ell).$$

When the financing need is large, effort incentives become a primary consideration. As I approaches $\bar{\mu}$ under levered equity, the threshold D_ℓ approaches zero. The entrepreneur's retained claim $\min\{q, D_\ell\}$ is then flat over almost the entire output range and varies only in very low-output states. It therefore provides little incentive to increase the probability of high output. In the limit, the entrepreneur receives no project payoff, chooses zero effort, and obtains utility $u(\bar{W})$.

Debt and unlevered equity allocate the entrepreneur's remaining stake differently. Under debt, he retains the upper-tail residual $\max\{q - D_d, 0\}$; under unlevered equity, he retains

the fraction $(1 - \alpha)q$. Both claims continue to reward high output, where the likelihood ratio is greatest, and therefore preserve effort incentives more effectively. Under the stated conditions, each can give the investor $\bar{\mu}$ while leaving the entrepreneur a strictly positive payoff in some states. Hence, both debt and unlevered equity dominate levered equity for financing needs sufficiently close to $\bar{\mu}$. The proposition does not rank debt against unlevered equity; it establishes that levered equity falls below both.

We now determine whether the priority structure of the optimal contract changes at the financing limit. The small-financing results required only limited liability. To characterize the contract at the financing limit, we additionally impose bilateral monotonicity. This requires both the investor's payment $s(q)$ and the entrepreneur's retained output $q - s(q)$ to be nondecreasing in output:

$$0 \leq s(q_2) - s(q_1) \leq q_2 - q_1 \text{ for all } q_1 \leq q_2.$$

The restriction captures the natural requirement that neither party becomes worse off when realized output increases, otherwise it would “burn” output. Let $\mathcal{S}_m \subset \mathcal{S}_{LL}$ denote the set of contracts satisfying limited liability (3) and bilateral monotonicity, which includes levered equity, unlevered equity, and debt.

Consider the program:

$$\max_{s \in \mathcal{S}_m, e \in [0, \bar{e}]} \int_0^{\bar{q}} s(q) f(q|e) dq \quad \text{s.t. (4).} \quad (10)$$

Let $\hat{s}$ and $\hat{e}$ denote solutions to this program.⁶ Thus, this program considers the opposite case of a very high financing need, so the entrepreneur needs to give as much of the output as possible to the investor. Proposition 4 characterises the priority structure of an optimal contract at the financing limit.

A natural approach would be to solve the optimal-contract problem for I close to the largest feasible financing amount. However, this upper bound is itself endogenous: the amount that can be pledged depends on the effort induced by the contract. Giving the investor more output can increase her payment directly but may also weaken effort and reduce expected output.

We therefore consider the dual problem of choosing the contract that maximizes the investor's expected payment subject to incentive compatibility. Its value identifies the fi-

⁶Under bilateral monotonicity, the payoffs must satisfy $w(0) = 0$ and $0 \leq \dot{w}(q) \leq 1$, so the set of admissible payoffs is compact in the sup norm by the Arzelà-Ascoli theorem. By Lemma 1 and the maximum theorem, the effort induced is unique and continuous in $w(\cdot)$, so the investor's expected payoff is a continuous function of $w(\cdot)$, and the Weierstrass theorem applies, which guarantees existence of a solution.

financing limit, and its solution describes the contract used when the entrepreneur must pledge as much output as possible.

Proposition 4 (Investor is paid first at the financing limit). *There exists $\phi > 0$ such that $s(q) = q$ for all $q \leq \phi$.*

At the financing limit, the investor receives as much of the project's output as possible while the contract still provides the entrepreneur with incentives to exert effort. Under MLRP, compensation paid to the entrepreneur in high-output states is particularly effective at providing incentives, because those states have the highest likelihood ratios. His limited compensation should therefore be concentrated in such states.

Consequently, the entrepreneur's limited-liability constraint binds at low outputs: he receives zero and the investor receives the entire output. The entrepreneur holds a junior, equity-like claim, while the investor holds a senior, debt-like claim. This is the mirror image of the small- I result. When the financing need is small, risk sharing dominates and the investor's limited-liability constraint binds in low-output states. At the financing limit, incentive provision dominates and the entrepreneur's limited-liability constraint binds in those states.

The small- and large-financing results reveal a reversal in the determinants of security design. When the financing need is small, the entrepreneur remains close to the residual claimant, so effort adjustments have only a second-order value effect and risk sharing is the first-order concern: the entrepreneur is paid first, and levered equity is approximately optimal.

At the financing limit, incentive provision becomes central: the entrepreneur is compensated only in high-output states, and the investor is paid first. Separately, as the financing need approaches levered equity's capacity $\bar{\mu}$, both debt and unlevered equity dominate levered equity because they preserve the entrepreneur's exposure to high-output states and therefore provide stronger incentives. Thus, the financing contract favored by the canonical moral-hazard model depends fundamentally on the amount that must be raised.

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Appendix

Proof of Lemma 1: The FOA is valid if the objective function $\int_0^{\bar{q}} u(\bar{W} + w(q))f(q|e)dq - C(e)$ is concave in e : Thus, the FOA is valid if:

$$\int_0^{\bar{q}} u(\bar{W} + w(q)) \frac{\partial^2 f}{\partial e^2}(q|e)dq < C''(e) \quad \forall e. \quad (11)$$

From bilateral limited liability, $w(q) \in [0, q]$ for all q . We have:

$$\begin{aligned} & \int_0^{\bar{q}} u(\bar{W} + w(q)) \frac{\partial^2 f}{\partial e^2}(q|e)dq \\ & \leq \int_0^{\bar{q}} \max \left\{ (u(\bar{W}) + u'(\bar{W}) \max\{q, 0\}) \frac{\partial^2 f}{\partial e^2}(q|e), u(\bar{W}) \frac{\partial^2 f}{\partial e^2}(q|e) \right\} dq \end{aligned}$$

since u is monotonically increasing and concave. Therefore, a sufficient condition for equation (11) is:

$$u(\bar{W}) \int_0^{\bar{q}} \frac{\partial^2 f}{\partial e^2}(q|e)dq + u'(\bar{W}) \int_0^{\bar{q}} q \max \left\{ \frac{\partial^2 f}{\partial e^2}(q|e), 0 \right\} dq < C''(e) \quad \forall e. \quad (12)$$

Rewriting using $\int_0^{\bar{q}} \frac{\partial^2 f}{\partial e^2}(q|e)dq = \frac{\partial^2}{\partial e^2} \int_0^{\bar{q}} f(q|e)dq = 0$, the definition of K_e gives Lemma 1. ■

Proof of Lemma 3: Under (1), the entrepreneur's effort maximization problem is strictly concave, so the optimal effort is unique and, by the maximum theorem, it varies continuously with the parameter of each of the three one-parameter contract families. Therefore the investor's expected payoff is continuous in each parameter.

For levered equity, the investor's expected payoff is zero at $D_\ell = \bar{q}$ and strictly positive at any $D_\ell < \bar{q}$. For unlevered equity, it is zero at $\alpha = 0$ and strictly positive at any $\alpha > 0$. For debt, it is zero at $D_d = 0$ and strictly positive at any $D_d > 0$. Hence, by the intermediate value theorem, each parameter can be chosen to raise any sufficiently small positive amount I .

Next, we establish uniqueness. By condition (1), continuity, and compactness, there exists $m > 0$ such that

$$U_{ee}(s, e) \leq -m < 0 \quad \text{for every } s \in \mathcal{S}_{LL} \text{ and } e \in [0, \bar{e}]. \quad (13)$$

Let

$$M_e := \max_{(q,e) \in [0,\bar{q}] \times [0,\bar{e}]} |f_e(q|e)| < \infty.$$

The bounds established above imply that every contract in any of the three families that raises I converges to the zero contract as $I \downarrow 0$. Since the induced effort is continuous in

each family parameter and the zero contract induces the interior effort $e^* \in (0, \bar{e})$, after reducing $\bar{I}$ if necessary, every contract in one of the three families satisfying $\Pi(s) = I$ with $I \in (0, \bar{I})$ induces an interior effort.

We first establish a comparison that applies to all three families. Let s_1 and s_2 be two contracts in the same family satisfying $\Pi(s_1) = \Pi(s_2) = I$, and suppose that

$$s_1(q) \leq s_2(q) \quad \text{for every } q,$$

with strict inequality on a set of positive Lebesgue measure. Let $e_i := e(s_i)$, for $i = 1, 2$, and define

$$\Delta := \int_0^{\bar{q}} (s_2(q) - s_1(q)) dq > 0.$$

Given that e_1 and e_2 are interior, they satisfy

$$U_e(s_1, e_1) = U_e(s_2, e_2) = 0.$$

Applying the mean-value theorem to $U_e(s_2, \cdot)$, for some $\tilde{e}$ between e_1 and e_2 ,

$$\begin{aligned} 0 &= U_e(s_2, e_2) - U_e(s_1, e_1) \\ &= U_{ee}(s_2, \tilde{e})(e_2 - e_1) + U_e(s_2, e_1) - U_e(s_1, e_1). \end{aligned}$$

Consequently, using (13),

$$\begin{aligned} m|e_2 - e_1| &\leq |U_e(s_2, e_1) - U_e(s_1, e_1)| \\ &= \left| \int_0^{\bar{q}} [u(\bar{W} + q - s_2(q)) - u(\bar{W} + q - s_1(q))] f_e(q|e_1) dq \right| \\ &\leq u'(\bar{W}) M_e \int_0^{\bar{q}} (s_2(q) - s_1(q)) dq. \end{aligned}$$

Thus,

$$|e_2 - e_1| \leq \frac{u'(\bar{W}) M_e}{m} \Delta. \tag{14}$$

We can now compare the investor's expected payments. We have

$$\begin{aligned} \Pi(s_2) - \Pi(s_1) &= \int_0^{\bar{q}} (s_2(q) - s_1(q)) f(q|e_2) dq \\ &\quad + \int_0^{\bar{q}} s_1(q) [f(q|e_2) - f(q|e_1)] dq. \end{aligned}$$

Since $f(q|e) \geq \delta$, the first term is at least $\delta\Delta$. By the mean-value theorem in effort and (14), the absolute value of the second term is bounded above by

$$M_e |e_2 - e_1| \int_0^{\bar{q}} s_1(q) dq \leq \frac{u'(\bar{W}) M_e^2}{m} \Delta \int_0^{\bar{q}} s_1(q) dq.$$

Moreover, because $\Pi(s_1) = I$ and $f \geq \delta$,

$$\int_0^{\bar{q}} s_1(q) dq \leq \frac{I}{\delta}.$$

It follows that

$$\Pi(s_2) - \Pi(s_1) \geq \left(\delta - \frac{u'(\bar{W})M_e^2}{m\delta} I \right) \Delta. \quad (15)$$

Shrinking $\bar{I}$ once more if necessary, we may ensure that

$$\delta - \frac{u'(\bar{W})M_e^2}{m\delta} I > 0 \quad \text{for every } I \in (0, \bar{I}).$$

Equation (15) then implies

$$\Pi(s_2) > \Pi(s_1),$$

contradicting $\Pi(s_1) = \Pi(s_2) = I$.

Finally, each of the three families is strictly ordered pointwise by its parameter:

$$\alpha_2 > \alpha_1 \implies s_u^{\alpha_2}(q) \geq s_u^{\alpha_1}(q) \quad \text{for every } q,$$

with strict inequality for every $q > 0$;

$$D_{d,2} > D_{d,1} \implies s_d^{D_{d,2}}(q) \geq s_d^{D_{d,1}}(q) \quad \text{for every } q,$$

with strict inequality on a set of positive measure; and

$$D_{\ell,2} > D_{\ell,1} \implies s_{\ell}^{D_{\ell,2}}(q) \leq s_{\ell}^{D_{\ell,1}}(q) \quad \text{for every } q,$$

again with strict inequality on a set of positive measure. Hence, two distinct parameters in any one of the three families cannot both give the investor the same sufficiently small expected payment I . This proves that D_{ℓ}^I , α^I , and D_d^I are unique.

It remains to show that any such contract is close to the zero-financing contract. Since $f(q|e) \geq \delta > 0$ for all q, e , any levered-equity contract that raises I satisfies

$$I = \int_0^{\bar{q}} (q - D_{\ell}^I)^+ f(q|e(s_{\ell})) dq \geq \delta \int_{D_{\ell}^I}^{\bar{q}} (q - D_{\ell}^I) dq = \frac{\delta}{2} (\bar{q} - D_{\ell}^I)^2 \geq 0.$$

Thus $D_{\ell}^I \rightarrow \bar{q}$ as $I \rightarrow 0$. Similarly, for unlevered equity,

$$I = \int_0^{\bar{q}} \alpha^I q f(q|e(s_u)) dq \geq \delta \alpha^I \int_0^{\bar{q}} q dq = \frac{\delta \alpha^I \bar{q}^2}{2} \geq 0,$$

so $\alpha^I \rightarrow 0$ as $I \rightarrow 0$. Finally, for debt,

$$I = \int_0^{\bar{q}} \min\{q, D_d^I\} f(q|e(s_d)) dq \geq \delta \int_0^{\bar{q}} \min\{q, D_d^I\} dq.$$

Since

$$\int_0^{\bar{q}} \min\{q, D_d^I\} dq = D_d^I \bar{q} - \frac{(D_d^I)^2}{2} \geq \frac{\bar{q} D_d^I}{2}$$

we have

$$I \geq \frac{\delta \bar{q} D_d^I}{2},$$

and therefore $D_d^I \rightarrow 0$ as $I \rightarrow 0$. ■

Proof of Lemma 2: Since $f(q|e) \geq \delta$ and $s(q) \geq 0$, the investor's break-even constraint implies

$$I = \int s(q) f(q|e(s)) dq \geq \delta \int s(q) dq \quad \Rightarrow \quad \int_0^{\bar{q}} s(q) dq \leq \frac{I}{\delta} \quad (16)$$

Since the utility function is continuously differentiable on $[\bar{W}, \bar{W} + \bar{q}]$, we have $u'(\bar{W}) = \max_w u'(w)$. Likewise, define $\bar{f} := \max_{q,e} f(q|e)$. We have $0 \leq u(\bar{W} + q) - u(\bar{W} + q - s(q)) \leq u'(\bar{W}) s(q)$ for every q (since u is increasing and $s(q) \geq 0$ for any q), so for every e ,

$$U(\mathbf{0}, e) - U(s, e) \leq u'(\bar{W}) \int_0^{\bar{q}} s(q) f(q|e) dq \leq u'(\bar{W}) \bar{f} \int_0^{\bar{q}} s(q) dq \leq u'(\bar{W}) \bar{f} \frac{I}{\delta}, \quad (17)$$

and $U(s, \cdot) \rightarrow U(\mathbf{0}, \cdot)$ uniformly in $e \in [0, \bar{e}]$ and $s \in \mathcal{S}_{LL}$ as $I \rightarrow 0$. By the condition in Lemma 1, $U(s, \cdot)$ is strictly concave in e for every $s \in \mathcal{S}_{LL}$, so its maximizer $e(s)$ is unique. In particular, $U(\mathbf{0}, \cdot)$ is maximized by $e^* \in (0, \bar{e})$, where $U_{ee}(\mathbf{0}, e^*) < 0$. The convergence is uniform and e^* is the unique maximizer of $U(\mathbf{0}, \cdot)$, so that we have $e(s) \rightarrow e^*$ uniformly in s as $I \rightarrow 0$. So there is $\bar{I} > 0$ such that $e(s) \in (0, \bar{e})$ whenever $I < \bar{I}$.

Differentiating the entrepreneur's objective function with respect to effort:

$$U_e(s, e) = \int_0^{\bar{q}} u(\bar{W} + q - s(q)) f_e(q|e) dq - C'(e).$$

Since e^* is the interior maximizer of $U(\mathbf{0}, \cdot)$, we have $U_e(\mathbf{0}, e^*) = 0$, and $U_{ee}(\mathbf{0}, e^*) < 0$ by the condition in Lemma 1. Likewise, since $e(s) \in (0, \bar{e})$ is the interior maximizer of $U(s, \cdot)$, it is described by the first-order condition (FOC) $U_e(s, e(s)) = 0$. Subtracting the two FOCs and applying the mean value theorem to $U_e(s, \cdot)$ between e^* and $e(s)$:

$$0 = U_e(s, e(s)) - U_e(\mathbf{0}, e^*) = U_e(s, e^*) - U_e(\mathbf{0}, e^*) + U_{ee}(s, \tilde{e}) (e(s) - e^*),$$

for some $\tilde{e} \in (\min\{e^*, e(s)\}, \max\{e^*, e(s)\})$, where, letting $M_e := \sup_{q,e} |f_e(q|e)|$ (which is finite since f_e is bounded):

$$\begin{aligned} U_e(\mathbf{0}, e^*) - U_e(s, e^*) &= \int_0^{\bar{q}} (u(\bar{W} + q) - u(\bar{W} + q - s(q))) f_e(q|e^*) dq \\ \Rightarrow |U_e(\mathbf{0}, e^*) - U_e(s, e^*)| &\leq u'(\bar{W}) M_e \int_0^{\bar{q}} s(q) dq \leq u'(\bar{W}) M_e \frac{I}{\delta}, \end{aligned}$$

From the condition in Lemma 1, $|U_{ee}(s, \tilde{e})| \geq m > 0$ uniformly in $s \in \mathcal{S}_{LL}$ and $\tilde{e} \in [0, \bar{e}]$, so that:

$$|e(s) - e^*| = \frac{|U_e(s, e^*) - U_e(\mathbf{0}, e^*)|}{|U_{ee}(s, \tilde{e})|} \leq \frac{u'(\bar{W})M_e I}{m} \frac{1}{\delta} =: K_1 I. \quad (18)$$

This establishes part (i).

The effort $e(s)$ is an interior maximizer of $U(s, \cdot)$, which implies $U_e(s, e(s)) = 0$. A second-order Taylor expansion of $U(s, e)$ around $e(s)$ gives, for some $\hat{e}$ in-between e^* and $e(s)$,

$$U(s, e^*) = U(s, e(s)) + \underbrace{U_e(s, e(s))}_{=0} (e^* - e(s)) + \frac{1}{2} U_{ee}(s, \hat{e}) (e^* - e(s))^2, \quad (19)$$

where $U(s, e(s)) \geq U(s, e^*)$ since the entrepreneur chooses effort $e(s)$ optimally. In addition, $U_{ee}(s, e) = \int u(\bar{W} + q - s(q)) f_{ee}(q|e) dq - C''(e)$ is uniformly bounded (since u bounded on the compact wealth interval, f_{ee} bounded, C'' continuous on the compact set $[0, \bar{e}]$), so that there is a positive finite constant $\bar{U}$ such that $U_{ee}(s, e) \geq -\bar{U}$. In sum, combining with equations (18) and (19):

$$0 \leq U(s, e(s)) - U(s, e^*) \leq \frac{1}{2} \bar{U} (K_1 I)^2 = \frac{1}{2} \bar{U} K_1^2 I^2 =: K_2 I^2.$$

This establishes part (ii). ■

Proof of Theorem 1: We first prove the ranking for the contracts normalized at the benchmark effort, i.e. each parameter is chosen so that $\int_0^{\bar{q}} s(q) f(q|e^*) dq = I$, and then show that the same strict ranking $V(s_\ell) > V(s_u) > V(s_d)$ holds when the contract is such that each break-even constraint is satisfied at the equilibrium effort: $\int_0^{\bar{q}} s(q) f(q|e(s)) dq = I$.

We denote by $\tilde{s}$ the benchmark-normalized version of a given contract s . Using $|e(s) - e^*| \leq K_1 I$ and $\int_0^{\bar{q}} s(q) dq \leq \frac{I}{\delta}$ from Lemma 2 and its proof:

$$\left| \int_0^{\bar{q}} s(q) (f(q|e^*) - f(q|e(s))) dq \right| \leq M_e |e(s) - e^*| \int_0^{\bar{q}} s(q) dq = O(I^2),$$

where M_e is defined in the proof of Lemma 2, so the expected payoffs of s and $\tilde{s}$ at e^* differ by $O(I^2)$. The only difference between the two contracts is the value of a parameter, and each contract is monotone in that parameter, so $s(q) - \tilde{s}(q)$ has the same sign for all q . Thus,

$$\int_0^{\bar{q}} |s(q) - \tilde{s}(q)| f(q|e^*) dq = \left| \int_0^{\bar{q}} (s(q) - \tilde{s}(q)) f(q|e^*) dq \right| = O(I^2),$$

and since $f(q|e) \geq \delta$ and $|u(a) - u(b)| \leq u'(\bar{W})|a - b|$ on $[\bar{W}, \bar{W} + \bar{q}]$, the objective functions satisfy $|V(s) - V(\tilde{s})| = O(I^2)$. Since the ranking gaps established below are of order I , they

dominate this $O(I^2)$ discrepancy, so the strict ranking proven under the benchmark normalization also holds for the contracts considered. Accordingly, we set contract parameters so that $\int_0^{\bar{q}} s(q) f(q|e^*) dq = I$.

Define the expected output for a self-financed entrepreneur as $\mu := \int_0^{\bar{q}} q f(q|e^*) dq$. For a small I , the contract parameters are interior: $\alpha = \frac{I}{\mu} \in (0, 1)$, while $\lim_{I \rightarrow 0} D_\ell = \bar{q}$ and $\lim_{I \rightarrow 0} D_d = 0$, and these two thresholds are continuous and monotone in I . Define:

$$\Phi(s) := U(\mathbf{0}, e^*) - U(s, e^*) = \int_0^{\bar{q}} (u(\bar{W} + q) - u(\bar{W} + q - s(q))) f(q|e^*) dq$$

From Lemma 2, $V(s) = U(s, e(s)) = U(\mathbf{0}, e^*) - \Phi(s) + \rho(s)$, where $\rho(s) := U(s, e(s)) - U(s, e^*)$ is such that

$$0 \leq \rho(s) \leq K_2 I^2 \quad (20)$$

by part (ii) of Lemma 2. Since u is three times continuously differentiable on the compact interval $[\bar{W}, \bar{W} + \bar{q}]$, a second-order Taylor expansion gives:

$$\Phi(s) = \int_0^{\bar{q}} u'(\bar{W} + q) s(q) f(q|e^*) dq - \underbrace{\frac{1}{2} \int_0^{\bar{q}} u''^2 f(q|e^*) dq}_{R(s)} \quad (21)$$

for some $\xi(q) \in (\bar{W} + q - s(q), \bar{W} + q)$, where $R(s)$ is nonnegative and such that:

$$R(s) \leq \frac{1}{2} \left(\max_w |u''(w)| \right) \int_0^{\bar{q}} (s(q))^2 f(q|e^*) dq.$$

Each contract has $\max_q s(q) \rightarrow 0$ as $I \rightarrow 0$, so that:

$$\int (s(q))^2 f(q|e^*) dq \leq \max_q s(q) \int s(q) f(q|e^*) dq = \max_q s(q) I = o(I)$$

This implies $R(s) = o(I)$. Thus, for any two of the contracts s_1, s_2 , we have $\int (s_1(q) - s_2(q)) f(q|e^*) dq = 0$ and:

$$\begin{aligned} \Phi(s_1) - \Phi(s_2) &= \int_0^{\bar{q}} u'(\bar{W} + q) (s_1(q) - s_2(q)) f(q|e^*) dq + o(I) \\ &= \text{cov}(u'(\bar{W} + q), s_1(q) - s_2(q)) + o(I). \end{aligned} \quad (22)$$

Applying the equality on the first line of equation (22) separately to the s_u and s_ℓ pair and the s_d and s_u pair, substituting $s_u(q) = \frac{I}{\mu} q$, and using the continuity of marginal utility with $\lim_{I \rightarrow 0} D_\ell = \bar{q}$ and $\lim_{I \rightarrow 0} D_d = 0$,

$$\Phi(s_u) - \Phi(s_\ell) = \frac{I}{\mu} \int_0^{\bar{q}} u'(\bar{W} + q) q f(q|e^*) dq - I u'(\bar{W} + \bar{q}) + o(I), \quad (23)$$

$$\Phi(s_d) - \Phi(s_u) = \int_0^{\bar{q}} u'(\bar{W} + q) \left(\min\{q, D_d\} - \frac{I}{\mu} q \right) f(q|e^*) dq + o(I). \quad (24)$$

Dividing by I and using $\lim_{I \rightarrow 0} D_\ell = \bar{q}$, $\lim_{I \rightarrow 0} D_d = 0$ (so that $\int u'(\bar{W}+q) \min\{q, D_d\} f(q|e^*) dq = I \int u'(\bar{W}+q) f(q|e^*) dq + o(I)$ as $I \rightarrow 0$) and continuous marginal utility:

$$\frac{\Phi(s_u) - \Phi(s_\ell)}{I} \xrightarrow{I \rightarrow 0} \frac{1}{\mu} \int u'(\bar{W}+q) q f(q|e^*) dq - u'(\bar{W} + \bar{q}) \quad (25)$$

$$\frac{\Phi(s_d) - \Phi(s_u)}{I} \xrightarrow{I \rightarrow 0} -\frac{1}{\mu} \text{cov}(u'(\bar{W}+q), q) \quad (26)$$

The expression on the RHS of equation (25) is strictly positive because $u'(\bar{W}+q) \geq u'(\bar{W}+\bar{q})$. The expression on the RHS of equation (26) is strictly positive because marginal utility is strictly decreasing. It follows that there exist $\kappa > 0$ and $\bar{I}_0 > 0$ such that $\Phi(s_u) - \Phi(s_\ell) \geq \kappa I$ and $\Phi(s_d) - \Phi(s_u) \geq \kappa I$ for $I \in (0, \bar{I}_0)$.

In sum, $V(s_\ell) - V(s_u) = \Phi(s_u) - \Phi(s_\ell) + \rho(s_\ell) - \rho(s_u) \geq \kappa I - K_2 I^2$, which is positive for $I < \min\{\bar{I}_0, \kappa/K_2\} =: \bar{I}$, and likewise $V(s_u) - V(s_d) > 0$; so $V(s_\ell) > V(s_u) > V(s_d)$ for $I \in (0, \bar{I})$. $\blacksquare$

Proof of Proposition 1: Any second-best contract s with induced effort $e(s)$ is feasible when effort is contractible so $V^{FB} \geq V^*$. A levered equity financing contract s_ℓ is feasible at the second-best. Applying equation (41) to contract s_ℓ and using $\rho(s_\ell) \geq 0$ from equation (20):

$$V^* \geq V(s_\ell) = U(\mathbf{0}, e^*) - \Phi(s_\ell) + \rho(s_\ell) \geq U(\mathbf{0}, e^*) - \Phi(s_\ell).$$

The rest of the proof will derive the upper bound:

$$V^{FB} \leq U(\mathbf{0}, e^*) - \Phi(s_\ell) + O(I^2), \quad (27)$$

which then implies $V^{FB} - V^* \leq \hat{K} I^2$ for some finite $\hat{K}$, and $\frac{V^{FB} - V^*}{I} \leq \hat{K} I \rightarrow 0$ as $I \rightarrow 0$.

From Proposition 5, for each effort e , the feasible contract that minimizes $U(\mathbf{0}, e) - U(s, e)$ is a levered equity contract s_ℓ^e . Writing $\Phi^e(s_\ell^e) := U(\mathbf{0}, e) - U(s_\ell^e, e) \geq 0$ for this minimized cost,

$$V^{FB} = \max_{e \in [0, \bar{e}]} U(s_\ell^e, e) = \max_{e \in [0, \bar{e}]} \{U(\mathbf{0}, e) - \Phi^e(s_\ell^e)\}. \quad (28)$$

The envelope theorem applied to s_ℓ^e , with Lagrange multiplier associated with the investor's break-even constraint for given effort e of $\nu_e = u'(\bar{W} + D_\ell^e) \leq u'(\bar{W})$, gives:

$$\frac{d}{de} \Phi^e(s_\ell^e) = \int_0^{\bar{q}} (u(\bar{W}+q) - u(\bar{W}+q-s_\ell^e(q))) f_e(q|e) dq - \nu_e \int_0^{\bar{q}} s_\ell^e(q) f_e(q|e) dq. \quad (29)$$

Using $|u(\bar{W}+q) - u(\bar{W}+q-s(q))| \leq u'(\bar{W}) s(q)$, $\nu_e \leq u'(\bar{W})$, $|f_e(q|e)| \leq M_e := \sup_{q,e} |f_e(q|e)|$, and the size bound $\int_0^{\bar{q}} s_\ell^e(q) dq \leq \frac{I}{\delta}$, each of the two terms on the RHS of equation (29) is lower than $u'(\bar{W}) M_e \frac{I}{\delta}$, so

$$\left| \frac{d}{de} \Phi^e(s_\ell^e) \right| \leq 2u'(\bar{W}) M_e \frac{I}{\delta} = O(I). \quad (30)$$

The objective $U(\mathbf{0}, \cdot)$ is strictly concave, reaches its peak at e^* ($U_e(\mathbf{0}, e^*) = 0$), and satisfies $U_{ee}(\mathbf{0}, \cdot) \leq -m < 0$ (cf. condition in Lemma 1), so $U(\mathbf{0}, e) - U(\mathbf{0}, e^*) \leq -\frac{m}{2}(e - e^*)^2$. Subtracting the value at e^* and using equation (30), for every e ,

$$\begin{aligned} U(\mathbf{0}, e) - \Phi^e(s_\ell^e) - [U(\mathbf{0}, e^*) - \Phi^{e^*}(s_\ell^{e^*})] &\leq -\frac{m}{2}(e - e^*)^2 + 2u'(\bar{W})M_e \frac{I}{\delta} |e - e^*| \\ &= \frac{(2u'(\bar{W})M_e \frac{I}{\delta})^2}{2m} - \frac{m}{2} \left(|e - e^*| - 2u'(\bar{W})M_e \frac{I}{m\delta} \right)^2 \\ &\leq \frac{(2u'(\bar{W})M_e \frac{I}{\delta})^2}{2m} = O(I^2). \end{aligned} \quad (31)$$

Also using equation (28):

$$V^{FB} = \max_{e \in [0, \bar{e}]} \{U(\mathbf{0}, e) - \Phi^e(s_\ell^e)\} \leq U(\mathbf{0}, e^*) - \Phi^{e^*}(s_\ell^{e^*}) + O(I^2). \quad (32)$$

The contract $s_\ell^{e^*}$ has benchmark cost $\hat{I}(s_\ell^{e^*}) = I$, whereas s_ℓ has $\hat{I}(s_\ell) = I + O(I^2)$. Applying (39) to $s_\ell^{e^*}$,

$$\Phi^{e^*}(s_\ell^{e^*}) = \Phi(s_\ell^{e^*}) \geq \Phi(s_\ell) + u'(\bar{W} + D_\ell)(\hat{I}(s_\ell^{e^*}) - \hat{I}(s_\ell)) = \Phi(s_\ell) - O(I^2),$$

That is, $\Phi(s_\ell) \leq \Phi^{e^*}(s_\ell^{e^*}) + O(I^2)$. Combining with equation (32) gives (27). Also using $V^* \geq U(\mathbf{0}, e^*) - \Phi(s_\ell)$ established above, we get:

$$V^{FB} - V^* \leq \hat{K}I^2$$

for a finite constant $\hat{K}$, therefore $\frac{V^{FB} - V^*}{I} \leq \hat{K}I \rightarrow 0$ as $I \rightarrow 0$. ■

Proof of Corollary 1: A Taylor expansion of u around $\bar{W} + q$ gives:

$$u(\bar{W} - I + q) = u(\bar{W} + q) - Iu'(\bar{W} + q) + O(I^2) \quad (33)$$

uniformly in q . Given $U_e(\mathbf{0}, e^*) = 0$, the change in the optimal level of effort only has a second order effect on the objective function with self-financing, which is:

$$U(\mathbf{0}, e^*) - I \int_0^{\bar{q}} u'(\bar{W} + q) f(q|e^*) dq + O(I^2). \quad (34)$$

Using Theorem 2 and equation (21) in the proof of Theorem 1 (where $R(s) = o(I)$):

$$V^* = U(\mathbf{0}, e^*) - \Phi(s_\ell) + O(I^2) = U(\mathbf{0}, e^*) - \int u'(\bar{W} + q) s_\ell(q) f(q|e^*) dq + o(I). \quad (35)$$

Combining equations (34) and (35), and using $\int s_\ell(q) f(q|e^*) dq = I + O(I^2)$, the difference between the entrepreneur's objective function at the second-best with external funding and with internal funding is:

$$I \int u'(\bar{W} + q) f(q|e^*) dq - \int u'(\bar{W} + q) s_\ell(q) f(q|e^*) dq + o(I) = -\text{cov}(u'(\bar{W} + q), s_\ell(q)) + o(I). \quad (36)$$

Since marginal utility is strictly decreasing and $s_\ell(q) = \max\{q - D_\ell, 0\}$ is nondecreasing and nonconstant, the covariance in equation (36) is strictly negative. To determine the order of this covariance, let

$$g(q) := u'(W + q)$$

and let $\mathbb{E}_*$ and cov_* denote expectation and covariance under the density $f(q|e^*)$. Write $s_\ell^I(q) = (q - D_\ell^I)^+$ and define

$$m_I := \mathbb{E}_*[s_\ell^I(q)].$$

As established above,

$$m_I = I + O(I^2), \tag{36a}$$

and Lemma 3 implies that $D_\ell^I \rightarrow \bar{q}$ as $I \downarrow 0$.

Since $s_\ell^I(q) = 0$ for $q < D_\ell^I$, we have

$$\begin{aligned} \left| \frac{\mathbb{E}_*[g(q)s_\ell^I(q)]}{m_I} - g(\bar{q}) \right| &= \left| \frac{\mathbb{E}_*[(g(q) - g(\bar{q}))s_\ell^I(q)]}{m_I} \right| \\ &\leq \sup_{q \in [D_\ell^I, \bar{q}]} |g(q) - g(\bar{q})|. \end{aligned}$$

The function g is continuous and $D_\ell^I \rightarrow \bar{q}$, so that the RHS converges to zero. Thus,

$$\mathbb{E}_*[g(q)s_\ell^I(q)] = g(\bar{q})m_I + o(m_I) = g(\bar{q})I + o(I). \tag{36b}$$

It follows that

$$\begin{aligned} -\text{cov}_*(g(q), s_\ell^I(q)) &= \mathbb{E}_*[g(q)]m_I - \mathbb{E}_*[g(q)s_\ell^I(q)] \\ &= m_I(\mathbb{E}_*[g(q)] - g(\bar{q})) + o(m_I) \\ &= I(\mathbb{E}_*[g(q)] - g(\bar{q})) + o(I). \end{aligned} \tag{36c}$$

Moreover,

$$c := \mathbb{E}_*[g(q)] - g(\bar{q}) = \int_0^{\bar{q}} [u'(W + q) - u'(W + \bar{q})] f(q|e^*) dq > 0,$$

because u' is strictly decreasing and $f(q|e^*) > 0$ on $[0, \bar{q}]$. Therefore,

$$-\text{cov}_*(u'(W + q), s_\ell^I(q)) = cI + o(I). \tag{36d}$$

Finally, using $m_I = I + O(I^2)$,

$$\begin{aligned} I\mathbb{E}_*[g(q)] - \mathbb{E}_*[g(q)s_\ell^I(q)] &= -\text{cov}_*(g(q), s_\ell^I(q)) + (I - m_I)\mathbb{E}_*[g(q)] \\ &= cI + o(I). \end{aligned}$$

Thus, including the $o(I)$ remainder already present in equation (36), the difference between the entrepreneur's value under external and internal financing is

$$cI + o(I), \quad c > 0.$$

In particular, it is strictly positive for all sufficiently small I and is of order I . $\blacksquare$

Proof of Theorem 2: The investor payoff with a levered equity contract is: $s_\ell(q) = \max\{q - D_\ell, 0\}$. For a small I , the threshold D_ℓ that satisfies the investor's break-even constraint is such that $D_\ell \in (0, \bar{q})$, with $D_\ell \nearrow \bar{q}$ as $I \rightarrow 0$. As in the proof of Theorem 1, define $\Phi(s) := U(\mathbf{0}, e^*) - U(s, e^*) \geq 0$ and $\rho(s) := U(s, e(s)) - U(s, e^*)$, so that $V(s) = U(\mathbf{0}, e^*) - \Phi(s) + \rho(s)$ with $0 \leq \rho(s) \leq K_2 I^2$ by part (ii) of Lemma 2. Given a contract s , let $\hat{I}(s) := \int_0^{\bar{q}} s(q) f(q|e^*) dq$ be its expected cost evaluated at the benchmark effort.

We will show that for every q and every $s(q) \in [0, q]$,

$$u(\bar{W} + q - s(q)) \leq u(\bar{W} + q - s_\ell(q)) + u'(\bar{W} + D_\ell)(s_\ell(q) - s(q)). \quad (37)$$

where $u'(\bar{W} + D_\ell) \in (0, u'(\bar{W}))$. The argument is based on a supporting-hyperplane inequality for $s_\ell(q) = \max\{q - D_\ell, 0\}$. With a concave utility function, we have $u(y) \leq u(x) + u'(x)(y - x)$. Applying this inequality to $x = \bar{W} + q - s_\ell(q)$ and $y = \bar{W} + q - s(q)$ gives

$$u(\bar{W} + q - s(q)) \leq u(\bar{W} + q - s_\ell(q)) + u'(\bar{W} + q - s_\ell(q))(s_\ell(q) - s(q)),$$

Thus, a sufficient condition for (37) is:

$$(u'(\bar{W} + q - s_\ell(q)) - u'(\bar{W} + D_\ell))(s_\ell(q) - s(q)) \leq 0. \quad (38)$$

There are two cases. First, if $q \geq D_\ell$ then $s_\ell(q) = q - D_\ell$ and the entrepreneur's wealth is $\bar{W} + q - s_\ell(q) = \bar{W} + D_\ell$, so the LHS of equation (38) is zero. Second, if $q < D_\ell$ then $s_\ell(q) = 0$, the entrepreneur's wealth is $\bar{W} + q < \bar{W} + D_\ell$, so $u'(\bar{W} + q) - u'(\bar{W} + D_\ell) > 0$ (marginal utility is decreasing), while $s_\ell(q) - s(q) = -s(q) \leq 0$, so the LHS of equation (38) is nonpositive. This establishes (37).

Integrating (37) and using $U(s, e^*) = U(\mathbf{0}, e^*) - \Phi(s)$ gives, for any contract such that $s(q) \in [0, q]$,

$$\Phi(s) \geq \Phi(s_\ell) + u'(\bar{W} + D_\ell) (\hat{I}(s) - \hat{I}(s_\ell)). \quad (39)$$

At equal benchmark cost, $\hat{I}(s) = \hat{I}(s_\ell)$, inequality (39) reduces to $\Phi(s) \geq \Phi(s_\ell)$: levered equity minimizes the risk-transfer cost Φ among all contracts in $\mathcal{S}_{LL}$ with the same benchmark cost. More generally, (39) is a linear lower bound on Φ that holds with equality at contract s_ℓ : it is a supporting hyperplane to Φ at s_ℓ with slope $u'(\bar{W} + D_\ell)$. It allows to extend the equal-cost minimization to contracts of arbitrary benchmark cost.

The contract's expected cost evaluated at the benchmark effort e^* differs from its cost at the induced effort $e(s)$ (which equals I by feasibility) only at the second order. For any feasible contract s (with $\int s(q)f(q|e(s))dq = I$), $\int_0^{\bar{q}} s(q)dq \leq \frac{I}{\delta}$ and $|e(s) - e^*| \leq K_1 I$ by Lemma 2, so

$$|\hat{I}(s) - I| = \left| \int s(q)[f(q|e^*) - f(q|e(s))]dq \right| \leq M_e |e(s) - e^*| \int_0^{\bar{q}} s(q)dq \leq \frac{M_e K_1}{\delta} I^2,$$

where $M_e = \sup_{q,e} |f_e(q|e)|$. Since s_ℓ is feasible, the same bound gives $|\hat{I}(s_\ell) - I| \leq \frac{M_e K_1}{\delta} I^2$. Using the triangle inequality,

$$|\hat{I}(s) - \hat{I}(s_\ell)| \leq |\hat{I}(s) - I| + |I - \hat{I}(s_\ell)| \leq \frac{2M_e K_1}{\delta} I^2.$$

Combined with $u'(\bar{W} + D_\ell) \leq u'(\bar{W})$, inequality (39) then yields:

$$\Phi(s) \geq \Phi(s_\ell) - u'(\bar{W} + D_\ell) |\hat{I}(s) - \hat{I}(s_\ell)| \geq \Phi(s_\ell) - 2u'(\bar{W}) \frac{M_e K_1}{\delta} I^2. \quad (40)$$

For part (i), since (39) holds for every $s \in \mathcal{S}_{LL}$, any feasible contract s is such that:

$$V(s) = U(\mathbf{0}, e^*) - \Phi(s) + \rho(s) \leq U(\mathbf{0}, e^*) - \Phi(s_\ell) + \left(2u'(\bar{W}) \frac{M_e K_1}{\delta} + K_2 \right) I^2. \quad (41)$$

In particular,

$$V^* \leq U(\mathbf{0}, e^*) - \Phi(s_\ell) + \left(2u'(\bar{W}) \frac{M_e K_1}{\delta} + K_2 \right) I^2.$$

Since $V(s_\ell) = U(\mathbf{0}, e^*) - \Phi(s_\ell) + \rho(s_\ell) \geq U(\mathbf{0}, e^*) - \Phi(s_\ell)$ (using $\rho(s_\ell) \geq 0$ from part (ii) of Lemma 2), combining these inequalities gives $V^* - V(s_\ell) \leq K I^2$, where $K := 2u'(\bar{W}) \frac{M_e K_1}{\delta} + K_2$.

For part (ii), the proof of Theorem 1 established for benchmark-normalized contracts that, for small I , $\Phi(s_u) - \Phi(s_\ell) \geq \kappa I$ and $\Phi(s_d) - \Phi(s_u) \geq \kappa I$ for some $\kappa > 0$. Adding these two inequalities gives $\Phi(s_d) - \Phi(s_\ell) \geq 2\kappa I$. For the contracts such that $\int s(q)f(q|e(s))dq = I$, the normalization step at the beginning of the proof of Theorem 1 shows that $\int_0^{\bar{q}} |s(q) - \tilde{s}(q)|f(q|e^*)dq = O(I^2)$, where $\tilde{s}$ denotes the benchmark-normalized contract from the same family. Since $|u(\bar{W} + q - \tilde{s}(q)) - u(\bar{W} + q - s(q))| \leq u'(\bar{W}) |s(q) - \tilde{s}(q)|$ pointwise, taking the expectation at the benchmark effort gives $|\Phi(s) - \Phi(\tilde{s})| \leq u'(\bar{W}) O(I^2) = O(I^2)$, which does not reverse the first-order differences between the contracts.

In sum, there exists $\kappa_0 > 0$ such that $\Phi(s_u) - \Phi(s_\ell) \geq \kappa_0 I$ and $\Phi(s_d) - \Phi(s_\ell) \geq \kappa_0 I$ for small I . Since $V^* \geq V(s_\ell) \geq U(\mathbf{0}, e^*) - \Phi(s_\ell)$ while $V(s_u) \leq U(\mathbf{0}, e^*) - \Phi(s_u) + K_2 I^2$,

$$V^* - V(s_u) \geq \Phi(s_u) - \Phi(s_\ell) - K_2 I^2 \geq \kappa_0 I - K_2 I^2 \geq \kappa I$$

for small I and a suitable $\kappa > 0$; the same holds for debt. Choosing $\bar{I}$ small enough completes the proof. $\blacksquare$

Proof of Proposition 3: For any contract, the entrepreneur's payoff is $w(q) = q - s(q)$ and his objective function is:

$$V(s) = \max_{e \in [0, \bar{e}]} \left\{ \int_0^{\bar{q}} u(\bar{W} + w(q)) f(q|e) dq - C(e) \right\}.$$

The entrepreneur's objective function is strictly concave in effort (Lemma 1), so the effort induced is unique. Since the entrepreneur's payoff at any q is continuous in each contract parameter (D_ℓ, D_d, α) and uniformly bounded, the Maximum Theorem implies that the induced effort, V , and Π are continuous functions of each contract parameter.

With levered equity financing, the entrepreneur's payoff is $w_\ell(q) = \min\{q, D_\ell\}$ and the investor's payoff is $s_\ell(q) = \max\{q - D_\ell, 0\}$. The latter is decreasing in D_ℓ , so an increase in I lowers D_ℓ . The entrepreneur's payoff is capped at D_ℓ . At $D_\ell = 0$, the LHS of the IC is such that:

$$\frac{d}{dD_\ell} \int_0^{\bar{q}} u(\bar{W} + w_\ell(q)) f_e(q|e) dq = u'(\bar{W}) \int_0^{\bar{q}} f_e(q|e) dq = 0,$$

This means that, at $D_\ell = 0$, a marginal increase in D_ℓ does not change the entrepreneur's optimally chosen effort. At the same time, at $D_\ell = 0$, the investor's payoff decreases one-for-one with a marginal increase in D_ℓ :

$$\frac{d\Pi}{dD_\ell}(s_\ell) = - \int_0^{\bar{q}} f(q|e) dq = -1.$$

Since $\Pi(s_\ell)$ is continuous and strictly decreasing in D_ℓ , with $\Pi(s_\ell) = \bar{\mu}$ at $D_\ell = 0$ (where $w_\ell(q) = 0$ for all q , so that $e(s_\ell) = 0$) and $\Pi(s_\ell) = 0$ at $D_\ell = \bar{q}$, every $I \in (0, \bar{\mu})$ is raised by a unique threshold $D_\ell(I)$, and $D_\ell(I) \rightarrow 0$ as $I \nearrow \bar{\mu}$. Since V is continuous in D_ℓ and $C(0) = 0$, the entrepreneur's objective function is such that, as $I \nearrow \bar{\mu}$:

$$V(s_\ell) \longrightarrow u(\bar{W}) \quad (42)$$

With debt financing, the entrepreneur's payoff is $w_d(q) = \max\{q - D_d, 0\}$. Fix an effort level e , and consider two contracts such that the entrepreneur has the same expected utility at this level of effort. The contract whose payoff is concentrated on high output is associated with a higher marginal benefit of effort, because:

$$\int_0^{\bar{q}} u(\bar{W} + w(q)) f_e(q|e) dq = \text{cov}(u(\bar{W} + w(q)), L(q|e)).$$

Indeed, the likelihood ratio $L(q|e)$ is increasing in q , while the difference $u(\bar{W} + w_d(q)) - u(\bar{W} + w_\ell(q))$ has mean zero under the same output distribution and single-crosses from negative to positive in q . It follows that $\text{cov}(u(\bar{W} + w_d(q)), L(q|e)) \geq \text{cov}(u(\bar{W} + w_\ell(q)), L(q|e))$,

so at a given effort and expected utility, debt financing provides stronger incentives than levered equity financing. An analogous argument holds for unlevered equity financing.

By assumption, $\max_{D_d} \Pi(s_d) > \bar{\mu}$ and $\max_{\alpha} \Pi(s_u) > \bar{\mu}$. Consider debt financing, and let $\hat{D}$ denote a maximizer of $\Pi(s_d)$, which exists because $\Pi(s_d)$ is continuous on the compact interval $[0, \bar{q}]$. We know that $\Pi(s_d) = 0$ at $D_d = 0$ (the investor gets zero for any output), $\Pi(s_d) = \bar{\mu}$ at $D_d = \bar{q}$ (the investor gets the entire output so that $w_d(q) = 0$ for all q and $e(s_d) = 0$), and $\Pi(s_d) > \bar{\mu}$ at $\hat{D}$ by assumption. This implies that $\hat{D} < \bar{q}$. Since $\Pi(s_d)$ is continuous on $[0, \hat{D}]$, with $\Pi(s_d) = 0$ at $D_d = 0$ and $\Pi(s_d) > \bar{\mu}$ at $D_d = \hat{D}$, we have $\tilde{D} \in (0, \hat{D})$ from the intermediate value theorem with $\Pi(s_d) = \bar{\mu}$ at $D_d = \tilde{D}$. It follows that the smallest face value D^* that raises external funding $\bar{\mu}$ is such that $D^* \leq \tilde{D} < \hat{D} < \bar{q}$. Applying the same argument to unlevered equity, using $\Pi(s_u) = 0$ at $\alpha = 0$ and $\Pi(s_u) = \bar{\mu}$ at $\alpha = 1$, shows that the minimum investor stake α^* that raises $\bar{\mu}$ is such that $\alpha^* < 1$. At D^* and α^* , the entrepreneur's payoff is strictly positive on a set of positive measure: $w_d(q) = q - D^* > 0$ for $q > D^*$, and $w_u(q) = (1 - \alpha^*)q > 0$ for $q > 0$.

Since the entrepreneur chooses effort, he is weakly better off under his optimally chosen effort level than under zero effort, so for $k \in \{d, u\}$ at $I = \bar{\mu}$,

$$\begin{aligned} V(s_k) &= \max_{e \in [0, \bar{e}]} \left\{ \int_0^{\bar{q}} u(\bar{W} + w_k(q)) f(q|e) dq - C(e) \right\} \\ &\geq \int_0^{\bar{q}} u(\bar{W} + w_k(q)) f(q|0) dq - C(0) \\ &> u(\bar{W}), \end{aligned} \tag{43}$$

where the last inequality is strict because $w_k(q) > 0$ on a set of positive measure and $u' > 0$.

Finally, we show that these strict inequalities hold for all I sufficiently close to $\bar{\mu}$. Let $I \in (0, \bar{\mu})$. Since $\Pi(s_d) = 0$ at $D_d = 0$ and $\Pi(s_d) = \bar{\mu} > I$ at $D_d = D^*$, from the intermediate value theorem there is a face value in $(0, D^*)$ such that the investor breaks even, so $D_d(I) < D^*$. Likewise, $\alpha(I) < \alpha^*$. Thus, the entrepreneur's payoff is weakly higher, for every q , than at D^* and α^* :

$$\begin{aligned} w_d(q) &= \max\{q - D_d(I), 0\} \geq \max\{q - D^*, 0\} \\ w_u(q) &= (1 - \alpha(I))q \geq (1 - \alpha^*)q \end{aligned}$$

so (43), evaluated at D^* and at α^* , bounds $V(s_d)$ and $V(s_u)$ below by a constant strictly greater than $u(\bar{W})$, uniformly in $I \in (0, \bar{\mu})$, while $V(s_\ell) \rightarrow u(\bar{W})$ from equation (42). In sum, there exists $\underline{I} < \bar{\mu}$ such that $V(s_d) > V(s_\ell)$ and $V(s_u) > V(s_\ell)$ for all $I \in (\underline{I}, \bar{\mu})$. ■

Proof of Proposition 4: For convenience, define the entrepreneur's payoff as $w(q) \equiv q - s(q)$. Note that bilateral monotonicity and bilateral limited liability hold if and only

if bilateral monotonicity and $w(0) = 0$ hold. To see this, first note that evaluating limited liability at $q = 0$, one finds that $w(0) = 0$. Moreover, any w that starts at zero and has slope between 0 and 1 cannot fall below zero and cannot exceed q .

We start by addressing corner cases. First, suppose that the equilibrium involves zero effort. Then the investor's payoff is $\int_0^{\bar{q}} (q - w(q)) f(q|0) dq \leq \bar{\mu}$, with equality if and only if $w(q) = 0$ almost everywhere, since $w(q) \geq 0$ and $f(q|e) \geq \delta > 0$. Moreover, $w(q) = 0$ for all q induces zero effort, because the entrepreneur's objective is then $u(\bar{W}) - C(e)$, which is maximized at $e = 0$. Thus, the optimal contract is such that $w(q) = 0$ almost everywhere, i.e., $s(q) = q$ for all q , and the Proposition holds for any $\phi > 0$. Second, effort $e = \bar{e}$ is not incentive-compatible for any feasible contract because of condition (2). We henceforth restrict attention to interior levels of effort ($e \in (0, \bar{e})$). Then, the incentive constraint can be written as the entrepreneur's FOC (see Lemma 1).

Therefore, the program can be written as:

$$\max_{w(\cdot), x(\cdot), e} \int_0^{\bar{q}} [q - w(q)] f(q|e) dq \quad (44)$$

$$\text{s.t.} \quad \int_0^{\bar{q}} u(\bar{W} + w(q)) f_e(q|e) dq = C'(e) \quad (45)$$

$$0 \leq \dot{w}(q) \leq 1 \quad \forall q \quad (46)$$

$$w(0) = 0 \quad (47)$$

Define the likelihood ratio $L(q|e) := \frac{f_e(q|e)}{f(q|e)}$, so we can write the program as:

$$\max_{w(\cdot), x(\cdot), e} \int_0^{\bar{q}} [q - w(q)] f(q|e) dq \quad (48)$$

$$\text{s.t.} \quad \int_0^{\bar{q}} u(\bar{W} + w(q)) L(q|e) f(q|e) dq = C'(e) \quad (49)$$

$$0 \leq x(q) \leq 1 \quad \forall q \quad (50)$$

$$\dot{w}(q) = x(q), \quad (51)$$

$$w(0) = 0. \quad (52)$$

Set up the Hamiltonian:

$$H = [q - w(q) + \lambda u(\bar{W} + w(q)) L(q|e)] f(q|e) + \mu(q) x(q).$$

• Pontryagin condition:

$$x(q) = \begin{cases} 0 & \text{if } \mu(q) < 0 \\ 1 & \text{if } \mu(q) > 0 \end{cases}.$$

- ODE of co-state:

$$\dot{\mu}(q) = [1 - \lambda u'(\bar{W} + w(q))L(q|e)] f(q|e).$$

- Transversality: $\mu(\bar{q}) = 0$.
- FOC with respect to an interior level of effort:

$$\int_0^{\bar{q}} [q - w(q)] f_e(q|e) dq + \lambda \underbrace{\left[\int_0^{\bar{q}} u(\bar{W} + w(q)) f_{ee}(q|e) dq - C'''(e) \right]}_{\leq 0 \text{ by FOA}} = 0. \quad (53)$$

The proof of our main result will use the fact that the Lagrange multiplier associated with IC is non-negative:

Claim 1. $\lambda \geq 0$.

Proof. Use the definition of L to write:

$$\int_0^{\bar{q}} [q - w(q)] f_e(q|e) dq = \int_0^{\bar{q}} [q - w(q)] L(q|e) f(q|e) dq.$$

Since $h(q) := q - w(q)$ and $L(q|e)$ are both increasing functions of q , it follows by Chebyshev inequality for monotone functions that

$$\int_0^{\bar{q}} [q - w(q)] L(q|e) f(q|e) dq \geq \int_0^{\bar{q}} [q - w(q)] f(q|e) dq \cdot \underbrace{\int_0^{\bar{q}} L(q|e) f(q|e) dq}_{=0} = 0,$$

where the equality at the end follows from the fact that L has mean zero:

$$\int_0^{\bar{q}} L(q|e) f(q|e) dq = \int_0^{\bar{q}} f_e(q|e) dq = \frac{d}{de} \underbrace{\left[\int_0^{\bar{q}} f(q|e) dq \right]}_{=1} = 0.$$

Substituting in (53), establishes that $\lambda \geq 0$. □

We now prove the Proposition. By Pontryagin's condition and $w(0) = 0$, it suffices to show that $\mu(0) < 0$. We now show that, in fact, $\mu(0) \leq -1$.

Integrate ODE and use transversality:

$$\underbrace{\mu(\bar{q})}_{=0} - \mu(0) = \int_0^{\bar{q}} [1 - \lambda u'(\bar{W} + w(q))L(q|e)] f(q|e) dq$$

$$\therefore \mu(0) = -1 + \lambda \int_0^{\bar{q}} u'(\bar{W} + w(q))L(q|e) f(q|e) dq. \quad (54)$$

Let $g(q) := u'(\overline{W} + w(q))$ and note that g is a (weakly) decreasing function. As shown before, $L(q|e)$ is an increasing function of q with mean zero. By Chebyshev inequality for anti-monotone functions, we have

$$\int_0^{\bar{q}} g(q)L(q|e)f(q|e)dq \leq \int_0^{\bar{q}} g(q)f(q|e)dq \cdot \underbrace{\int_0^{\bar{q}} L(q|e)f(q|e)dq}_{=0} = 0.$$

Substituting back in (54), gives

$$\mu(0) = -1 + \lambda \int_0^{\bar{q}} u'(\overline{W} + w(q))L(q|e)f(q|e)dq \leq -1.$$

■

Online Appendix

At the first-best, the entrepreneur can commit to effort at the contracting stage. In this sense, there is no moral hazard. Proposition 5 derives the first-best contract when the Let $\bar{I}^{FB} := \max_{e \in [0, \bar{e}]} \int_0^{\bar{q}} q f(q|e) dq$ be the largest amount that can be financed when effort is contractible.

Proposition 5 (First-best contract). *For every $I \in (0, \bar{I}^{FB})$, levered equity financing is first-best optimal: there exist a first-best effort $e^{FB} \in [0, \bar{e}]$ and a threshold $D_\ell^{FB} \in [0, \bar{q}]$ such that:*

$$V^{FB} = U(s_\ell^{FB}, e^{FB}), \quad s_\ell^{FB}(q) = \max\{q - D_\ell^{FB}, 0\}.$$

When effort is contractible it is chosen at the contracting stage, so the contract only matters for risk sharing. Unconstrained risk sharing would involve full insurance of the risk averse entrepreneur by the risk neutral investor, but bilateral limited liability ($s(q) \geq 0$) prevents the investor from paying the entrepreneur. The best allocation maximizes the entrepreneur's payoff in low output states, where his marginal utility is highest, and only gives the investor a positive payoff in high output states. Thus, the entrepreneur's claim on output is capped at D_ℓ^{FB} . That is, the investor holds a levered equity claim.

This is the same contract that is approximately optimal at the second-best when I is small (Proposition 2): moral hazard does not change the contract, and only distorts the effort it induces, an effect of second order in I (Lemma 2). On the contrary, when I is large I , the second-best optimal financing contract is debt-like (Proposition 4), which is the polar opposite of levered equity. In this case, the agency cost is first order: for incentive purposes, the entrepreneur must have a highly risk claim on firm output.

Proof of Proposition 5: Consider a contractible level of effort e . The first-best optimal contract solves

$$\max_{0 \leq s(q) \leq q} \int_0^{\bar{q}} u(\bar{W} + q - s(q)) f(q|e) dq \quad \text{s.t.} \quad \int_0^{\bar{q}} s(q) f(q|e) dq = I.$$

Consider the pointwise optimization. The objective is concave in s and the constraints are linear, so the Kuhn-Tucker conditions are necessary and sufficient. The entrepreneur's marginal utility $u'(\bar{W} + q - s(q))$ is constant across all states at which $s(q) \in (0, q)$. At low output, the constraint $s(q) \geq 0$ binds and we have $s(q) = 0$. The optimal contract is therefore levered equity (a.e.):

$$s_\ell^e(q) = \max\{q - D_\ell^e, 0\} \quad \text{for Lebesgue almost every } q,$$

where $D_\ell^e \in [0, \bar{q})$ is determined by the break-even constraint $\int_0^{\bar{q}} \max\{q - D_\ell^e, 0\} f(q|e) dq = I$ (the LHS is higher than I at $D_\ell^e = 0$, equal to 0 at $D_\ell^e = \bar{q}$, and continuous and strictly decreasing in D_ℓ^e).

The feasible effort set is nonempty (as $I < \bar{I}^{FB}$) and compact, and the mapping from effort to $U(s_\ell^e, e)$ is continuous, so it reaches its maximum at some effort e^{FB} . So $V^{FB} = U(s_\ell^{e^{FB}}, e^{FB})$ and the first-best contract is levered equity. ■

Proof of Proposition 2: We will use the first derivative of the contract:

Fix $q_0 < \bar{q}$, and let s be a second-best optimal contract that raises I . We show that if I is sufficiently small, then s cannot be positive on a positive-measure subset of $[0, q_0]$.

We first calculate how a small change in the contract affects effort, the entrepreneur's value, and the investor's expected payment. The proof of the effort-distortion lemma uses only bilateral limited liability. It therefore implies that, uniformly over all feasible contracts and all sufficiently small I , the induced effort $e(s)$ is interior and

$$U_{ee}(s, e(s)) \leq -m < 0$$

for some constant m . It also gives

$$\int_0^{\bar{q}} s(q) dq \leq \frac{I}{\delta}.$$

For any bounded measurable function h , consider the perturbed contract $s_\varepsilon = s + \varepsilon h$. The implicit-function theorem applies to the entrepreneur's first-order condition (his effort is interior and his objective function is uniformly strictly concave in effort). Writing $e = e(s)$, differentiation at $\varepsilon = 0$ gives

$$De(s)[h] = \frac{\int_0^{\bar{q}} u'(\bar{W} + q - s(q)) h(q) f_e(q|e) dq}{U_{ee}(s, e)}.$$

Since u' and f_e are bounded and $U_{ee}(s, e) \leq -m$, there is a constant K , independent of s , I , and h , such that

$$|De(s)[h]| \leq K \int_0^{\bar{q}} |h(q)| dq.$$

The envelope theorem gives

$$DV(s)[h] = - \int_0^{\bar{q}} u'(\bar{W} + q - s(q)) h(q) f(q|e) dq.$$

Differentiating the investor's expected payment gives

$$D\Pi(s)[h] = \int_0^{\bar{q}} h(q) f(q|e) dq + De(s)[h] \int_0^{\bar{q}} s(q) f_e(q|e) dq.$$

Since f_e is bounded and $\int s(q) dq \leq I/\delta$,

$$\left| \int_0^{\bar{q}} s(q) f_e(q|e) dq \right| = O(I).$$

Consequently,

$$D\Pi(s)[h] = \int_0^{\bar{q}} h(q) f(q|e) dq + O(I) \int_0^{\bar{q}} |h(q)| dq, \quad (\text{A.1})$$

where the $O(I)$ bound is uniform over all feasible contracts.

Suppose, toward a contradiction, that s is positive on a positive-measure subset of $[0, q_0]$. There then exist $\eta > 0$ and a measurable set $A \subseteq [0, q_0]$ of positive measure such that

$$s(q) \geq \eta \quad \text{for every } q \in A.$$

Choose $q_0 < q_1 < q_2 < \bar{q}$ and set $d := \frac{q_1 - q_0}{2}$. Consider the set

$$B := \{q \in [q_1, q_2] : s(q) \leq d\}.$$

Since

$$d |[q_1, q_2] \setminus B| \leq \int_0^{\bar{q}} s(q) dq \leq \frac{I}{\delta},$$

the set B has positive measure whenever I is sufficiently small. Moreover, on B ,

$$q - s(q) \geq q_1 - d = \frac{q_0 + q_1}{2} > q_0.$$

Thus payments can be reduced slightly on A and increased slightly on B without violating bilateral limited liability.

Normalize the two perturbations by defining

$$a(q) := \frac{\mathbf{1}_A(q)}{\int_A f(t|e) dt}, \quad b(q) := \frac{\mathbf{1}_B(q)}{\int_B f(t|e) dt}.$$

Then

$$\int_0^{\bar{q}} a(q) f(q|e) dq = \int_0^{\bar{q}} b(q) f(q|e) dq = 1.$$

Furthermore, since $f(q|e) \geq \delta$,

$$\int_0^{\bar{q}} a(q) dq \leq \frac{1}{\delta}, \quad \int_0^{\bar{q}} b(q) dq \leq \frac{1}{\delta}.$$

For small $\alpha, \beta \geq 0$, define $s_{\alpha, \beta} := s - \alpha a + \beta b$. By the definitions of A and B , this contract satisfies $0 \leq s_{\alpha, \beta}(q) \leq q$ when α and β are sufficiently small. Applying (A.1) to a and b gives

$$D\Pi(s)[a] = 1 + O(I), \quad D\Pi(s)[b] = 1 + O(I).$$

It follows from the implicit-function theorem that, for small α , there is a differentiable function $\beta(\alpha)$ with $\beta(0) = 0$ such that

$$\Pi(s_{\alpha, \beta(\alpha)}) = I$$

and

$$\beta'(0) = \frac{D\Pi(s)[a]}{D\Pi(s)[b]} = 1 + O(I).$$

Thus the perturbation preserves the investor's break-even constraint exactly.

Using the envelope formula, the effect on the entrepreneur's value is

$$\begin{aligned} \left. \frac{d}{d\alpha} V(s_{\alpha, \beta(\alpha)}) \right|_{\alpha=0} &= \int_0^{\bar{q}} a(q) u'(\bar{W} + q - s(q)) f(q|e) dq \\ &\quad - \beta'(0) \int_0^{\bar{q}} b(q) u'(\bar{W} + q - s(q)) f(q|e) dq. \end{aligned}$$

On A , the entrepreneur's retained output satisfies

$$q - s(q) \leq q \leq q_0,$$

whereas on B ,

$$q - s(q) \geq \frac{q_0 + q_1}{2}.$$

Since u' is strictly decreasing and the two perturbations were normalized to integrate to one under $f(\cdot|e)$, we obtain

$$\left. \frac{d}{d\alpha} V(s_{\alpha, \beta(\alpha)}) \right|_{\alpha=0} \geq u'(\bar{W} + q_0) - (1 + O(I)) u' \left(\bar{W} + \frac{q_0 + q_1}{2} \right).$$

Strict concavity of u implies $u'(\bar{W} + q_0) > u'(\bar{W} + \frac{q_0 + q_1}{2})$, so the derivative above is strictly positive for all sufficiently small I . Hence, for a sufficiently small $\alpha > 0$, the contract $s_{\alpha, \beta(\alpha)}$ satisfies bilateral limited liability, gives the investor the same expected payment I , and gives the entrepreneur strictly greater utility. This contradicts the optimality of s .

Therefore, for sufficiently small I , every second-best optimal contract satisfies $s(q) = 0$ for almost every $q \in [0, q_0]$. ■

Lemma 4 (Effort productivity). *If $C'(0) = 0$ and*

$$u'(\bar{W}) \operatorname{cov}(q, L(q|0)) \min \left\{ L(\bar{q}|0), \frac{\operatorname{cov}(q, L(q|0))}{\bar{\mu}} \right\} > C''(0), \quad (\text{A.2})$$

where the covariance is taken under the output distribution with zero effort, then

$$\max_{d \in [0, \bar{q}]} \Pi(s_d) > \bar{\mu} \quad \text{and} \quad \max_{\alpha \in [0, 1]} \Pi(s_\alpha) > \bar{\mu}.$$

Proof of Lemma 4: We show each inequality by perturbing the relevant contract away from the zero-effort corner.

First, consider debt financing with $s_{\delta_d}(q) = \min\{q, \bar{q} - \delta_d\}$, which gives the entrepreneur a payoff $w_{\delta_d} = \max\{q - \bar{q} + \delta_d, 0\} \leq \delta_d$. With the FOA, the equilibrium effort is such that $\int u(\bar{W} + w_{\delta_d})f_e(q|e)dq = C'(e)$; expanding $u(\cdot)$ around $\bar{W}$ and using $\int f_e(q|e)dq = 0$ shows that the LHS is equal to $\frac{1}{2}u'(\bar{W})f_e(\bar{q}|0)\delta_d^2 + o(\delta_d^2)$. With $C'(0) = 0$, this gives $e(\delta_d) = \frac{u'(\bar{W})L(\bar{q}|0)f(\bar{q}|0)}{2C''(0)}\delta_d^2 + o(\delta_d^2)$. Then

$$\begin{aligned}\Pi(s_{\delta_d}) - \bar{\mu} &= \text{cov}(q, L(q|0))e(\delta_d) - \frac{1}{2}f(\bar{q}|0)\delta_d^2 + o(\delta_d^2) \\ &= \frac{f(\bar{q}|0)}{2} \left[\frac{u'(\bar{W})L(\bar{q}|0)\text{cov}(q, L(q|0))}{C''(0)} - 1 \right] \delta_d^2 + o(\delta_d^2),\end{aligned}\tag{A.3}$$

which is strictly positive for small δ_d under condition (A.2).

Second, consider unlevered equity financing with $\alpha = 1 - \delta_\alpha$, so the entrepreneur gets a fraction $\delta_\alpha q$ of output. The IC gives $e(\delta_\alpha) = \frac{u'(\bar{W})\text{cov}(q, L(q|0))}{C''(0)}\delta_\alpha + o(\delta_\alpha)$, and

$$\Pi(s_u) - \bar{\mu} = \delta_\alpha \left[\frac{u'(\bar{W}) (\text{cov}(q, L(q|0)))^2}{C''(0)} - \bar{\mu} \right] + o(\delta_\alpha),$$

which is strictly positive for small δ_α under condition (A.2). ■

Define

$$I^* := \int \hat{s}(q)f(q|\hat{e})dq$$

where $\hat{s}$ and $\hat{e}$ are the solutions to the program in equation (10), as the highest amount that an entrepreneur can raise within $\mathcal{S}_m$. In Remark 1, we also assume that effort increases pledgeable income, in the sense that the highest possible amount I that the firm can raise externally exceeds expected output under zero effort, i.e. $I^* > \bar{\mu}$.

Remark 1 (Levered equity cannot finance a large I). *If $I^* > \bar{\mu}$, then the highest possible value of the investor's claim under levered equity financing is strictly less than I^* .*

Remark 1 and Proposition 4 imply that a large financing need (when I is sufficiently high but less than I^*) can be financed by the debt-like optimal contract of Proposition 4, but not by levered equity.⁷ The reason is that the highest pledgeable income is obtained with a financing contract that gives the investor a senior claim and the entrepreneur a junior claim on firm output, so that the entrepreneur is paid zero for low output and is the residual claimant for high output. Levered equity financing does the opposite: the

⁷At $I = I^*$, the contract characterized in Proposition 4 is the only feasible financing instrument.

entrepreneur is paid first, with a senior claim, and his payoff is capped for high output. This lowers effort incentives, the expected value of the project, and pledgeable income. When the financing need is sufficiently high, levered equity financing is not feasible even though debt-like financing is possible.

Proof of Remark 1: Given D_ℓ , the levered equity contract gives the entrepreneur a payoff $w_\ell(q) = \min\{q, D_\ell\}$ which is feasible for the program of Proposition 4 in equations (44)-(47), so $\Pi(s_\ell) := \int_0^{\bar{q}} s_\ell(q) f(q|e(s_\ell)) dq \leq I^*$ for every D_ℓ . With the FOA, $\Pi(s_\ell)$ is continuous in D_ℓ in the interval $[0, \bar{q}]$, so it reaches its maximum at some D_ℓ^* .

If $D_\ell^* > 0$, then $w_\ell(q) = q > 0$ for $0 < q \leq D_\ell^*$. We know from Proposition 4 that this contract does not maximize the amount of funding that can be raised, i.e. $\Pi(s_\ell) < I^*$.

If $D_\ell^* = 0$, the contract is such that $s(q) = q$ for all q , so that $w(q) = 0$ for all q , incentive compatibility gives $e = 0$, and $\Pi(s_\ell) = \bar{\mu}$, which is strictly less than I^* by assumption.

In any case, we have $\max_{D_\ell \in [0, \bar{q}]} \Pi(s_\ell) < I^*$. ■